Book A First Course in Abstract Algebra 6^{th} Edition, Exercises

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0.4 Complex and Matrix Algebra

Example 1. addition in $\mathbb{C}$
Compute $(2+3i)+(4-7i)$

Solution. Observe that $(2+3i)+(4-7i) = 2+3i+4-7i = 6-4i$. □

Example 2. multiplication in $\mathbb{C}$
Compute $(2-5i)(8+3i)$

Solution. Observe that $(2-5i)(8+3i) = 16+6i-40i-15i^2 = 16-34i-15(-1) = 31-34i$. □

Example 3. Find all solutions in $\mathbb{C}$ of the given equation.
$z^4 = -16$.

Solution. Let S be the solution set to the equation $z^4 = -16$.
Then $S = \{z \in \mathbb{C} : z^4 = -16\}$.
Let $z \in S$.
Then $z \in \mathbb{C}$ and $z^4 = -16$.
Since $z \in \mathbb{C}$, then there exist $|z| \in \mathbb{R}$ and $\theta \in \mathbb{R}$ such that $z = |z|cis(\theta)$.
Observe that

$$\begin{aligned} 16 \cdot cis(\pi) &= -16 \\ &= z^4 \\ &= (|z|cis(\theta))^4 \\ &= |z|^4(cis(\theta))^4 \\ &= |z|^4(cis(4\theta)). \end{aligned}$$

Hence, $16 \cdot cis(\pi) = |z|^4(cis(4\theta))$, so $|z|^4 = 16$, and the angles π and 4θ differ by an integer multiple of 2π.
Since $|z|^4 = 16$, then $|z|^2 = 4$, so $|z| = 2$.
Since the angles π and 4θ differ by an integer multiple of 2π, then $4\theta - \pi = 2n\pi$, so $4\theta = 2n\pi + \pi$.

Hence, $\theta = \frac{n\pi}{2} + \frac{\pi}{4}$ for any integer n.

On the interval $[0, 2\pi)$, $\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$.

Since $z = |z|cis(\theta)$, and $|z| = 2$, and either $\theta = \frac{\pi}{4}$ or $\theta = \frac{3\pi}{4}$ or $\theta = \frac{5\pi}{4}$ or $\theta = \frac{7\pi}{4}$, then either $z = 2 \cdot cis(\frac{\pi}{4})$ or $z = 2 \cdot cis(\frac{3\pi}{4})$ or $z = 2 \cdot cis(\frac{5\pi}{4})$ or $z = 2 \cdot cis(\frac{7\pi}{4})$.

Observe that $2 \cdot cis(\frac{\pi}{4}) = 2 \cdot [\cos(\frac{\pi}{4}) + i\sin(\frac{\pi}{4})] = 2(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i) = \sqrt{2}(1+i)$.

Observe that $2 \cdot cis(\frac{3\pi}{4}) = 2 \cdot [\cos(\frac{3\pi}{4}) + i\sin(\frac{3\pi}{4})] = 2 \cdot (\frac{-1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i) = \sqrt{2}(-1+i)$.

Observe that $2 \cdot cis(\frac{5\pi}{4}) = 2 \cdot [\cos(\frac{5\pi}{4}) + i\sin(\frac{5\pi}{4})] = 2 \cdot (\frac{-1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i) = \sqrt{2}(-1-i)$.

Observe that $2 \cdot cis(\frac{7\pi}{4}) = 2 \cdot [\cos(\frac{7\pi}{4}) + i\sin(\frac{7\pi}{4})] = 2 \cdot (\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i) = \sqrt{2}(1-i)$.

Thus, either $z = \sqrt{2}(1+i)$ or $z = \sqrt{2}(-1+i)$ or $z = \sqrt{2}(-1-i)$ or $z = \sqrt{2}(1-i)$, so $z \in \{\sqrt{2}(1+i), \sqrt{2}(-1+i), \sqrt{2}(-1-i), \sqrt{2}(1-i)\}$.

