Book Calculus by Larson Exercises

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Exercise 1. P dollars invested at r percent simple interest for t years grows to an amount $A = P + Prt$.

If an investment of \$1000 is to grow to an amount greater than \$1250 in two years, then the interest rate must be greater than what percentage?

Solution. Let P be the principal invested.
Let t be the time in years.
Let r be the interest rate.
Let A be the amount to which the investment has grown.
Then $A = P + Prt$.
We are given $P = 1000$ and $t = 2$ and $A > 1250$.
Since $P+Prt = A$ and $A > 1250$, then $P+Prt > 1250$, so $P(1+rt) > 1250$.
Thus, $1 + rt > \frac{1250}{P}$, so $rt > \frac{1250}{P} - 1$.
Hence, $r > \frac{1250}{Pt} - \frac{1}{t} = \frac{1250-P}{Pt} = \frac{1250-1000}{1000*2} = 0.125$, so $r > 0.125$.
Therefore, r must be greater than 12.5%. $\square$

Exercise 2. A utility company has a fleet of vans for which the annual operating cost per van is estimated to be $C = 0.32m+2300$, where C is measured in dollars and m is measured in miles.

If the company wants the annual operating cost per van to be less than $\$10,000$, then m must be less than what value?

Solution. Let m be the number of miles driven per van.
Let C be the annual operating cost per van.
Then $C = 0.32m + 2300$.
Since $C < 10000$, then $0.32m + 2300 < 10000$.
Hence, $0.32m < 7700$, so $m < 24062.5$.
Therefore, m must be less than 24062.5 miles. $\square$

Exercise 3. To determine if a coin is fair, an experimenter tosses it 100 times and records the number of heads, x.

Through statistical theory, the coin is declared unfair if $|\frac{x-50}{5}| \geq 1.645$.

For what values of x will the coin be declared unfair?

Solution. Let x represent the number of heads.
Then x is an integer between 0 and 100.
Let S be the set of all x such that the coin is unfair.

Then $S = \{x \in \mathbb{Z} : |\frac{x-50}{5}| \geq 1.645\}$.

Thus, $x \in S$ iff $|\frac{x-50}{5}| \geq 1.645$ iff $\frac{|x-50|}{5} \geq 1.645$ iff $|x-50| \geq 8.225$ iff either $x - 50 \geq 8.225$ or $x - 50 \leq -8.225$ iff either $x \geq 58.225$ or $x \leq 41.775$.

Hence, $x \in S$ iff either $x \geq 58.225$ or $x \leq 41.775$.

Since x is an integer between 0 and 100, then this implies $x \geq 59$ or $x \leq 41$.

Therefore, $S = \{x \in \mathbb{Z} : x \leq 41 \text{ or } x \geq 59\}$.

We conclude that nonnegative integer values of x that are less than or equal to 41 or greater than or equal to 59 will cause the coin to be declared unfair. $\square$

Exercise 4. What is the equation for the path of a communications satellite in a circular orbit 22000 miles above the surface of the earth? Assume the radius of the earth is 4000 miles.

Solution. Let the center of the earth be the origin of the xy coordinate system.

Then the equation of the circular orbit is the equation of a circle given by $x^2 + y^2 = r^2$, where r is the radius of the circle.

Thus, r = radius of the earth + distance between surface of earth and the satellite $= 4000 + 22000 = 26000$, so $r = 26000$.

Therefore, $x^2 + y^2 = 26000^2$ is the equation of the path of the satellite in its circular orbit around the earth. $\square$

Exercise 5. Determine the equation of the line giving the relationship between the temperature in degrees Celsius and degrees Fahrenheit.

Water freezes at 0°Celsius (32°Fahrenheit) and boils at 100°Celsius (212°Fahrenheit).

Solution. Let F represent the temperature in degrees Fahrenheit.

Let C represent the temperature in degrees Celsius.

