Book Abstract Algebra theory and applications, Annual Edition 2022, Exercises

Jason Sass

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Judson Chapter 1

Exercise 1. Let $x \in \mathbb{Z}$.
Show that the statement '$2x = 6$ exactly when $x = 4$' is false.

Proof. Let $x = 4$.
Then $2x = 2(4) = 8 \neq 6$, so $2x \neq 6$.

Conversely, suppose $2x = 6$.
We divide by 2 to obtain $x = 3$, so $x \neq 4$.
Therefore, the statement '$2x = 6$ exactly when $x = 4$' is false. □

Proposition 2. ***quadratic formula to find zeros of quadratic polynomial***
Let $a, b, c, x \in \mathbb{R}$.
If $ax^2 + bx + c = 0$ and $a \neq 0$, then $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

Proof. Suppose $ax^2 + bx + c = 0$ and $a \neq 0$.
Since $a \neq 0$, then divide each side of the equation $ax^2 + bx + c = 0$ by a.

Observe that

$$\begin{aligned}
ax^2 + bx + c &= 0 \\
x^2 + \frac{b}{a}x + \frac{c}{a} &= 0 \\
x^2 + \frac{b}{a}x &= -\frac{c}{a} \\
x^2 + \frac{b}{a}x + (\frac{b}{2a})^2 &= -\frac{c}{a} + (\frac{b}{2a})^2 \\
(x + \frac{b}{2a})^2 &= -\frac{c}{a} + \frac{b^2}{4a^2} \\
(x + \frac{b}{2a})^2 &= \frac{b^2 - 4ac}{4a^2} \\
x + \frac{b}{2a} &= \frac{\pm\sqrt{b^2 - 4ac}}{2a} \\
x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.
\end{aligned}$$

Therefore, $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$. $\square$

Exercise 3. Let $A = \{4, 7, 9\}$.
Let $B = \{2, 4, 5, 8, 9\}$.
Show that A is not a subset of B.

Proof. Since $7 \in A$, but $7 \notin B$, then A is not a subset of B, so $A \not\subseteq B$. $\square$

Exercise 4. Let $A = \{1, 3, 5\}$.
Let $B = \{1, 2, 3, 9\}$.
Compute $A \cup B$ and $A \cap B$.

Solution. Observe that $A \cup B = \{1, 2, 3, 5, 9\}$ and $A \cap B = \{1, 3\}$. $\square$

Exercise 5. Let A be the set of all even integers.
Let B be the set of all odd integers.
Show that A and B are disjoint sets.

Proof. Suppose $A \cap B \neq \emptyset$.
Then there exists an element in $A \cap B$.
Let n be an element of $A \cap B$.
Then $n \in A \cap B$, so $n \in A$ and $n \in B$.
Since $n \in A$, then n is an even integer, so $n = 2a$ for some integer a.
Since $n \in B$, then n is an odd integer, so $n = 2b + 1$ for some integer b.
Hence, $2a = n = 2b + 1$, so $2a = 2b + 1$.
Thus, $1 = 2a - 2b = 2(a - b)$.
Since $a - b \in \mathbb{Z}$ and $1 = 2(a - b)$, then 1 is even, so $1 = 2c$ for some integer c.

Consequently, $c = \dfrac{1}{2}$.

But, $\dfrac{1}{2} \notin \mathbb{Z}$, so this contradicts the fact that $c \in \mathbb{Z}$.

Therefore, $A \cap B = \emptyset$, so A and B are disjoint sets. $\square$

Exercise 6. Let $\mathbb{R}$ be the universal set.
Let $A = \{x \in \mathbb{R} : 0 < x \leq 3\}$.
Let $B = \{x \in \mathbb{R} : 2 \leq x < 4\}$.
Compute $A \cap B$ and $A \cup B$ and $A - B$ and $\overline{A}$.

Solution. Observe that $A \cap B = \{x \in \mathbb{R} : 2 \leq x \leq 3\} = [2, 3]$.
Observe that $A \cup B = \{x \in \mathbb{R} : 0 < x < 4\} = (0, 4)$.
Observe that $A - B = \{x \in \mathbb{R} : 0 < x < x\} = (0, 2)$.
Observe that $\overline{A} = \{x \in \mathbb{R} : x \notin A\} = \{x \in \mathbb{R} : x \leq 0 \text{ or } x > 3\} = (-\infty, 0] \cup (3, \infty)$. $\square$

Exercise 7. Let A and B be sets.
Then $(A - B) \cap (B - A) = \emptyset$.

