Group Theory Exercises 6

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Homomorphisms

Exercise 1. Let $(G, *)$ be an abelian group.
Let $n \in \mathbb{Z}$ be fixed.
Let $f_n : G \to G$ be defined by $f(g) = g^n$ for all $g \in G$.
Then f_n is a group homomorphism.

Proof. Clearly, f_n is a function.
Let $a, b \in G$.
Then

$$\begin{aligned} f(ab) &= (ab)^n \\ &= a^n b^n \\ &= f(a)f(b). \end{aligned}$$

Therefore, $f(ab) = f(a)f(b)$, so f is a group homomorphism. □

Exercise 2. Let $(G, *)$ be an abelian group.
Let $h : G \to G$ be defined by $h(g) = g^{-1}$ for all $g \in G$.
Then h is a group homomorphism.

Proof. Clearly, h is a function.
Let $a, b \in G$.
Observe that

$$\begin{aligned} h(ab) &= (ab)^{-1} \\ &= b^{-1}a^{-1} \\ &= a^{-1}b^{-1} \\ &= h(a)h(b). \end{aligned}$$

Therefore, $h(ab) = h(a)h(b)$, so h is a group homomorphism. □

Exercise 3. Let $\phi : G \to G'$ be a group homomorphism.
If G' is finite, then $\phi(G)$ is finite and $|\phi(G)|$ divides $|G'|$.

Solution. We should realize that to each element $\phi(g)$ of $\phi(G)$ there corresponds a set, the left coset gH with representative g, where $H = \ker(\phi)$. □

Proof. Suppose G is finite.

Let $\phi(g) \in \phi(G)$.

Then $g \in G$.

Let $K = \ker(\phi)$.

Then $K < G$.

The preimage of $\phi(g)$ is the left coset of K in G with representative g.

Thus, $\phi^{-1}(\phi(g)) = \{x \in G : \phi(x) = \phi(g)\} = gK$.

Hence, for each element of $\phi(G)$ there corresponds exactly one left coset.

Therefore, the number of elements in $\phi(G)$ is the number of left cosets of K in G.

Since G is finite, then there exist a finite number of subsets of G.

In particular, there exist a finite number of left cosets of K in G, so $[G : K]$ is finite.

Thus, $[G : K] = |\phi(G)|$, so $\phi(G)$ is finite.

Since G is a finite group, by LaGrange's theorem, $|G| = |K| * [G : K]$.

Therefore, $[G : K]$ divides $|G|$, so $|\phi(G)|$ divides $|G|$. □

Proof. Suppose G' is finite.

The image of a homomorphism is a subgroup of G', so $\phi(G) < G'$.

Thus, $\phi(G) \subset G'$.

Since every subset of a finite set is finite and G' is finite, then $\phi(G)$ is finite.

Since G' is a finite group and $\phi(G)$ is a subgroup of G', then by LaGrange's theorem, the order of $\phi(G)$ divides the order of G'.

Therefore, $|\phi(G)|$ divides $|G'|$. □

Exercise 4. Let G be an abelian group and $n \in \mathbb{N}$.

Then $\phi : G \to G$ defined by $g \to g^n$ is a group homomorphism.

Proof. Clearly, ϕ is a function.

Let $a, b \in G$.

Then

$$\begin{aligned} \phi(ab) &= (ab)^n \\ &= a^n b^n \\ &= \phi(a)\phi(b). \end{aligned}$$

Hence, ϕ is a group homomorphism.

The kernel is $\ker(\phi) = \{g \in G : g^n = e\} < G$ and the image of ϕ is $Im(\phi) = \{g^n : g \in G\} < G$. □

Exercise 5. Let G be a group of prime order.

If $\phi : G \to G'$ is a group homomorphism, then either ϕ is the trivial homomorphism or ϕ is injective.

Proof. Suppose $\phi : G \to G'$ is a group homomorphism.

Since G is a group of prime order, then G is cyclic, so the only subgroups of G are G itself and the trivial group.

Let $K = \ker(\phi)$.

Since ϕ is a homomorphism, then $K < G$.

Hence, either $K = G$ or $K = \{e\}$, where e is the identity of G.

We consider these cases separately.

Case 1: Suppose $K = G$.

Then ϕ maps every element of G to the identity of G'.

