Combinatorics Notes

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May 27, 2023

Combinatorics

Study of sets(finite) and set systems(ie,graphs, codes, geometries,groups,rings,fields,topologies,algebras, vector spaces,etc)

Existence, enumeration, analysis, optimization of discrete structures.

- Existence: Arrange objects in a set to satisfy certain conditions. Does an arrangement exist?
- Enumeration/Classification: If an arrangement exists, how many ways can this be done? isomorphism? classify into types, what is the probability of getting this arrangement?
- Analysis: Study an arrangement and develop structure theorems
- Optimize: Are there 'optimal' arrangements? (map coloring, network-flow,etc)

Addition Principle

Let event P occur in p different ways.

Let distinct event Q occur in q different ways.

Then P **OR** Q can occur in $p + q$ different ways.

OR means ADD.

Generalized Addition Principle

Let $k =$ the number of distinct events,$k \geq 1$.

Let P_i be a distinct event, $i = 1..k$.

Let P_i occur in p_i different ways.

Then P_1 or P_2 or ... or P_k can occur in $\sum_{i=1}^{k} p_i$ different ways.

Let S be a finite set.

Let $\{S_1, S_2, ..., S_k\}$ be a **partition** of S.

Then $|S| = \sum_{i=1}^{k} |S_i|$.

Decompose into **mutually exclusive** cases.

Multiplication Principle

Let event P occur in p different ways.
Let distinct event Q occur in q different ways.
Then P **AND** Q can occur in pq different ways.

Equivalently,
if first task has p outcomes, and no matter the outcome,
there are q ways to do a second task, then the whole procedure has pq possible outcomes.
AND means multiply.
Decompose into sequential tasks for a procedure.

General Problem Solving Strategy

1. Does order matter?
2. Is repetition allowed?
3. Make most restrictive choices first.
4. Apply counting principles.

Definition 1. factorial function
The factorial of $n \in \mathbb{Z}^+$, denoted $n!$, is defined by

$$n! = \begin{cases} 1 & \text{if } n = 0 \\ n \cdot (n-1)! & \text{if } n > 0 \end{cases}$$

Therefore, $0! = 1$.

The factorial is a function $\mathbb{Z}^+ \cup \{0\} \to \mathbb{Z}^+$.

Proposition 2. *Let $n \in \mathbb{Z}^+$.*
Then $n! = 1 \cdot 2 \cdot 3 \cdot ... \cdot (n-1) \cdot n$.

Definition 3. binomial coefficient
Let $n, k \in \mathbb{Z}$ and $n \geq 0$.
The **binomial coefficient** $\binom{n}{k}$ is defined by

$$\binom{n}{k} = \begin{cases} \frac{n!}{(n-k)!k!} & \text{if } 0 \leq k \leq n \\ 0 & \text{otherwise} \end{cases}$$

Let $n, k \in \mathbb{Z}$.
If $k < 0$ or $k > n$, then $\binom{n}{k} = 0$.

The binomial coefficient $\binom{n}{k}$ is the number of k element subsets of an n element set.

It is the number of ways of selecting k unordered outcomes from n possible outcomes (n choose k).

Proposition 4. ***properties of binomial coefficients***

Let $n \in \mathbb{Z}^+$.

1. $\binom{0}{0} = 1$.

2. $\binom{n}{1} = n$.

3. $\binom{n}{0} = 1$.

4. $\binom{n}{n} = 1$.

5. Let $k \in \mathbb{Z}^+$.

If $k \leq n$, then $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$ *(Pascal's Recursion Rule)*

6. Let $k \in \mathbb{Z}$.

If $0 \leq k \leq n$, then $\binom{n}{k} = \binom{n}{n-k}$. *(Symmetry)*

Theorem 5. ***Binomial Theorem***

Let $a, b \in \mathbb{R}$.

Then $(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$ *for all $n \in \mathbb{Z}^+$.*

Observe that $2^n = (1+1)^n = \sum_{k=0}^{n} \binom{n}{k}$.

Definition 6. Permutation of Set

permutation of set = ordered arrangement, no repetition

k**-permutation of** n**-set** = ordered sequence of k distinct elements from set of n distinct elements

$P(n,k)$ = permutation of n things taken k at a time

= the number of different k permutations of an n-set

= the number of different ordered selections of k distinct objects from set of n distinct objects.

$P(n,k) = 0$ if $k > n$.

$P(n,n) = n!$ = the number of permutations of n distinct objects

$P(n,k) = \frac{n!}{(n-k)!} = n(n-1)(n-2)...(n-k+1), 0 < k \leq n$

There are $(n-1)!$ circular permutations of a set of n distinct objects.

There are $P(n,k)/k$ circular k-permutations of n distinct objects.

Definition 7. Permutation of MultiSet

permutation of multiset = ordered arrangement with repetition

Notes: How many

Definition 8. Combination of Set

combination of set = unordered arrangement , no repetition

k-**combination of** n**-set** (k-subset) = unordered selection of k distinct elements from set of n distinct elements

$\binom{n}{k}$ = combination of n things taken k at a time
= n choose k
= the number of different k combinations of an n-set
= the number of different k-subsets of an n-set (binomial coefficient)
= the number of different unordered selections of k distinct objects from set of n distinct objects.

Definition 9. Combination of MultiSet

combination of multiset = unordered arrangement , with repetition