The relationship between F and C is linear and $F = 32$ when $C = 0$ and $F = 212$ when $C = 100$.

If we plot F vs C, then the slope is $\frac{\Delta F}{\Delta C} = \frac{212-32}{100-0} = \frac{180}{100} = \frac{9}{5}$ and the y intercept is 32.

Therefore, the equation of the line is $F = \frac{9C}{5} + 32$. $\square$

Exercise 6. A small business purchases a piece of equipment for \$875. After five years the equipment will be outdated and have no value.

Write a linear equation giving the value V of the equipment during the five years it will be used.

Solution. Let V be the value of the equipment.

Let t be the time in years.

Then $0 \leq t \leq 5$.

We assume a linear relationship between V and t.

We are given that $V = 875$ when $t = 0$ and $V = 0$ when $t = 5$.

If we plot V vs t, then the slope is $m = \frac{\Delta V}{\Delta t} = \frac{0-875}{5-0} = \frac{-875}{5} = -175$ and the y intercept is $b = 875$.

Thus, $V = -175t + 875 = 875 - 175t$.

Therefore, the equation of the line is $V = 875 - 175t$, with $0 \leq t \leq 5$. $\square$

Exercise 7. A real estate office handles an apartment complex with 50 units. When the rent is \$280 per month, all 50 units are occupied. However, when the rent is \$325, the average number of occupied units drops to 47. Assume the relationship between the monthly rent p and the demand x is linear.

a. Determine the equation of the line giving the demand x in terms of the rent p.

b. Use this equation to predict the number of units occupied if the rent is raised to \$355.

c. Predict the number of units occupied if the rent is lowered to \$295.

Solution. Let x be the demand which is measured by the number of occupied units.

Let p be the monthly rent.

It is given that the relationship between x and p is linear and $x = 50$ when $p = 280$ and $x = 47$ when $p = 325$.

If we plot x vs p, then the slope is $\frac{\Delta x}{\Delta p} = \frac{47-50}{325-280} = \frac{-3}{45} = \frac{-1}{15}$.

One of the points on the line is $(280, 50)$, so $\frac{-1}{15} = \frac{x-50}{p-280}$.

Thus, $x = \frac{-1}{15}(p-280)+50 = -\frac{p}{15}+\frac{280}{15}+50 = -\frac{p}{15}+\frac{280+50\cdot 15}{15} = -\frac{p}{15}+\frac{1030}{15} = \frac{1030}{15} - \frac{p}{15} = \frac{1}{15}(1030 - p)$.

Therefore, the equation of the line is $x = \frac{1}{15}(1030 - p)$.

b. If $p = 355$, then $x = \frac{1}{15}(1030 - 355) = 45$.

Therefore, if the rent is raised to \$355, then the number of units occupied is 45.

c. If $p = 295$, then $x = \frac{1}{15}(1030 - 295) = 49$.

Therefore, if the rent is lowered to \$295, then the number of units occupied is 49. □

Exercise 8. Let $f : \mathbb{R} \to \mathbb{R}$ be a function defined by

$$f(x) = \begin{cases} x + 3 & \text{if } x \leq 2 \\ cx + 6 & \text{if } x > 2. \end{cases}$$

Determine the value of c that makes f continuous.

Solution. The function f is continuous when f is continuous at 2.

Thus, we must have $\lim_{x\to 2} f = f(2)$.

Hence, $5 = 2 + 3 = f(2) = \lim_{x\to 2} f(x) = \lim_{x\to 2}(cx + 6) = 2c + 6$, so $5 = 2c + 6$.

Therefore, $2c = -1$, so $c = -\frac{1}{2}$. □

Exercise 9. The height s at time t of a coin dropped from a building is given by $s(t) = -16t^2 + 1350$, where s is measured in feet and t is measured in seconds.

a. Compute the average velocity on the interval $[1, 2]$.

b. Compute the instantaneous velocity at $t = 1$ and $t = 2$.

c. How long will it take the coin to hit the ground?

d. Compute the velocity of the coin when it hits the ground.