Hence, $z \in S$ implies $z \in \{\sqrt{2}(1+i), \sqrt{2}(-1+i), \sqrt{2}(-1-i), \sqrt{2}(1-i)\}$, so $S \subset \{\sqrt{2}(1+i), \sqrt{2}(-1+i), \sqrt{2}(-1-i), \sqrt{2}(1-i)\}$.

We prove $\{\sqrt{2}(1+i), \sqrt{2}(-1+i), \sqrt{2}(-1-i), \sqrt{2}(1-i)\} \subset S$.

Let $\alpha = \sqrt{2}(1+i)$ and $\beta = \sqrt{2}(-1+i)$ and $\gamma = \sqrt{2}(-1-i)$ and $\delta = \sqrt{2}(1-i)$.

Then $\alpha, \beta, \gamma, \delta \in \mathbb{C}$.

We must prove $\alpha \in S$ and $\beta \in S$ and $\gamma \in S$ and $\delta \in S$.

Observe that

$$\begin{aligned}
\alpha^4 &= [\sqrt{2}(1+i)]^4 \\
&= [2 \cdot cis(\frac{\pi}{4})]^4 \\
&= 2^4 \cdot [cis(\frac{\pi}{4})]^4 \\
&= 16 \cdot cis(\pi) \\
&= 16(-1) \\
&= -16.
\end{aligned}$$

Since $\alpha \in \mathbb{C}$ and $\alpha^4 = -16$, then $\alpha \in S$.

Observe that

$$\begin{aligned}
\beta^4 &= [\sqrt{2}(-1+i)]^4 \\
&= [2 \cdot cis(\frac{3\pi}{4})]^4 \\
&= 2^4 \cdot [cis(\frac{3\pi}{4})]^4 \\
&= 16 \cdot cis(3\pi) \\
&= 16(-1) \\
&= -16.
\end{aligned}$$

Since $\beta \in \mathbb{C}$ and $\beta^4 = -16$, then $\beta \in S$.
Observe that

$$\begin{aligned}
\gamma^4 &= [\sqrt{2}(-1-i)]^4 \\
&= [2 \cdot cis(\frac{5\pi}{4})]^4 \\
&= 2^4 \cdot [cis(\frac{5\pi}{4})]^4 \\
&= 16 \cdot cis(5\pi) \\
&= 16(-1) \\
&= -16.
\end{aligned}$$

Since $\gamma \in \mathbb{C}$ and $\gamma^4 = -16$, then $\gamma \in S$.
Observe that

$$\begin{aligned}
\delta^4 &= [\sqrt{2}(1-i)]^4 \\
&= [2 \cdot cis(\frac{7\pi}{4})]^4 \\
&= 2^4 \cdot [cis(\frac{7\pi}{4})]^4 \\
&= 16 \cdot cis(7\pi) \\
&= 16(-1) \\
&= -16.
\end{aligned}$$

Since $\delta \in \mathbb{C}$ and $\delta^4 = -16$, then $\delta \in S$.

Since $\alpha \in S$ and $\beta \in S$ and $\gamma \in S$ and $\delta \in S$, then $\{\sqrt{2}(1+i), \sqrt{2}(-1+i), \sqrt{2}(-1-i), \sqrt{2}(1-i)\} \subset S$.

Since $S \subset \{\sqrt{2}(1+i), \sqrt{2}(-1+i), \sqrt{2}(-1-i), \sqrt{2}(1-i)\}$ and $\{\sqrt{2}(1+i), \sqrt{2}(-1+i), \sqrt{2}(-1-i), \sqrt{2}(1-i)\} \subset S$, then $S = \{\sqrt{2}(1+i), \sqrt{2}(-1+i), \sqrt{2}(-1-i), \sqrt{2}(1-i)\}$.