Proof. Observe that

$$\begin{aligned}
(A - B) \cap (B - A) &= (A \cap \overline{B}) \cap (B \cap \overline{A}) \\
&= A \cap (\overline{B} \cap B) \cap \overline{A} \\
&= A \cap \overline{A} \cap (\overline{B} \cap B) \\
&= (A \cap \overline{A}) \cap (\overline{B} \cap B) \\
&= (A \cap \overline{A}) \cap (B \cap \overline{B}) \\
&= \emptyset \cap \emptyset \\
&= \emptyset.
\end{aligned}$$

Therefore, $(A - B) \cap (B - A) = \emptyset$. $\square$

Exercise 8. Let $A = \{x, y\}$.
Let $B = \{1, 2, 3\}$.
Let $C = \emptyset$.
Compute $A \times B$ and $A \times C$.

Solution. Observe that $A \times B = \{(x, 1), (x, 2), (x, 3), (y, 1), (y, 2), (y, 3)\}$ and $A \times C = \emptyset$. $\square$

Exercise 9. Let $f : \mathbb{Z} \to \mathbb{Q}$ be defined by $f(n) = \dfrac{n}{1}$.
Then f is injective, but not surjective.

Proof. Observe that $f : \mathbb{Z} \to \mathbb{Q}$ is a map.
Let $a \in \mathbb{Z}$ and $b \in \mathbb{Z}$ such that $f(a) = f(b)$.
Then $\dfrac{a}{1} = f(a) = f(b) = \dfrac{b}{1}$, so $\dfrac{a}{1} = \dfrac{b}{1}$.
Therefore, $a = b$, so f is injective. $\square$

Proof. We prove f is not surjective.

Let $a \in \mathbb{Z}$.

Then $f(a) = \frac{a}{1} = a$.

Since $a \in \mathbb{Z}$ and $\frac{1}{2} \notin \mathbb{Z}$, then $a \neq \frac{1}{2}$, so $f(a) \neq \frac{1}{2}$.

Hence, $f(a) \neq \frac{1}{2}$ for every $a \in \mathbb{Z}$.

Since $\frac{1}{2} \in \mathbb{Q}$ and $f(a) \neq \frac{1}{2}$ for every $a \in \mathbb{Z}$, then $\frac{1}{2} \in \mathbb{Q}$ and there is no $a \in \mathbb{Z}$ such that $f(a) = \frac{1}{2}$, so f is not surjective. $\square$

Exercise 10. Let $g : \mathbb{Q} \to \mathbb{Z}$ be defined by $g(\frac{p}{q}) = p$, where $\gcd(p, q) = 1$ and $q \in \mathbb{Z}^+$.

Then f is surjective, but not injective.

Proof. Observe that $g : \mathbb{Q} \to \mathbb{Z}$ is a map, since g is well-defined.

Since $\frac{1}{2} \in \mathbb{Q}$ and $\frac{1}{3} \in \mathbb{Q}$ and $\frac{1}{2} \neq \frac{1}{3}$, but $g(\frac{1}{2}) = 1 = g(\frac{1}{3})$, then g is not injective. $\square$

Proof. Let $b \in \mathbb{Z}$.

Since $b \in \mathbb{Z}$ and $1 \in \mathbb{Z}$ and $1 \neq 0$, then $\frac{b}{1} \in \mathbb{Q}$.

Since $b \in \mathbb{Z}$, then $\gcd(b, 1) = 1$.

Since $\frac{b}{1} \in \mathbb{Q}$ and $\gcd(b, 1) = 1$ and $1 \in \mathbb{Z}^+$, then $g(\frac{b}{1}) = b$, so there is $\frac{b}{1} \in \mathbb{Q}$ such that $g(\frac{b}{1}) = b$.

Therefore, for every $b \in \mathbb{Z}$, there is $\frac{b}{1} \in \mathbb{Q}$ such that $g(\frac{b}{1}) = b$, so g is surjective. $\square$

Judson 1.3 Reading Questions

Exercise 11. What do relations and mappings have in common?

Solution. Both relations and maps are sets of ordered pairs.

A relation from set A to set B is a subset of the Cartesian product $A \times B$.

Similarly, a map(function) from set A to set B is a subset of the Cartesian product $A \times B$. $\square$

Exercise 12. What makes relations and mappings different?

Solution. A relation is any set of ordered pairs, but a map is a special type of relation.

In a map $f : A \to B$, if $f(a) = b$ and $f(a) = b'$, then $b = b'$.