Hence, ϕ is the trivial homomorphism.

Case 2: Suppose $K = \{e\}$.

Since ϕ is injective iff $K = \{e\}$ and $K = \{e\}$, then ϕ is injective.

Hence, in all cases either ϕ is the trivial homomorphism or ϕ is injective. □

Exercise 6. For groups $(\mathbb{R}^+, \cdot)$ and $(\mathbb{R}, +)$, the function $\phi : \mathbb{R}^+ \to \mathbb{R}$ defined by $\phi(x) = \log x$ for all $x \in \mathbb{R}^+$ is a group homomorphism.

Solution. Let $x, y \in \mathbb{R}^+$.

Then $\phi(xy) = \log(xy) = \log(x) + \log(y) = \phi(x) + \phi(y)$.

Therefore, ϕ is a group homomorphism.

Observe that $\phi(1) = \log 1 = 0$, so ϕ preserves the group identity. In other words, the multiplicative identity $1 \in \mathbb{R}^+$ maps to the additive identity $0 \in \mathbb{R}$.

Let $a \in \mathbb{R}^+$.

Then $\phi(a^{-1}) = \log(a^{-1}) = \log(\frac{1}{a}) = \log 1 - \log a = 0 - \log a = -\log a = -\phi(a)$, so ϕ preserves inverses. In other words, the multiplicative inverse of $a \in \mathbb{R}^+$ maps to the additive inverse of the image of a.

Let $a \in \mathbb{R}^+$ and $k \in \mathbb{Z}$.

Then $\phi(a^k) = \log(a^k) = k \log a = k\phi(a)$, so ϕ preserves powers of $a \in \mathbb{R}^+$. In other words, powers of $a \in \mathbb{R}^+$ map to multiples of the image of a.

The image of ϕ is the set $\phi(\mathbb{R}^+) = \{\phi(x) \in \mathbb{R} : x \in \mathbb{R}^+\} = \{\log x \in \mathbb{R} : x \in \mathbb{R}^+\} = \mathbb{R}$.

The kernel of ϕ is the set $\ker(\phi) = \{x \in \mathbb{R}^+ : \phi(x) = 0\} = \{x \in \mathbb{R}^+ : \log x = 0\} = \{1\}$.

Since log is one to one and onto, then log is bijective, so ϕ is bijective.

Therefore, ϕ is a group isomorphism. □

Exercise 7. Let $(\mathbb{Z}, +)$ be the additive group of integers.

Let $\phi : \mathbb{Z} \to \mathbb{Z}$ be defined by $\phi(n) = 7n$.

Is ϕ a homomorphism?

Solution. Clearly, ϕ is a function.

Let $a, b \in \mathbb{Z}$.

Then $\phi(a+b) = 7(a+b) = 7a+7b = \phi(a)+\phi(b)$, so ϕ is a group homomorphism.

Thus, ϕ maps the identity 0 to 0, so $\phi(0) = 0$.

Hence, $0 \in \ker(\phi)$.

Suppose $g \in \ker(\phi)$.

Then $g \in \mathbb{Z}$ and $\phi(g) = 0$, so $7g = 0$.

Thus, $g = 0$, so $g \in \ker(\phi)$ implies $g \in \{0\}$.

Therefore, $\ker(\phi) \subset \{0\}$.

Since $\ker(\phi) \subset \{0\}$ and $0 \in \ker(\phi)$, then $\ker(\phi) = \{0\}$.

Hence, ϕ is injective.

Observe that $Im(\phi) = \phi(\mathbb{Z}) = \{\phi(g) : g \in \mathbb{Z}\} = \{7g : g \in \mathbb{Z}\} = 7\mathbb{Z} = \langle 7 \rangle$.

Since ϕ is injective, then $\mathbb{Z} \cong \phi(\mathbb{Z})$, so $\mathbb{Z} \cong 7\mathbb{Z}$. $\square$

Exercise 8. Let $f : GL_2(\mathbb{R}) \to \mathbb{R}$ be defined by

$$f \begin{bmatrix} a & b \\ c & d \end{bmatrix} = a + d.$$

Is f a homomorphism?

Solution. Clearly, f is a function.

Suppose f is a homomorphism.

Then f maps the multiplicative identity of $GL_2(\mathbb{R})$ to the additive identity of $\mathbb{R}$.