Therefore, the solution set to the equation $z^4 = -16$ is $\{\sqrt{2}(1+i), \sqrt{2}(-1+i), \sqrt{2}(-1-i), \sqrt{2}(1-i)\}$. □

Example 4. division in $\mathbb{C}$

Compute $\frac{2+3i}{1-5i}$.

Solution. Observe that

$$\begin{aligned}\frac{2+3i}{1-5i} &= \frac{2+3i}{1-5i}\cdot\frac{1+5i}{1+5i} \\ &= \frac{(2+3i)(1+5i)}{(1-5i)(1+5i)} \\ &= \frac{2+13i+15i^2}{1-25i^2} \\ &= \frac{2+13i+15(-1)}{1-25(-1)} \\ &= \frac{-13+13i}{26} \\ &= \frac{13(-1+i)}{13\cdot 2} \\ &= \frac{-1+i}{2} \\ &= \frac{-1}{2}+\frac{1}{2}i.\end{aligned}$$

Therefore, $\frac{2+3i}{1-5i} = \frac{-1}{2}+\frac{1}{2}i$. ☐

Fraleigh exercises 0.4

Exercise 5. Compute the given arithmetic expression and give the answer in the form $a+bi$ for $a,b\in\mathbb{R}$.

$(2-3i)+(4+5i)$

Solution. Observe that $(2-3i)+(4+5i) = 2-3i+4+5i = 6+2i$. ☐

Exercise 6. Compute the given arithmetic expression and give the answer in the form $a+bi$ for $a,b\in\mathbb{R}$.

$i+(5-3i)$

Solution. Observe that $i+(5-3i) = i+5-3i = 5-2i$. ☐

Exercise 7. Compute the given arithmetic expression and give the answer in the form $a+bi$ for $a,b\in\mathbb{R}$.

$(5+7i)-(3-2i)$

Solution. Observe that $(5+7i)-(3-2i) = 5+7i-3+2i = 2+9i$. ☐

Exercise 8. Compute the given arithmetic expression and give the answer in the form $a+bi$ for $a,b\in\mathbb{R}$.

$(1-3i)-(-4+2i)$

Solution. Observe that $(1-3i)-(-4+2i) = 1-3i+4-2i = 5-5i$. □

Exercise 9. Compute the given arithmetic expression and give the answer in the form $a+bi$ for $a, b \in \mathbb{R}$.
i^3

Solution. Observe that $i^3 = i^2 \cdot i = (-1)i = -i = 0-i$. □

Exercise 10. Compute the given arithmetic expression and give the answer in the form $a+bi$ for $a, b \in \mathbb{R}$.
i^4

Solution. Observe that $i^4 = (i^2)^2 = (-1)^2 = 1 = 1+0i$. □

Exercise 11. Compute the given arithmetic expression and give the answer in the form $a+bi$ for $a, b \in \mathbb{R}$.
i^{23}

Solution. Observe that

$$\begin{aligned} i^{23} &= i^{22+1} \\ &= i^{22} \cdot i \\ &= (i^2)^{11} \cdot i \\ &= (-1)^{11} \cdot i \\ &= (-1)(i) \\ &= -i \\ &= 0-i. \end{aligned}$$

Therefore, $i^{23} = 0-i$. □

Exercise 12. Compute the given arithmetic expression and give the answer in the form $a+bi$ for $a, b \in \mathbb{R}$.
$(4-i)(5+3i)$

Solution. Observe that

$$\begin{aligned} (4-i)(5+3i) &= 20+12i-5i-3(i^2) \\ &= 20+7i-3(-1) \\ &= 23+7i. \end{aligned}$$

□

Exercise 13. Compute the given arithmetic expression and give the answer in the form $a+bi$ for $a, b \in \mathbb{R}$.
$(8+2i)(3-i)$

Solution. Observe that

$$\begin{aligned}(8+2i)(3-i) &= 24-8i+6i-2i^2\\ &= 24-2i-2(-1)\\ &= 26-2i.\end{aligned}$$