Solution. The height s at time t of the coin is given by the function $s(t) = -16t^2 + 1350$.

We have $\frac{\Delta s}{\Delta t} = \frac{s(2)-s(1)}{2-1} = s(2)-s(1) = [-16(2^2)+1350]-[-16(1^2)+1350] = (-64+1350)-(-16+1350)) = -64+16 = -48$.

Since $\frac{\Delta s}{\Delta t} = -48$, then the average velocity on the time interval $[1,2]$ is -48 ft/s.

Since $s(t) = -16t^2 + 1350$, then $v(t) = s'(t) = -32t$.

Thus, $v(1) = -32(1) = -32$ and $v(2) = -32(2) = -64$.

Therefore, the instantaneous velocity at $t = 1$ is -32 ft/s and the instantaneous velocity at $t = 2$ is -64 ft/s.

The coin will hit the ground when $s(t) = 0$.

Since $0 = s(t) = -16t^2 + 1350$ iff $16t^2 = 1350$ iff $t^2 = \frac{1350}{16}$ if $t = \frac{\sqrt{1350}}{4}$ iff $t = \frac{15\sqrt{6}}{4}$, then the coin will hit the ground when $t = \frac{15\sqrt{6}}{4} \approx 9.2$.

Therefore, the coin will hit the ground after 9.2 seconds.

Since $v(\frac{15\sqrt{6}}{4}) = -32(\frac{15\sqrt{6}}{4}) \approx -293.9$, then the velocity of the coin when it hits the ground is -293.9 ft/s. $\square$

Exercise 10. A projectile is shot upward from the surface of the earth with an initial velocity of 384 ft/s.

Assume free-fall motion and acceleration due to gravity is constant, where $g = -32$ ft/s^2.

Compute the velocity of the projectile after 5 seconds and after 10 seconds.

Solution. Since we have one dimensional linear motion with constant acceleration due to gravity, then the position function of the projectile is given by $s(t) = s_0 + v_0t + \frac{gt^2}{2} = s_0 + v_0t - \frac{32t^2}{2} = s_0 + v_0t - 16t^2$.

Thus, $s(t) = -16t^2 + v_0t + s_0$.

We are given $s_0 = 0$ and $v_0 = 384$, so $s(t) = -16t^2 + 384t$.

Hence, $v(t) = s'(t) = -32t + 384$, so $v(5) = -32(5) + 384 = 224$ and $v(10) = -32(10) + 384 = 64$.

Therefore, the velocity of the projectile after 5 seconds is 224 ft/s and the velocity after 10 seconds is 64 ft/s. $\square$

Exercise 11. A pebble is dropped from a height of 600 ft.

Assume free-fall motion and acceleration due to gravity is constant, where $g = -32$ ft/s^2.

Compute the velocity of the pebble when it hits the ground.

Solution. Since we have one dimensional linear motion with constant acceleration due to gravity, then the position function of the pebble is given by $s(t) = s_0 + v_0t + \frac{gt^2}{2} = s_0 + v_0t - \frac{32t^2}{2} = s_0 + v_0t - 16t^2$.

Thus, $s(t) = -16t^2 + v_0t + s_0$.

We are given $s_0 = 600$ and $v_0 = 0$, so $s(t) = -16t^2 + 600$.

Hence, $v(t) = s'(t) = -32t$.

The pebble hits the ground when $s(t) = 0$.

Since $0 = s(t) = -16t^2 + 600$ iff $16t^2 = 600$ iff $t^2 = \frac{600}{16}$ if $t = \frac{\sqrt{600}}{4} = \frac{5\sqrt{6}}{2}$, then the pebble hits the ground when $t = \frac{5\sqrt{6}}{2}$.

Thus, $v(\frac{5\sqrt{6}}{2}) = -32(\frac{5\sqrt{6}}{2}) = -80\sqrt{6} \approx -195.96$.

