Combinatorics Exercises

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Exercise 1. Either a math faculty or math major can be chosen as a committee rep. How many different choices exist if there are 37 faculty members and 83 majors?

Solution. We must find the number of different choices to select a committee rep.

Let F = the set of math faculty. Then $|F| = 37$.

Let M = the set of math majors. Then $|M| = 83$.

Let n = the number of different choices to select a committee rep = the number of different ways to choose a committee rep from either F or M = the number of ways to choose a different committee rep from EITHER F OR M = the number of ways to choose a different committee rep from $F \cup M$ = the number of different committee reps that can be selected from $F \cup M$ = the size of $F \cup M = |F \cup M|$.

How are F and M related? No math faculty member is a math major and no math major is a member of the math faculty. Thus, $F \cap M = \emptyset$.

Let the universal set be $S = F \cup M$. Then $\{F, M\}$ is a partition of S. Hence, we can use the addition principle to count the different choices.

Thus, $|F \cup M| = |F| + |M| = 37 + 83 = 120$.

In general, $|F \cup M| = |F| + |M| - |F \cap M| = 37 + 83 - 0 = 120$.

We can prove why $F \cap M = \emptyset$. The proof is based upon two facts:

1. A math faculty member is not a math major.
2. A math major is not a member of the math faculty.

Let $x \in F$. Since a math faculty member is not a math major, then $x \notin M$. Thus, $x \in F \to x \notin M$. Since x is arbitrary, then $\forall x \in F. x \notin M$.

Let $y \in M$. Since a math major is not a member of the math faculty, then $y \notin F$. Thus, $y \in M \to y \notin F$. Since y is arbitrary, then $\forall y \in M. y \notin F$.

Suppose $a \in F \cap M$. Then $a \in F$ and $a \in M$. Since $\forall x \in F. x \notin M$ and $a \in F$, then $a \notin M$. Since $\forall y \in M. y \notin F$ and $a \in M$, then $a \notin F$. Thus, $a \notin F$ and $a \notin M$. Hence, $a \notin F \cap M$. But, $a \in F \cap M$ by assumption. Thus, $\nexists a \ni a \in F \cap M$.

Hence, $F \cap M$ must be empty, so $F \cap M = \emptyset$. $\square$

Exercise 2. A bookshelf holds 6 in English, 8 in Spanish, 10 in German, and 2 in Japanese. How many ways exist to select one book in any language?

Solution. We must find the number of different ways to select one book in any language.

Let E = the set of books in English. Then $|E| = 6$.

Let S = the set of books in Spanish. Then $|S| = 8$.

Let G = the set of books in German. Then $|G| = 10$.

Let J = the set of books in Japanese. Then $|J| = 2$.

Let n = the number of different ways to select one book in any language = the number of ways to select a different book in any language = the number of different books in any language = the number of different books in English, Spanish, German, OR Japanese

How are E, S, G and J related? Any pair of them is disjoint.

Let the universal set be the bookshelf $B = E \cup S \cup G \cup J$. Then $\{E, S, G, J\}$ is a partition of B. Hence, we can use the addition principle to count the different choices.

Thus, $n = |B| = |E \cup S \cup G \cup J| = |E| + |S| + |G| + |J| = 6+8+10+2 = 26$. □

Exercise 3. The chairs of an auditorium are to be labeled with a letter and a positive integer not exceeding 100. What is the largest number of chairs that can be labeled differently?

Solution. We must find the number of different chairs that can be labeled differently.

Let n = the number of different chairs that can be labeled differently.

Let $C = \{c : c \text{ is a chair that can be labeled differently }\}$.

Then $n = |C|$.

For example, let c be a chair that is labeled $G, 79$.

Then $c \in C$, so $C \neq \emptyset$.

So, how many chairs exist in C ?

he number of different chairs that can be labeled differently = the number of different ways in which one chair can be labeled differently = the number of different ways to label one chair.

Thus, n = the number of different ways to label one chair.

What does it mean to label a chair?

To label a chair means to choose a letter AND to choose a number.

Let t be the task to label a chair.

Let $|t|$ = the number of different ways to label one chair.

Task t can be decomposed into subtasks.

Let t_1 = choose a letter.

Let t_2 = choose a number.

Are these subtasks independent? Yes, because choosing a particular letter does not depend on choosing a particular number, and vice-versa. Thus, we can use the multiplication principle to count the number of chairs.

Hence, $n = |t| = |t_1| * |t_2|$ where

$|t_1|$ = the number of different ways to choose a letter

$|t_2|$ = the number of different ways to choose a number.