□

Exercise 14. Compute the given arithmetic expression and give the answer in the form $a+bi$ for $a, b \in \mathbb{R}$.
$(2-3i)(4+i)+(6-5i)$

Solution. Observe that

$$\begin{aligned}(2-3i)(4+i)+(6-5i) &= 8+2i-12i-3i^2+6-5i\\ &= 14-15i-3(-1)\\ &= 17-15i.\end{aligned}$$

□

Exercise 15. Compute the given arithmetic expression and give the answer in the form $a+bi$ for $a, b \in \mathbb{R}$.
$(1+i)^3$

Solution. Observe that $1+i=\sqrt{2}cis(\frac{\pi}{4})$.
Using the DeMoivre formula, we have

$$\begin{aligned}(1+i)^3 &= (\sqrt{2}cis(\frac{\pi}{4}))^3\\ &= (\sqrt{2})^3 cis(\frac{3\pi}{4})\\ &= (\sqrt{2})^3(-\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}})\\ &= 2(-1+i)\\ &= -2+2i.\end{aligned}$$

□

Exercise 16. Compute the given arithmetic expression and give the answer in the form $a+bi$ for $a, b \in \mathbb{R}$.
$(1-i)^5$

Solution. Since $\mathbb{C}$ is a field, then $\mathbb{C}$ is a commutative division ring, so $\mathbb{C}$ is a commutative ring.
Thus, we can use the binomial theorem.

Observe that

$$\begin{aligned}
(1-i)^5 &= 1(1^5)(-i)^0 + 5(1)^4(-i) + 10(1^3)(-i)^2 + 10(1^2)(-i)^3 + 5(1)(-i)^4 + 1(1^0)(-i)^5 \\
&= 1 - 5i - 10 + 10i + 5 - i^5 \\
&= -4 + 5i - i^5 \\
&= -4 + 5i - i \\
&= -4 + 4i.
\end{aligned}$$

Therefore, $(1-i)^5 = -4+4i$. $\square$

Solution. Let $z = 1 - i$.

Then $z = |z|cis(\frac{7\pi}{4}) = \sqrt{2}cis(\frac{7\pi}{4})$.

Since z is in polar form, we use the DeMoivre formula.

Since $\frac{35\pi}{4} = 8\frac{3\pi}{4} = 8\pi + \frac{3\pi}{4} = 4(2\pi) + \frac{3\pi}{4}$, then the angles $\frac{35\pi}{4}$ and $\frac{3\pi}{4}$ are co-terminal.

$$\begin{aligned}
z^5 &= [\sqrt{2}cis(\frac{7\pi}{4})]^5 \\
&= (\sqrt{2})^5 cis(\frac{35\pi}{4}) \\
&= (\sqrt{2})^5 cis(\frac{3\pi}{4}) \\
&= (\sqrt{2})^5[\cos(\frac{3\pi}{4}) + i\sin(\frac{3\pi}{4})] \\
&= (\sqrt{2})^5(-\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}) \\
&= -(\sqrt{2})^4 + (\sqrt{2})^4 i \\
&= -4 + 4i.
\end{aligned}$$

Therefore, $(1-i)^5 = -4+4i$. $\square$

Exercise 17. Compute the given arithmetic expression and give the answer in the form $a + bi$ for $a, b \in \mathbb{R}$.

$\frac{7-5i}{1+6i}$

Solution. Observe that

$$\begin{aligned}
\frac{7-5i}{1+6i} &= \frac{7-5i}{1+6i}\cdot\frac{1-6i}{1-6i} \\
&= \frac{(7-5i)(1-6i)}{(1+6i)(1-6i)} \\
&= \frac{7-47i+30(i^2)}{1-36(i^2)} \\
&= \frac{7-47i-30}{1-36(-1)} \\
&= \frac{-23-47i}{37} \\
&= -\frac{23}{37}-\frac{47i}{37}.
\end{aligned}$$