Therefore, the velocity of the pebble when it hits the ground is -195.96 ft/s. $\square$

Exercise 12. To estimate the height of a building, a stone is dropped from the top of the building into a pool of water at ground level.

Assume free-fall motion and acceleration due to gravity is constant, where $g = -32$ ft/s^2.

How high is the building if the splash is seen 6.8 seconds after the stone is dropped?

Solution. Since we have one dimensional linear motion with constant acceleration due to gravity, then the position function of the stone is given by $s(t) = s_0 + v_0 t + \frac{gt^2}{2} = s_0 + v_0 t - \frac{32t^2}{2} = s_0 + v_0 t - 16t^2$.

We are given $v_0 = 0$ and $s(6.8) = 0$.

We must find s_0, the height of the building.

Since $v_0 = 0$, then $s(t) = s_0 - 16t^2$.

Since $0 = s(6.8) = s_0 - 16(6.8)^2$, then $s_0 = 16(6.8)^2 \approx 740$.

Therefore, the building is 740 feet high. $\square$

Exercise 13. Let s be the length of the side of a square.

Compute the rate of change of its area with respect to s when $s = 4$.

Solution. Let A be the area of the square with side s.

Then $A = s^2$, so A is a function of s.

Therefore, $A(s) = s^2$, so the rate of change of A with respect to s is $A'(s) = \frac{dA}{ds} = 2s$.

Hence, $A'(4) = 2(4) = 8$.

The rate of change of the area of the square with respect to s when $s = 4$ is 8. $\square$

Exercise 14. A spherical balloon is inflated with gas at a rate of 20 cubic feet per minute. How fast is the radius of the balloon increasing when the radius is 1 ft, 2ft?

Solution. Let V be the volume of the balloon.

Let r be the radius of the balloon.

Then $V = \frac{4\pi r^3}{3}$.

We are given $\frac{dV}{dt} = 20$ cubic feet per minute.

We must compute the rate of change of the balloon's radius $\frac{dr}{dt}$.

Since $V = \frac{4\pi r^3}{3}$, then $\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$.

Since $\frac{dV}{dt} = 20$, then $20 = 4\pi r^2 \frac{dr}{dt}$, so $\frac{dr}{dt} = \frac{5}{\pi r^2}$.

If $r = 1$, then $\frac{dr}{dt} = \frac{5}{\pi 1^2} = \frac{5}{\pi}$.

If $r = 2$, then $\frac{dr}{dt} = \frac{5}{\pi 2^2} = \frac{5}{4\pi}$.

Therefore, the radius of the balloon is increasing at $\frac{5}{\pi}$ feet per minute when the radius is 1 foot and the radius of the balloon is increasing at $\frac{5}{4\pi}$ feet per minute when the radius is 2 feet. $\square$

Exercise 15. Sand is falling onto a conical pile at a rate of 10 cubic feet per minute. The diameter of the base of the cone is three times the altitude. At what rate is the height of the pile changing when it is 15 feet high?

Solution. Let V be the volume of the cone with radius r and height h.

Then $V = \frac{\pi r^2 h}{3}$.

We are given $\frac{dV}{dt} = 10$ cubic feet per minute and $2r = 3h$.

We must find $\frac{dh}{dt}$ when $h = 15$.

Since $V = \frac{\pi r^2 h}{3} = \frac{\pi}{3}(\frac{3h}{2})^2 h = \frac{3\pi}{4}h^3$, then $V = \frac{3\pi}{4}h^3$.

Hence, $\frac{dV}{dt} = \frac{9\pi}{4}h^2\frac{dh}{dt}$, so $\frac{dh}{dt} = \frac{4}{9\pi h^2}\frac{dV}{dt}$.

Thus, $\frac{dh}{dt} = \frac{4}{9\pi 15^2}(10) = \frac{8}{405\pi}$.