Hence, $f \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 0$.

But,

$$f \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 1 + 1 = 2 \neq 0.$$

Hence, f cannot be a homomorphism. $\square$

Exercise 9. Find all possible homomorphisms ϕ from $(\mathbb{Z}_7, +)$ to $(\mathbb{Z}_{12}, +)$.

Solution. Suppose $\phi : \mathbb{Z}_7 \to \mathbb{Z}_{12}$ is a group homomorphism. Let $K = \ker(\phi)$. Then $K < \mathbb{Z}_7$. Since $\mathbb{Z}_7$ is a group of prime order, then $\mathbb{Z}_7$ is cyclic, so the only subgroups of $\mathbb{Z}_7$ are $\mathbb{Z}_7$ itself and the trivial group $\{0\}$. Hence, either $K = \mathbb{Z}_7$ or $K = \{0\}$.

Suppose ϕ is injective. Then $\mathbb{Z}_7 \cong \phi(\mathbb{Z}_7)$. Thus, $|\mathbb{Z}_7| = |\phi(\mathbb{Z}_7)|$, so $7 = |\phi(\mathbb{Z}_7)| = |Im(\phi)|$. Since ϕ is a homomorphism, then $Im(\phi) < \mathbb{Z}_{12}$. By Lagrange, $|Im(\phi)|$ divides $|\mathbb{Z}_{12}|$, so $7|12$, a contradiction. Hence, ϕ cannot be injective.

Since ϕ is injective iff $\ker(\phi) = \{0\}$, then ϕ is not injective iff $\ker(\phi) \neq \{0\}$. Since ϕ is not injective, then $\ker(\phi) \neq \{0\}$, so $K \neq \{0\}$.

Suppose $K = \mathbb{Z}_7$. Then ϕ is the trivial homomorphism.

Thus, there is only one homomorphism from $\mathbb{Z}_7$ to $\mathbb{Z}_{12}$, the trivial homomorphism. $\square$

Exercise 10. Find all possible homomorphisms from $(\mathbb{Z}_{24}, +)$ to $(\mathbb{Z}_{18}, +)$.

Solution. Let $\phi : \mathbb{Z}_{24} \to \mathbb{Z}_{18}$ be a group homomorphism. Let $K = \ker(\phi)$. Since ϕ is a homomorphism, then $K < \mathbb{Z}_{24}$ and $Im(\phi) < \mathbb{Z}_{18}$.

Since $|\mathbb{Z}_{24}| = 24 > 18 = |\mathbb{Z}_{18}|$, then ϕ cannot be injective by the pigeonhole principle. Since ϕ is injective iff $\ker(\phi) = \{0\}$, then ϕ is not injective iff $\ker(\phi) \neq \{0\}$. Since ϕ is not injective, then $\ker(\phi) \neq \{0\}$. Hence, the kernel of ϕ cannot be the trivial group.

The subgroups of $\mathbb{Z}_{24}$ have orders $1, 2, 3, 4, 6, 8, 12, 24$ and the subgroups of $\mathbb{Z}_{18}$ have orders $1, 2, 3, 6, 9, 18$. Since $K \neq \{0\}$, then K cannot have order 1. Thus, the possible orders of K are $2, 3, 4, 6, 8, 12, 24$.

Since $\mathbb{Z}_{24}$ is a finite group and ϕ is a homomorphism, then $|\mathbb{Z}_{24}| = |K| * |Im(\phi)|$. Hence, $24 = |K| * |Im(\phi)|$, so $|Im(\phi)|$ divides 24. Since $\mathbb{Z}_{18}$ is finite, then by LaGrange's theorem, $|Im(\phi)|$ divides $|\mathbb{Z}_{18}|$, so $|Im(\phi)|$ divides 18. Thus, $|Im(\phi)|$ divides 24 and $|Im(\phi)|$ divides 18, so $|Im(\phi)|$ is a common divisor of 24 and 18. Hence, the order of $Im(\phi)$ is either $1, 2, 3, 6$.

Every subgroup of a cyclic group is cyclic, so $K < \mathbb{Z}_{24}$ is cyclic and $Im(\phi) < \mathbb{Z}_{18}$ is cyclic. Hence $K = \langle k \rangle$ for some integer k and $Im(\phi) = \langle m \rangle$ for some integer m. The order of an element is the order of the cyclic subgroup generated by that element.

