Group Theory Exercises 5

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Normal Subgroups

Exercise 1. Let G be a group.
If $H < G$ and $[G : H] = 2$, then $H \lhd G$.

Proof. Suppose $H < G$ and $[G : H] = 2$.
Since $[G : H] = 2$, then there are two distinct left cosets of H in G and there are two distinct right cosets of H in G.
One of the cosets must be H and the other coset is $G - H$.
Thus, the set of left cosets is $\{H, G - H\}$ and the set of right cosets is $\{H, G - H\}$.
Let $g \in G$.
Either $g \in H$ or $g \notin H$.
We consider these cases separately.
Case 1: Suppose $g \in H$.
Since $g \in gH$ and $g \in H$, then $gH = H$.
Since $g \in Hg$ and $g \in H$, then $Hg = H$.
Therefore, $gH = H = Hg$, so $gH = Hg$.
Case 2: Suppose $g \notin H$.
Since $g \in G$ and $g \notin H$, then $g \in G - H$.
Since $g \in gH$ and $g \in G - H$, then $gH = G - H$.
Since $g \in Hg$ and $g \in G - H$, then $Hg = G - H$.
Therefore, $gH = G - H = Hg$, so $gH = Hg$.
Hence, in all cases, $gH = Hg$.
Since g is arbitrary, then $gH = Hg$ for all $g \in G$,
Therefore, $H \lhd G$. □

Exercise 2. If G has exactly one subgroup H of order k, then $H \lhd G$.

Solution. Let G be a group.
Suppose there exists a unique subgroup H of G of order k.
We must prove $H \lhd G$. □

Proof. Let G be a group.
Suppose there exists a unique subgroup H of G of order k.
□

Exercise 3. Let p be prime.

If $H < G$ and $|H| = p$ and $|G| = p^2$, then $H \lhd G$.

Moreover, G is abelian.

Solution. Suppose $H < G$ and $|H| = p$ and $|G| = p^2$.

We must prove $H \lhd G$ and G is abelian. □

Proof. Suppose $H < G$ and $|H| = p$ and $|G| = p^2$. Since $H < G$ and $|H| = p$, then H is a group of prime order. Every group of prime order is cyclic, so H is cyclic. Hence, there exists $h \in H$ such that $H = \{h^k : k \in \mathbb{Z}\}$.

Let $a \in G$. Since $a \in aH$, then $a = ah'$ for some $h' \in H$. Since H is cyclic, then $h' = h^k$ for some integer k. Thus, $a = ah^k$.

Since $a \in Ha$, then $a = h''a$ for some $h'' \in H$. Since H is cyclic, then $h'' = h^m$ for some integer m. Thus, $a = h^m a$.

Hence, $ah^k = a = h^m a$, so $ah^k = h^m a$. Therefore, $ah^k a^{-1} = h^m$. Since $h^m \in H$, then $ah^k a^{-1} \in H$. Therefore, $H \lhd G$.

Hence, the quotient group $\frac{G}{H}$ exists and has order $[G : H] = \frac{|G|}{|H|} = \frac{p^2}{p} = p$. Thus, $\frac{G}{H}$ is a group of prime order, so $\frac{G}{H}$ is cyclic. Therefore, there exists $gH \in \frac{G}{H}$ such that $\frac{G}{H} = \{(gH)^k : k \in \mathbb{Z}\}$. Hence, there exists $g \in G$ such that $\frac{G}{H} = \{g^k H : k \in \mathbb{Z}\}$.

Let $aH, bH \in \frac{G}{H}$. Then $a, b \in G$ and there exist integers m and n such that $aH = g^m H$ and $bH = g^n H$.

Every cyclic group is abelian, so $\frac{G}{H}$ is abelian. □

Exercise 4. Is $A_4 \lhd S_4$?

Solution. Clearly, $A_4 < S_4$.

To prove $A_4 \lhd S_4$, we must prove for all $\alpha \in S_4$ and all $\beta \in A_4$, $\alpha\beta\alpha^{-1} \in A_4$.

Let $\alpha \in S_4$ and $\beta \in A_4$.

Since $\beta \in A_4$, then β is even.

Either α is odd or α is even.

We consider these cases separately.

Case 1: Suppose α is even.

The parity of a permutation is the same as the parity of its inverse, so α^{-1} is even.

The composition of permutations of the same parity is even, so $\alpha\beta$ is even.

Thus, $\alpha\beta\alpha^{-1}$ is even.

Case 2: Suppose α is odd.

The parity of a permutation is the same as the parity of its inverse, so α^{-1} is odd.

The composition of two permutations of opposite parity is odd, so $\alpha\beta$ is odd.

The composition of permutations of the same parity is even, so $\alpha\beta\alpha^{-1}$ is even.

Therefore, in all cases, $\alpha\beta\alpha^{-1}$ is even.

Since $\alpha\beta\alpha^{-1} \in S_4$ and $\alpha\beta\alpha^{-1}$ is even, then $\alpha\beta\alpha^{-1} \in A_4$.

Thus, $A_4 \lhd S_4$.