Therefore, the height of the pile is increasing at $\frac{8}{405\pi}$ feet per minute when the pile is 15 feet high. $\square$

Exercise 16. All edges of a cube are expanding at a rate of 3 centimeters per second. How fast is the volume changing when each edge is 1 cm, 10 cm ?

Solution. Let V be the volume of a cube with side s.

Then $V = s^3$.

We are given $\frac{ds}{dt} = 3$ centimeters per second.

We must find $\frac{dV}{dt}$ when $s = 1$ and when $s = 10$.

Since $V = s^3$, then $\frac{dV}{dt} = 3s^2\frac{ds}{dt}$.

If $s = 1$, then $\frac{dV}{dt} = 3(1)^2(3) = 9$.

If $s = 10$, then $\frac{dV}{dt} = 3(10)^2(3) = 900$.

Therefore, when the edge is 1 cm, the volume is changing at the rate of 9 cm^3/sec and when the edge is 10 cm, the volume is changing at the rate of 900 cm^3/sec. $\square$

Exercise 17. A point is moving along the graph of $y = x^2$ so that $\frac{dx}{dt} = 2$ centimeters per minute. Find $\frac{dy}{dt}$ when $x = 0$ and $x = 3$.

Solution. We are given $y = x^2$ and $\frac{dx}{dt} = 2$.

We must find $\frac{dy}{dt}$.

Since $y = x^2$, then $\frac{dy}{dt} = 2x\frac{dx}{dt}$.

If $x = 0$, then $\frac{dy}{dt} = 2(0)(2) = 0$.

If $x = 3$, then $\frac{dy}{dt} = 2(3)(2) = 12$.

Therefore, if $x = 0$, then $\frac{dy}{dt} = 0$ cm/min and if $x = 3$, then $\frac{dy}{dt} = 12$ cm/min. $\square$

Exercise 18. A ladder 25 feet long is leaning against a house wall. The base of the ladder is pulled away from the house wall at a rate of 2 feet per second. How fast is the top moving down the wall when the base of the ladder is $7, 15, 24$ feet from the wall?

Solution. The model is a right triangle with hypotenuse 25 and with horizontal distance x and vertical distance y.

Thus, $25^2 = x^2 + y^2$.

We are given $\frac{dx}{dt} = 2$ feet per sec.

We must find $\frac{dy}{dt}$ when $x = 7, 15, 24$ feet.

Since $x^2 + y^2 = 25^2$, then $2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$, so $2y\frac{dy}{dt} = -2x\frac{dx}{dt}$.

Hence, $\frac{dy}{dt} = -\frac{x}{y}\frac{dx}{dt} = -\frac{2x}{y}$.

Since $x^2 + y^2 = 25^2 = 625$, then $y = \sqrt{625 - x^2}$.

Thus, $\frac{dy}{dt} = -\frac{2x}{\sqrt{625-x^2}}$.

If $x = 7$, then $\frac{dy}{dt} = -\frac{2(7)}{\sqrt{625-7^2}} = -\frac{7}{12}$.

If $x = 15$, then $\frac{dy}{dt} = -\frac{2(15)}{\sqrt{625-15^2}} = -\frac{3}{2}$.

If $x = 24$, then $\frac{dy}{dt} = -\frac{2(24)}{\sqrt{625-24^2}} = -\frac{48}{7}$.

When the base of the ladder is 7 feet from the wall, the top is moving down at a rate of $-\frac{7}{12}$ feet per sec. When the base of the ladder is 15 feet from the wall, the top is moving down at a rate of $-\frac{3}{2}$ feet per sec. When the base of the ladder is 24 feet from the wall, the top is moving down at a rate of $-\frac{48}{7}$ feet per sec. □

Exercise 19. An air traffic controller spots two planes at the same altitude converging on a point as they fly at right angles to one another. One plane is 150 miles from the point and is moving at 450 miles per hour. The other plane is 200 miles from the point and has a speed of 600 miles per hour.

a. At what rate is the distance between the planes decreasing?

b. How much time does the traffic controller have to get one of the planes on a different flight path?