Let $k \in \mathbb{Z}_{24}$. Then $|k| = |K| = \frac{|\mathbb{Z}_{24}|}{\gcd(m, |\mathbb{Z}_{24}|)} = \frac{24}{\gcd(m, 24)}$.

Let $m \in \mathbb{Z}_{18}$. Then $|m| = |Im(\phi)| = \frac{|\mathbb{Z}_{18}|}{\gcd(m, |\mathbb{Z}_{18}|)} = \frac{18}{\gcd(m, 18)}$.

If $|Im(\phi)| = 1$, then $Im(\phi) = \langle 0 \rangle = \{0\}$ and $|K| = 24$, so $K = \mathbb{Z}_{24}$. This corresponds to the trivial homomorphism.

If $|Im(\phi)| = 2$, then $Im(\phi) = \langle 9 \rangle = \{0, 9\}$ and $|K| = 12$, so $K = \langle 2 \rangle = \{0, 2, 4, 6, 8, 10, 12, 14, 16, 18, 20, 22\}$.

If $|Im(\phi)| = 3$, then $Im(\phi) = \langle 6 \rangle = \{0, 6, 12\}$ and $|K| = 8$, so $K = \langle 3 \rangle = \{0, 3, 6, 9, 12, 15, 18, 21\}$.

If $|Im(\phi)| = 6$, then $Im(\phi) = \langle 3 \rangle = \{0, 3, 6, 9, 12, 15\}$ and $|K| = 4$, so $K = \langle 6 \rangle = \{0, 6, 12, 18\}$.

Thus, the possible homomorphisms ϕ are:

$K = \mathbb{Z}_{24}$ and $Im(\phi) = \{0\}$ (the trivial homomorphism) or

$K = \langle 2 \rangle = \{0, 2, 4, 6, 8, 10, 12, 14, 16, 18, 20, 22\}$ and $Im(\phi) = \langle 9 \rangle = \{0, 9\}$ or

$K = \langle 3 \rangle = \{0, 3, 6, 9, 12, 15, 18, 21\}$ and $Im(\phi) = \langle 6 \rangle = \{0, 6, 12\}$ or

$K = \langle 4 \rangle = \{0, 6, 12, 18\}$ and $Im(\phi) = \langle 3 \rangle = \{0, 3, 6, 9, 12, 15\}$. $\square$

Exercise 11. Find all possible homomorphisms from $(\mathbb{Z}, +)$ to $(\mathbb{Z}_{12}, +)$.

Solution. Let $\phi : \mathbb{Z} \to \mathbb{Z}_{12}$ be a group homomorphism. Let $K = \ker(\phi)$ and $Im(\phi) = \phi(\mathbb{Z})$. Then $K < \mathbb{Z}$ and $Im(\phi) < \mathbb{Z}_{12}$. The only subgroups of $\mathbb{Z}_{12}$ are the finite cyclic groups of order n such that $n|12$, by Lagrange. Hence, the subgroups of $\mathbb{Z}_{12}$ have order $1, 2, 3, 4, 6, 12$. Thus, $|Im(\phi)| = n = 1, 2, 3, 4, 6, 12$. Since $K \lhd \mathbb{Z}$, then the number of cosets of K in $\mathbb{Z}$ is $[\mathbb{Z} : K] = n$ and since $K < \mathbb{Z}$, then $K = \langle n \rangle$.

Thus, the possible homomorphisms ϕ are:

$K = \mathbb{Z}$ and $Im(\phi) = \{0\}$ or

$K = \langle 2 \rangle$ and $Im(\phi) = \langle 6 \rangle = \{0, 6\}$ or

$K = \langle 3 \rangle$ and $Im(\phi) = \langle 4 \rangle = \{0, 4, 8\}$ or

$K = \langle 4 \rangle$ and $Im(\phi) = \langle 3 \rangle = \{0, 3, 6, 9\}$ or
$K = \langle 6 \rangle$ and $Im(\phi) = \langle 2 \rangle = \{0, 2, 4, 6, 8, 10\}$ or
$K = \langle 12 \rangle$ and $Im(\phi) = \mathbb{Z}_{12}$. $\square$