We must determine $|t_1|$ and $|t_2|$.

$|t_1|$ = the number of different ways to choose a letter = the number of ways to choose a different letter from a set of letters = the number of different letters in a set of letters = the count of letters in a set of letters = the count of letters in the alphabet = the size of the alphabet = 26.

$|t_2|$ = the number of different ways to choose a number = the number of ways to choose a different number from a set of numbers = the number of different numbers in a set of numbers = the count of numbers in a set of numbers = the count of numbers in a set of numbers S = the size of $S = |S|$, where

S = the set of positive integers not exceeding 100 $= \{n \in \mathbb{Z}^+ : x \leq 100\} = \{1, 2, 3, ..., 100\}$. Thus, $|S| = 100$.

Hence, $n = |t| = |t_1| * |t_2| = 26 * 100 = 2600$.

Thus, there are 2600 different chairs that can be labeled differently.

If the 2601st chair were labeled, it would be a duplicate labeling of one of the 2600 chairs. Thus, the maximum number of chairs that can be labeled differently is 2600. □

Exercise 4. How many different bit strings exist of length 7?

Solution.

We must find the number of different bit strings of length 7.

Let n = the number of different bit strings of length 7 = the number of different 7 bit strings.

Let $S = \{s : s \text{ is a 7 bit string }\}$.

Then $n = |S|$. We know $1011101 \in S$, so obviously $S \neq \emptyset$.

So, how many bit strings exist in S? The number of bit strings in S is the number of different ways to create one 7 bit string.

Thus, n =the number of different ways to create a 7 bit string.

What does it mean to create a 7 bit string?

To create a 7 bit string means to choose the first bit, then choose the 2nd bit, then choose the 3rd bit, ... AND then choose the 7th bit.

Let t be the task to create a 7 bit string.

Let $|t|$ = the number of different ways to create a 7 bit string.

Task t can be decomposed into subtasks.

Let t_k = choose the k^{th} bit,

where $k = 1..7$.

Are these subtasks independent? Yes, because choosing a bit in the k^{th} position does not depend on the choosing of a bit in any other position. Thus, we can use the multiplication principle to count the number of 7 bit strings that can be created.

Hence, $n = |t| = \prod_{i=1}^{7} |t_k|$ where

$|t_k|$ = the number of different ways to choose the k^{th} bit.

We must determine $|t_k|, k = 1..7$.

The number of ways to choose the k^{th} bit is independent of its position. Thus, each of the $|t_k|$ is the same.

Let $k =$ any arbitrary position of the 7 bit string to be created.

$|t_k| =$ the number of different ways to choose the k^{th} bit $=$ the number of different ways to choose a bit (regardless of position) $=$ the number of choices to select a different bit from bit set $=$ the number of different bits in the bitset $=$ the count of bits in the bitset $B =$ the size of the bit set $B = |B|$,

where $B = \{0, 1\}$. Thus, $|t_k| = |B| = 2$.

Hence, $n = 2^7 = 128$.

Therefore, there are 128 different bit strings of length 7. □

Exercise 5. How many different license plates exist if each plate contains a sequence of 3 letters followed by 3 digits?

Solution. Let $n =$ the number of different license plates.

Let $S = \{s : s \text{ is a license plate that contains a sequence of 3 letters followed by 3 digits}\}$. Then $n = |S|$. For example, a license plate that contains $ABC483 \in S$, so obviously $S \neq \emptyset$.

So, how many license plates exist in S?

The number of license plates in $S =$ the number of different ways to create a single license plate. Thus, $n =$ the number of different ways to create a license plate.

What does it mean to create a license plate?

Let $t =$ task to create one license plate.

Let $|t| =$ the number of different ways to create one license plate.

Task t can be decomposed into subtasks:

t_1 : choose a letter in position 1
t_2 : choose a letter in position 2
t_3 : choose a letter in position 3
t_4 : choose a digit in position 4
t_5 : choose a digit in position 5
t_6 : choose a digit in position 6

Each $|t_i| =$ the number of different ways to choose a letter in position i, where i = 1 to 3.

Each $|t_i| =$ the number of different ways to choose a digit in position i, where i = 4 to 6.

Choosing a letter or digit is independent of choosing any other letter or digit. Hence, each subtask is independent of the others, so we can use the multiplication principle. Thus, $n = |t| = \prod_{i=1}^{6} |t_i|$.

We must now determine each $|t_i|$ for $i = 1..6$.

The number of ways to choose a letter is independent of the position. Thus, $|t_1| = |t_2| = |t_3|$.