Therefore, $\frac{7-5i}{1+6i} = -\frac{23}{37} - \frac{47i}{37}$. □

Exercise 18. Compute the given arithmetic expression and give the answer in the form $a+bi$ for $a, b \in \mathbb{R}$.
$\frac{i}{1+i}$

Solution. Observe that

$$\begin{aligned}
\frac{i}{1+i} &= \frac{i}{1+i}\cdot\frac{1-i}{1-i} \\
&= \frac{(i)(1-i)}{(1+i)(1-i)} \\
&= \frac{i-i^2}{1-i^2} \\
&= \frac{i-(-1)}{1-(-1)} \\
&= \frac{1+i}{2} \\
&= \frac{1}{2}+\frac{1}{2}i.
\end{aligned}$$

Therefore, $\frac{i}{1+i} = \frac{1}{2} + \frac{1}{2}i$. □

Solution. Let $z_1 = i = 0 + i = cis(\frac{\pi}{2})$.

Let $z_2 = 1 + i = \sqrt{2}cis(\frac{\pi}{4})$.

Observe that

$$\begin{aligned}
\frac{z_1}{z_2} &= \frac{1}{\sqrt{2}} \cdot cis(\frac{\pi}{2} - \frac{\pi}{4}) \\
&= \frac{1}{\sqrt{2}} \cdot cis(\frac{\pi}{4}) \\
&= \frac{1}{\sqrt{2}} \cdot [\cos(\frac{\pi}{4}) + i\sin(\frac{\pi}{4})] \\
&= \frac{1}{\sqrt{2}} \cdot (\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}) \\
&= \frac{1}{2} + \frac{1}{2}i.
\end{aligned}$$

Therefore, $\frac{i}{1+i} = \frac{1}{2} + \frac{1}{2}i.$ □

Exercise 19. Compute the given arithmetic expression and give the answer in the form $a + bi$ for $a, b \in \mathbb{R}$.

$\frac{1-i}{i}$

Solution. Observe that

$$\begin{aligned}
\frac{1-i}{i} &= \frac{1-i}{i} \cdot \frac{-i}{-i} \\
&= \frac{(1-i)(-i)}{(i)(-i)} \\
&= \frac{-i+i^2}{(-1)(-1)} \\
&= \frac{-i+(-1)}{1} \\
&= -i-1 \\
&= -1-i.
\end{aligned}$$

Therefore, $\frac{1-i}{i} = -1-i.$ □

Solution. Let $z_1 = 1 - i = \sqrt{2} \cdot cis(\frac{7\pi}{4})$.

Let $z_2 = i = 0 + i = cis(\frac{\pi}{2})$.

Observe that

$$\begin{aligned}
\frac{z_1}{z_2} &= \sqrt{2} \cdot cis(\frac{7\pi}{2} - \frac{\pi}{2}) \\
&= \sqrt{2} \cdot cis(\frac{3\pi}{4}) \\
&= \sqrt{2} \cdot [\cos(\frac{3\pi}{4}) + i\sin(\frac{3\pi}{4})] \\
&= \sqrt{2} \cdot (-\frac{1}{\sqrt{2}} + i \cdot \frac{1}{\sqrt{2}}) \\
&= -1 + i.
\end{aligned}$$

Therefore, $\dfrac{1-i}{i} = -1 - i$. □

Exercise 20. Compute the given arithmetic expression and give the answer in the form $a + bi$ for $a, b \in \mathbb{R}$.

$\dfrac{i(3+i)}{2-4i}$

Solution. Observe that

$$\begin{aligned}
\frac{i(3+i)}{2-4i} &= \frac{3i+i^2}{2(1-2i)} \\
&= \frac{1}{2} \cdot \frac{3i-1}{1-2i} \\
&= \frac{1}{2} \cdot \frac{-1+3i}{1-2i} \cdot \frac{1+2i}{1+2i} \\
&= \frac{1}{2} \cdot \frac{(-1+3i)(1+2i)}{(1-2i)(1+2i)} \\
&= \frac{1}{2} \cdot \frac{-1+i+6i^2}{1-4i^2} \\
&= \frac{1}{2} \cdot \frac{-1+i+6(-1)}{1-4(-1)} \\
&= \frac{1}{2} \cdot \frac{-7+i}{5} \\
&= \frac{1}{2} \cdot (-\frac{7}{5} + \frac{i}{5}) \\
&= -\frac{7}{10} + \frac{1}{10} i.
\end{aligned}$$