However, there is no such requirement for a relation. $\square$

Judson 1.4 Exercises End of Chapter 1

Exercise 13. Let $A = \{x : x \in \mathbb{N} \text{ and } x \text{ is even}\}$.
Let $B = \{x : x \in \mathbb{N} \text{ and } x \text{ is prime}\}$.
Let $C = \{x : x \in \mathbb{N} \text{ and } x \text{ is a multiple of } 5\ \}$.
Describe the below sets.
a. $A \cap B$
b. $B \cap C$
c. $A \cup B$
d. $A \cap (B \cup C)$

Solution. For part a.
Observe that $A \cap B = \{x : x \in A \text{ and } x \in B\}$.
Therefore, $A \cap B = \{2\}$, the set of all natural numbers that are even primes. $\square$

Solution. For part b.
Observe that $B \cap C = \{x : x \in B \text{ and } x \in C\}$.
Therefore, $B \cap C = \{5\}$, the set of all natural numbers that are prime and multiples of 5. $\square$

Solution. For part c.
Observe that $A \cup B = \{x : x \in A \text{ or } x \in B\}$.
Therefore, $A \cup B = \{2, 3, 4, 5, 6, 7, 8, 10, 11, 12, 13, 14, 16, 17, 18, 19, 20, ...\}$, the set of all natural numbers that are even or prime. $\square$

Solution. For part d.
Observe that $A \cap B = \{2\}$, the set of all natural numbers that are even primes.
Observe that $A \cap C = \{10, 20, 30, 40, 50, ...\}$, the set of all natural numbers that are multiples of 10.
Observe that

$$\begin{aligned} A \cap (B \cup C) &= (A \cap B) \cup (A \cap C) \\ &= \{2\} \cup \{10, 20, 30, 40, 50, ...\} \\ &= \{2, 10, 20, 30, 40, 50, ...\}. \end{aligned}$$

Therefore, $A \cap (B \cup C) = \{2, 10, 20, 30, 40, 50, ...\}$, the set of all natural numbers that are either 2 or multiples of 10. $\square$

Exercise 14. Let $A = \{a, b, c\}$.
Let $B = \{1, 2, 3, \}$.
Let $C = \{x\}$.
Let $D = \emptyset$.
Compute the following sets.
a. $A \times B$.
b. $B \times A$.
c. $A \times B \times C$.
d. $A \times D$.

Solution. For part a.

Observe that $A \times B = \{(a,1),(a,2),(a,3),(b,1),(b,2),(b,3),(c,1),(c,2),(c,3)\}$. $\square$

Solution. For part b.

Observe that $B \times A = \{(1,a),(1,b),(1,c),(2,a),(2,b),(2,c),(3,a),(3,b),(3,c)\}$. $\square$

Solution. For part c.

Observe that $A \times B \times C = \{(a,1,x),(a,2,x),(a,3,x),(b,1,x),(b,2,x),(b,3,x),(c,1,x),(c,2,x),(c,3,x)\}$. $\square$

Solution. For part d.

Observe that $A \times D = \emptyset$. $\square$

Exercise 15. Find an example of two nonempty sets A and B for which $A \times B = B \times A$.

Solution. Let $A = \{1,2\}$ and $B = \{1,2\}$.

Since $1 \in A$, then $A \neq \emptyset$.

Since $2 \in B$, then $B \neq \emptyset$.

Observe that $A \times B = \{(1,1),(1,2),(2,1),(2,2)\} = B \times A$. $\square$

Exercise 16. Let A and B be sets.

Then $(A \cap B) - B = \emptyset$.

Proof. Observe that

$$\begin{aligned}
(A \cap B) - B &= (A \cap B) \cap \overline{B} \\
&= A \cap (B \cap \overline{B}) \\
&= A \cap \emptyset \\
&= \emptyset
\end{aligned}$$

$\square$

Exercise 17. Let A and B be sets.

Then $(A \cup B) - B = A - B$.

Proof. Observe that

$$\begin{aligned}
(A \cup B) - B &= (A \cup B) \cap \overline{B} \\
&= (A \cap \overline{B}) \cup (B \cap \overline{B}) \\
&= (A \cap \overline{B}) \cup \emptyset \\
&= A \cap \overline{B} \\
&= A - B
\end{aligned}$$

$\square$

Proposition 18. *Let A, B, C be sets.*

Then $A \cap (B - C) = (A \cap B) - (A \cap C)$.

Proof. Observe that

$$\begin{aligned}
(A \cap B) - (A \cap C) &= (A \cap B) \cap \overline{A \cap C} \\
&= (A \cap B) \cap (\overline{A} \cup \overline{C}) \\
&= (A \cap B \cap \overline{A}) \cup (A \cap B \cap \overline{C}) \\
&= (A \cap \overline{A} \cap B) \cup (A \cap B \cap \overline{C}) \\
&= (\emptyset \cap B) \cup (A \cap B \cap \overline{C}) \\
&= \emptyset \cup (A \cap B \cap \overline{C}) \\
&= A \cap B \cap \overline{C} \\
&= A \cap (B \cap \overline{C}) \\
&= A \cap (B - C)
\end{aligned}$$

□

Exercise 19. Let $f : \mathbb{Q} \to \mathbb{Q}$ be defined by $f(\frac{p}{q}) = \frac{p+1}{p-2}$.

Determine if f is a map.

Solution. TODO: Start here. □