The number of ways to choose a digit is independent of the position. Thus, $|t_4| = |t_5| = |t_6|$.

Let $k =$ arbitrary position for 1..3.

How many different ways exist to choose a letter in position k?

The number of different ways to choose a letter in position k = the number of different ways to choose a letter (regardless of position) = the number of ways to choose a different letter from a set = the number of different letters from a set of letters = the count of letters in the alphabet = size of alphabet = 26.

Thus, $|t_k| = 26$ for $k = 1..3$. Hence, $|t_1| = |t_2| = |t_3| = 26$.

Let k = arbitrary position for 4..6.

How many different ways exist to choose a digit in position k?

The number of different ways to choose a digit in position k = the number of different ways to choose a digit (regardless of position) = the number of ways to choose a different digit from a set of digits = the number of different digits in a set of digits = the count of digits in the set of digits D = the size of the digit set D = $|D|$,

where $D = \{0, 1, 2, 3, ..., 9\}$.

Thus, $|t_k| = |D| = 10$ for $k = 4..6$. Hence, $|t_4| = |t_5| = |t_6| = 10$.

Thus, $n = |t| = 26^3 * 10^3 = 260^3 = 17576000$.

Hence, 17576000 different license plates exist. □

Exercise 6. A group contains n men and n women. How many ways exist to arrange them in a row if men and women must alternate?

Solution.

We must find the number of different ways to arrange the men and women.

Let x = the number of different ways to arrange the men and women.

An arrangement of alternating men and women in a row is an ordered arrangement of men and women. Thus, we must count the number of permutations of alternating men and women.

Let M = the set of n men.

Let W = the set of n women.

Then $|M| = |W| = n$.

Let S = the set of all permutations of alternating men and women = $\{P : P \text{ is a permutation of alternating men and women}\}$.

The number of different ways to arrange the men and women is the number of different arrangements of alternating men and women.

Thus, $x = |S|$.

Let P be an arrangement of alternating men and women.

The number of different arrangements in S = the number of different ways to create an arrangement. Thus, $|S|$ = the number of different ways to create an alternating arrangement of men and women.

What does it mean to create an alternating arrangement of men and women?

There are two choices to create an alternating arrangement of men and women: EITHER MWMWM... OR WMWMWM...

Let P_1 = create permutation P given that choice 1 is made.

Let $|P_1|$ = the number of P that can be created given that choice 1 is made.

Let P_2 = create permutation P given that choice 2 is made.

Let $|P_2|$ = the number of P that can be created given that choice 2 is made.

Since these two choices are mutually exclusive, we use the addition principle.

Thus, $|S| = |P| = |P_1| + |P_2|$.

We now must determine $|P_1|$ and $|P_2|$.

P_1 can be created by arranging the men first and then arranging the women. Similarly, P_2 can be created by arranging the men first and then arranging the women. Thus, regardless of the choice made, $|P_1| = |P_2|$.

Let $m =$ the task to arrange the men.

Let $|m| =$ the number of different ways to arrange the men.

Let $w =$ the task to arrange the women.

Let $|w| =$ the number of different ways to arrange the women.

By the multiplication principle, $|P_1| = |m| * |w|$.

Thus, $|P| = |P_1| + |P_1| = 2 * |P_1| = 2 * |m| * |w|$.

We must now determine $|m|$ and $|w|$.

Since there are n men to arrange from a set of n men, then $|m| = P(n, n) = n!$.

Since there are n women to arrange from a set of n women, then $|w| = P(n, n) = n!$.

Thus, $x = |S| = |P| = 2 * (n!) * (n!) = 2(n!)^2$.

□

Exercise 7. How many 4-permutations of positive integers not exceeding 100 contain exactly 3 consecutive integers(in the correct order) though they may be separated by other integers(ie, 65,54,55,67 is valid)?

Solution.

We must find the number of different 4-permutations of 3 consecutive positive integers satisfying the criteria.

Let $S =$ the set of all positive integers not exceeding $100 = \{n \in \mathbb{Z}^+ : n \leq 100\} = \{1, 2, 3, ...99, 100\}$.

Let $x =$ the number of different 4-permutations of 3 consecutive positive integers chosen from set $S =$ the number of different ways to create a single 4-permutation of 3 consecutive integers selected from S.

What does it mean to create a single 4-permutation of 3 consecutive integers chosen from S?

Let $P =$ a 4-permutation of 3 consecutive integers chosen from S.

Let $|P| =$ the number of different ways to create a single 4-permutation.

Each P can occur in exactly one of 4 mutually exclusive ways:

Either abc* OR ab*c OR a*bc OR *abc, where * is some integer in S.