Therefore, $\dfrac{i(3+i)}{2-4i} = -\dfrac{7}{10} + \dfrac{1}{10} i$. □

Exercise 21. Compute the given arithmetic expression and give the answer in the form $a + bi$ for $a, b \in \mathbb{R}$.

$\dfrac{3+7i}{(1+i)(2-3i)}$

Solution. Observe that

$$\begin{aligned}
\frac{3+7i}{(1+i)(2-3i)} &= \frac{3+7i}{2-i-3i^2} \\
&= \frac{3+7i}{2-i-3(-1)} \\
&= \frac{3+7i}{5-i} \cdot \frac{5+i}{5+i} \\
&= \frac{(3+7i)(5+i)}{(5-i)(5+i)} \\
&= \frac{15+38i+7i^2}{25-i^2} \\
&= \frac{15+38i+7(-1)}{25-(-1)} \\
&= \frac{8+38i}{26} \\
&= \frac{2(4+19i)}{2(13)} \\
&= \frac{4+19i}{13} \\
&= \frac{4}{13} + \frac{19}{13}i.
\end{aligned}$$

Therefore, $\frac{3+7i}{(1+i)(2-3i)} = \frac{4}{13} + \frac{19}{13}i$. □

Exercise 22. Compute the given arithmetic expression and give the answer in the form $a+bi$ for $a, b \in \mathbb{R}$.
$\frac{(1-i)(2+i)}{(1-2i)(1+i)}$

Solution. Observe that

$$\begin{aligned}
\frac{(1-i)(2+i)}{(1-2i)(1+i)} &= \frac{2-i-i^2}{1-i-2i^2} \\
&= \frac{2-i-(-1)}{1-i-2(-1)} \\
&= \frac{3-i}{3-i} \\
&= 1.
\end{aligned}$$

Therefore, $\frac{(1-i)(2+i)}{(1-2i)(1+i)} = 1$. □

Exercise 23. Compute the given arithmetic expression and give the answer in the form $a+bi$ for $a, b \in \mathbb{R}$.
$|3-4i|$

Solution. The modulus of the complex number $3-4i$ is $|3-4i| = \sqrt{3^2+(-4)^2} = 5$. ☐

Exercise 24. Compute the given arithmetic expression and give the answer in the form $a+bi$ for $a, b \in \mathbb{R}$.

$|6+4i|$

Solution. The modulus of the complex number $6+4i$ is $|6+4i| = \sqrt{6^2+4^2} = \sqrt{52} = 2\sqrt{13}$. ☐

Exercise 25. Write the given complex number z in the polar form $|z|(a+bi)$, where $|a+bi| = 1$ for real numbers a and b.

Let $z = x+yi$ for some real numbers x and y.

Then $x+yi = z = |z|(a+bi)$, so $x+yi = |z|a+|z|bi$.

Hence, $x = |z|a$ and $y = |z|b$, so $a = \dfrac{x}{|z|}$ and $b = \dfrac{y}{|z|}$.

$3-4i$

Solution. Let $z = 3-4i = x+yi$.

Then $x = 3$ and $y = -4$ and $|z| = \sqrt{3^2+(-4)^2} = 5$.

Hence, $a = \dfrac{x}{|z|} = \dfrac{3}{5}$ and $b = \dfrac{y}{|z|} = \dfrac{-4}{5}$, so $z = 5(\dfrac{3}{5}+\dfrac{-4}{5}i)$.