Hence, the quotient group, the group of cosets $\frac{S_4}{A_4}$ exists and $|\frac{S_4}{A_4}| = [S_4 : A_4] = \frac{|S_4|}{|A_4|} = \frac{|S_4|}{|S_4|/2} = 2$.

Therefore, $\frac{S_4}{A_4} = \{A_4, (12)A_4\}$. □

Exercise 5. Is $D_4 \lhd S_4$?

Solution. Since D_4 is isomorphic to a subgroup of S_4, then there exists $H < S_4$ such that $D_4 \cong H$. So, essentially, $D_4 < S_4$. The number of distinct left cosets of D_4 in S_4 is $[S_4 : D_4] = \frac{|S_4|}{|D_4|} = 24/8 = 3$. Hence, there are 3 distinct left cosets of D_4 in S_4, each containing 8 permutations. Computations show that the left cosets of D_4 in S_4 are:

$D_4 = \{(1), (1234), (13)(24), (1432), (12)(34), (14)(23), (24), (13)\}$

$L_1 = \{(34), (124), (1423), (132), (12), (1324), (234), (143)\}$

$L_2 = \{(23), (243), (123), (1243), (1342), (134), (142), (14)\}$.

Similarly, the right cosets of D_4 in S_4 are

$D_4 = \{(1), (1234), (13)(24), (1432), (12)(34), (14)(23), (24), (13)\}$

$R_1 = \{(34), (123), (1324), (142), (12), (1423), (243), (134)\}$

$R_2 = \{(23), (124), (1342), (314), (1243), (14), (234), (132)\}$.

Since $L_1 \neq R_1$, then $(12)D_4 \neq D_4(12)$, so D_4 is not a normal subgroup of S_4. □

Exercise 6. Prove or disprove:

If H is a normal subgroup of G such that H and $\frac{G}{H}$ are abelian, then G is abelian.

Solution. We can easily devise a counterexample to this assertion. Ie, this assertion is false.

We disprove this assertion.

Let $G = S_3 = \{(1), (12), (13), (23), (123), (132)\}$ and $H = A_3 = \{(1), (123), (132)$.

Clearly, $A_3 \lhd S_3$ since $A_n \lhd S_n$.

Thus, the quotient group $\frac{S_3}{A_3}$ exists and is $\{A_3, \{(12), (13), (23)\}\}$.

Hence, $\frac{S_3}{A_3}$ is a group of order 2.

Any group of order 2 is abelian, so $\frac{S_3}{A_3}$ is abelian.

Since $|A_3| = \frac{3!}{2} = 3$ and any group of order 3 is abelian, then A_3 is abelian.

However, S_3 is not abelian. □

Exercise 7. Prove or disprove:

If H and $\frac{G}{H}$ are cyclic, then G is cyclic.

Solution. We can easily devise a counterexample to this assertion.

Let $G = S_3 = \{(1), (12), (13), (23), (123), (132)\}$ and $H = A_3 = \{(1), (123), (132)$.

Observe that $A_3 \lhd S_3$, so the quotient group $\frac{S_3}{A_3}$ exists and is $\{A_3, \{(12), (13), (23)\}\}$.

Since A_3 is a group of order 3 and $\frac{S_3}{A_3}$ is a group of order 2 and any group of prime order is cyclic, then A_3 and $\frac{S_3}{A_3}$ are cyclic.

Every cyclic group is abelian, so if a group is cyclic, then it is abelian.

Hence, if a group is not abelian, then it cannot be cyclic.

Since S_3 is not abelian, then this implies S_3 is not cyclic.

Therefore, A_3 and $\frac{S_3}{A_3}$ are cyclic, but S_3 is not cyclic, disproving this assertion. □

Exercise 8. If G is an abelian group, then the center of the group is G.

Proof. Let G be an abelian group.

Let $Z(G) = \{x \in G : (\forall g \in G)(xg = gx)\}$ be the center of G.

We must prove $Z(G) = G$.

We prove $Z(G) \subset G$.

Since $Z(G) = \{x \in G : (\forall g \in G)(xg = gx)\}$, then $Z(G) \subset G$.

We prove $G \subset Z(G)$.

Let $x \in G$.

Let $g \in G$.

Since G is abelian and $x \in G$ and $g \in G$, then $xg = gx$, so $xg = gx$ for all $g \in G$.

Since $x \in G$ and $xg = gx$ for all $g \in G$, then $x \in Z(G)$.

Therefore, $x \in G$ implies $x \in Z(G)$, so $G \subset Z(G)$.

Since $Z(G) \subset G$ and $G \subset Z(G)$, then $Z(G) = G$, as desired. □

Exercise 9. Compute the center of the symmetric group $(S_3, \circ)$.

Solution. The center of the group S_3 is the set $Z(S_3) = \{f \in S_3 : (\forall g \in S_3)(fg = gf)\}$.

Since the identity permutation $(1) \in S_3$ satisfies $(1)g = g(1)$ for all $g \in S_3$, then $(1) \in Z(S_3)$.

By inspecting the Cayley table for S_3, we see that no other permutation of S_3 is in the center of S_3.

Therefore, $Z(S_3) = \{(1)\}$. □