Thus, using the addition principle, we have $x = 4 * |P|$.

How do we create a single 4-permutation? We apply the most restrictive criteria first. Therefore, we first choose 3 consecutive integers and then choose the 4th integer.

Let $p_1 =$ choose 3 consecutive integers from S.

Let $|p_1| =$ the number of different ways to choose 3 consecutive integers from S.

Let $p_2 =$ choose the 4th integer from S.

Let $|p_2|$ = the number of different ways to choose the 4th integer from S.

Subtasks p_1, p_2 are NOT independent because for each of the ways to choose 3 consecutive integers + 4th, we have 97 choices for the 4th. For the next choice, we will have counted a permutation twice based on the previous choice for any of the 4 possibilities.

For example, for first choice, we can choose 1,2,3 and 4th can be 4 to 100. For second choice, we can choose 2,3,4 and 4th can be 1,5-100. Thus, for the second choice we can have 2,3,4,1 which is one of the 4 possibilities that is counted from the first choice (1,2,3,4) for possibility abc*. For third choice,we can choose 3,4,5 and 4th can be 1,2,6-100. Thus, for 3rd choice we can have 3,4,5,2 which is one of the 4 possibilities that is counted for the 2nd choice (2,3,4,5). Therefore, we must subtract each occurrence of a 4-permutation of 4 consecutive integers to avoid counting a permutation twice.

Let p_3 = choose 4-permutation of 4 consecutive integers. Let $|p_3|$ = the number of 4-permutations of 4 consecutive integers. Then we can have 1,2,3,4 OR 2,3,4,5 OR 3,4,5,6 OR ... OR 97,98,99,100. Thus, $|p_3| = 97$.

We use the multiplication principle taking into account the duplicates.

Thus, $|P| = |p_1| * |p_2|$.

We now must determine $|p_1|$ and $|p_2|$.

How can we determine the number of different ways to choose 3 consecutive integers? One way is to choose 1,2,3. Another is 2,3,4. Another is 3,4,5 ... up to 98,99,100. Thus, $|p_1| = 98$.

How can we determine the number of different ways to choose the 4 integer? Well, since the 4-permutation must consist of 4 distinct integers and we've already chosen 3 consecutive integers for task p_1 from a set of $|S| = 100$, then this leaves only $100 - 3 = 97$ ways to choose the 4th integer. Thus, $|p_2| = 97$.

Hence, $x = 4 * |P| - |p_3| = 4 * |p_1| * |p_2| - |p_3| = 4 * 98 * 97 - 97 = 37927$.

Therefore, there are 38024 different 4-permutations of 3 consecutive positive integers not exceeding 100.

□

Exercise 8. How many ways can 6 men and 6 women be seated around a table so men and women alternate?

Solution.

We must find the number of different ways to arrange 6 men and 6 women around a table in alternating fashion.

Let n = the number of different ways to arrange men and women = number of different ways to arrange women followed by arranging the men

The arrangement of 6 women is a circular 6-permutation, so there are $P(6, 6)/6 = 6!/6 = 5!$ ways to arrange the women. How many ways to arrange the 6 men? Well, we have 6 ways to select the first man. Once he's selected there are 5 ways to select the 2nd man, and so on. Thus, there are 6! different ways to arrange the men once the women are chosen.

Hence, $n = 5! * 6!$ different ways. □

Exercise 9. There are 16 men and 22 women at a party. How many ways are there to form 10 couples consisting of one man and one woman?

Solution.

We must find the number of different ways to form 10 couples.

Let $M =$ the set of 16 men. Let $W =$ the set of 22 women. Then $|M| = 16$ and $|W| = 22$.

Let $n =$ the number of different ways to form 10 couples.

To form 10 couples choose 10 women and 10 men. The number of ways to choose 10 women from 22 is $\binom{22}{10}$ since order doesn't matter. Once the women are chosen choose 10 men from the 16 available. The number of ways to choose 10 men from 16 is $\binom{16}{10}$. Since order matters when determining a couple we must permute the men among the women. Thus, the number of ways to permute 10 men among the women is $\binom{16}{10} * 10!$.

Hence, $n = \binom{22}{10} * \binom{16}{10} * 10!$ different couples can be formed.

Equivalently, we could've chosen the men first. There are $\binom{16}{10}$ ways to choose 10 men from 16. Then we permute the 10 women chosen among the men. There are $\binom{22}{10} * 10!$ ways to permute 10 women chosen from 22. Hence, $n = \binom{16}{10} * \binom{22}{10} * 10!$. □