Observe that $|a+bi| = |\dfrac{3}{5}+\dfrac{-4}{5}i| = \sqrt{(\dfrac{3}{5})^2+(\dfrac{-4}{5})^2} = 1$. ☐

Exercise 26. Write the given complex number z in the polar form $|z|(a+bi)$, where $|a+bi| = 1$ for real numbers a and b.

Let $z = x+yi$ for some real numbers x and y.

Then $x+yi = z = |z|(a+bi)$, so $x+yi = |z|a+|z|bi$.

Hence, $x = |z|a$ and $y = |z|b$, so $a = \dfrac{x}{|z|}$ and $b = \dfrac{y}{|z|}$.

$-1+i$

Solution. Let $z = -1+i = x+yi$.

Then $x = -1$ and $y = 1$ and $|z| = \sqrt{(-1)^2+1^2} = \sqrt{2}$.

Hence, $a = \dfrac{x}{|z|} = \dfrac{-1}{\sqrt{2}}$ and $b = \dfrac{y}{|z|} = \dfrac{1}{\sqrt{2}}$.

Therefore, $z = \sqrt{2}(\dfrac{-1}{\sqrt{2}}+\dfrac{1}{\sqrt{2}}i)$.

Observe that $|a+bi| = |\dfrac{-1}{\sqrt{2}}+\dfrac{1}{\sqrt{2}}i| = \sqrt{(\dfrac{-1}{\sqrt{2}})^2+(\dfrac{1}{\sqrt{2}})^2} = 1$. ☐

Exercise 27. Write the given complex number z in the polar form $|z|(a+bi)$, where $|a+bi| = 1$ for real numbers a and b.

Let $z = x+yi$ for some real numbers x and y.

Then $x+yi = z = |z|(a+bi)$, so $x+yi = |z|a+|z|bi$.

Hence, $x = |z|a$ and $y = |z|b$, so $a = \dfrac{x}{|z|}$ and $b = \dfrac{y}{|z|}$.

$12+5i$

Solution. Let $z = 12 + 5i = x + yi$.

Then $x = 12$ and $y = 5$ and $|z| = \sqrt{(12)^2 + 5^2} = 13$.

Hence, $a = \frac{x}{|z|} = \frac{12}{13}$ and $b = \frac{y}{|z|} = \frac{5}{13}$.

Therefore, $z = 13(\frac{12}{13} + \frac{5}{13}i)$.

Observe that $|a + bi| = |\frac{12}{13} + \frac{5}{13}i| = \sqrt{(\frac{12}{13})^2 + (\frac{5}{13})^2} = 1$. $\square$

Exercise 28. Write the given complex number z in the polar form $|z|(a + bi)$, where $|a + bi| = 1$ for real numbers a and b.

Let $z = x + yi$ for some real numbers x and y.

Then $x + yi = z = |z|(a + bi)$, so $x + yi = |z|a + |z|bi$.

Hence, $x = |z|a$ and $y = |z|b$, so $a = \frac{x}{|z|}$ and $b = \frac{y}{|z|}$.

$-3 + 5i$

Solution. Let $z = -3 + 5i = x + yi$.

Then $x = -3$ and $y = 5$ and $|z| = \sqrt{(-3)^2 + 5^2} = \sqrt{34}$.

Hence, $a = \frac{x}{|z|} = \frac{-3}{\sqrt{34}}$ and $b = \frac{y}{|z|} = \frac{5}{\sqrt{34}}$.

Therefore, $z = \sqrt{34}(\frac{-3}{\sqrt{34}} + \frac{5}{\sqrt{34}}i)$.

Observe that $|a + bi| = |\frac{-3}{\sqrt{34}} + \frac{5}{\sqrt{34}}i| = \sqrt{(\frac{-3}{\sqrt{34}})^2 + (\frac{5}{\sqrt{34}})^2} = 1$. $\square$

Exercise 29. Find all solutions in $\mathbb{C}$ of the given equation.

$z^4 = 1$.

Solution. Let S be the solution set to the equation $z^4 = 1$.

Then $S = \{z \in \mathbb{C} : z^4 = 1\}$.

Let $z \in S$.

Then $z \in \mathbb{C}$ and $z^4 = 1$.

Observe that

$$\begin{aligned} 0 &= z^4 - 1 \\ &= (z^2 - 1)(z^2 + 1) \\ &= (z - 1)(z + 1)(z - i)(z + i). \end{aligned}$$

Hence, $(z - 1)(z + 1)(z - i)(z + i) = 0$, so either $z = 1$ or $z = -1$ or $z = i$ or $z = -i$.

Therefore, $z \in \{1, -1, i, -i\}$.

Consequently, $z \in S$ implies $z \in \{1, -1, i, -i\}$, so $S \subset \{1, -1, i, -i\}$.

We verify each solution.

Since $1 \in \mathbb{C}$ and $1^4 = 1$, then $1 \in S$.

Since $-1 \in \mathbb{C}$ and $(-1)^4 = 1^4 = 1$, then $-1 \in S$.

Since $i \in \mathbb{C}$ and $i^4 = (i^2)^2 = (-1)^2 = 1$, then $i \in S$.

Since $-i \in \mathbb{C}$ and $(-i)^4 = i^4 = 1$, then $-i \in S$.
Hence, $\{1, -1, i, -i\} \subset S$.

Since $S \subset \{1, -1, i, -i\}$ and $\{1, -1, i, -i\} \subset S$, then $S = \{1, -1, i, -i\}$.
Therefore, the solution set to the equation $z^4 = 1$ is $\{1, -1, i, -i\}$. $\square$

Exercise 30. Find all solutions in $\mathbb{C}$ of the given equation.
$z^4 = -1$.

Solution. Let S be the solution set to the equation $z^4 = -1$.
Then $S = \{z \in \mathbb{C} : z^4 = -1\}$.
Let $z \in S$.
Then $z \in \mathbb{C}$ and $z^4 = -1$.
TODO: Start here.
$\square$

Fraleigh exercises 5.1

Exercise 31. Compute the product $[12][16]$ in the ring $(\mathbb{Z}_{24}, +, \cdot)$.

Solution. Observe that

$$\begin{aligned}
[12][16] &= [12 \cdot 16] \\
&= [24 \cdot 8] \\
&= [24][8] \\
&= [0][8] \\
&= [0].
\end{aligned}$$

Therefore, $[12][16] = [0]$ in $\mathbb{Z}_{24}$. $\square$

Exercise 32. Compute the product $[16][3]$ in the ring $(\mathbb{Z}_{32}, +, \cdot)$.

Solution. Observe that

$$\begin{aligned}
[16][3] &= [16] \cdot [2 + 1] \\
&= [16] \cdot ([2] + [1]) \\
&= [16][2] + [16][1] \\
&= [16 \cdot 2] + [16 \cdot 1] \\
&= [32] + [16] \\
&= [0] + [16] \\
&= [16].
\end{aligned}$$

Therefore, $[16][3] = [16]$ in $\mathbb{Z}_{32}$. $\square$

Exercise 33. Compute the product $[11][-4]$ in the ring $(\mathbb{Z}_{15}, +, \cdot)$.

Solution. Observe that

$$\begin{aligned} [11][-4] &= [-4] \cdot [-4] \\ &= [(-4)(-4)] \\ &= [16] \\ &= [1]. \end{aligned}$$

Therefore, $[11][-4] = [1]$ in $\mathbb{Z}_{15}$. □

Exercise 34. Compute the product $[20][-8]$ in the ring $(\mathbb{Z}_{26}, +, \cdot)$.

Solution. Observe that

$$\begin{aligned} [20][-8] &= [-6] \cdot [-8] \\ &= [(-6)(-8)] \\ &= [48] \\ &= [22]. \end{aligned}$$

Therefore, $[20][-8] = [22]$ in $\mathbb{Z}_{26}$. □