Solution. We model the planes flying at right angles using xy coordinate system with the point of convergence at the origin.

Let s be the distance between the planes.

Then we have a right triangle with hypotenuse s and lengths x and y and right angle at the origin.

Thus, $x^2 + y^2 = s^2$.

We are given $\frac{dx}{dt} = -450$ miles per hour at $x = 150$ and $\frac{dy}{dt} = -600$ miles per hour and $y = 200$.

We must find $\frac{ds}{dt}$.

Since $s^2 = x^2 + y^2$, then $2s\frac{ds}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$, so $\frac{ds}{dt} = (x\frac{dx}{dt} + y\frac{dy}{dt})/s$.

Since $s^2 = x^2 + y^2$, then $s = \sqrt{x^2 + y^2}$.

Thus, $\frac{ds}{dt} = (x\frac{dx}{dt} + y\frac{dy}{dt})/\sqrt{x^2 + y^2}$.

Hence, $\frac{ds}{dt} = (150(-450) + 200(-600))/\sqrt{150^2 + 200^2} = -750$.

Therefore, the distance between the planes is decreasing at the rate -750 miles per hour.

The plane with $x = 150$ miles and speed of 450 miles per hour will reach the point of convergence at time $t_x = \frac{150}{450} = \frac{1}{3}$ hour, which is 20 minutes.

The plane with $y = 200$ miles and speed of 600 miles per hour will reach the point of convergence at time $t_y = \frac{200}{600} = \frac{1}{3}$ hour, which is 20 minutes.

The air traffic controller will have 20 minutes in which to divert one of the planes onto a different path in order to avoid a collision with the other plane. □

Exercise 20. A baseball diamond has the shape of a square with sides 90 feet long. A player is running to third base at a speed of 28 feet per second. At what rate is the player's distance from home plate changing when the player is 30 feet from third base?

Solution. Let s be the distance between the player and home plate.

Let x be the distance between the player and third base.

Then we have a right triangle with hypotenuse s and sides 90 and x.

Thus, $s^2 = x^2 + 90^2 = x^2 + 8100$.

We must find $\frac{ds}{dt}$ when $x = 30$.

We are given $\frac{dx}{dt} = 28$ feet per second.

Since $s^2 = x^2 + 8100$, then $2s\frac{ds}{dt} = 2x\frac{dx}{dt}$, so $\frac{ds}{dt} = \frac{x}{s}\frac{dx}{dt}$.

Since $s^2 = x^2 + 8100$, then $s = \sqrt{x^2 + 8100}$, so $\frac{ds}{dt} = \frac{x}{\sqrt{x^2+8100}}\frac{dx}{dt} = \frac{28x}{\sqrt{x^2+8100}}$.

If $x = 30$, then $\frac{ds}{dt} = \frac{28x}{\sqrt{x^2+8100}} = \frac{28(30)}{\sqrt{30^2+8100}} = \frac{28}{\sqrt{10}} \approx 8.85$.

Therefore, the rate of change is $\frac{ds}{dt} = \frac{28}{\sqrt{10}} \approx 8.85$ feet per second. □

Exercise 21. A man 6 feet tall walks at a rate of 5 feet per second away from a light that is 15 feet above the ground. At what rate is the tip of his shadow moving and at what rate is the length of his shadow changing?

Solution. Let x be the distance between the light base and the man.

Let s be the length of the man's shadow.

Then we have two similar right triangles, so $\frac{s}{6} = \frac{x+s}{15}$.

We are given $\frac{dx}{dt} = 5$ feet per second.

To find the rate at which the man's shadow length is changing, we must find $\frac{ds}{dt}$.

Since $\frac{s}{6} = \frac{x+s}{15}$, then $15s = 6x + 6s$, so $9s = 6x$.

Hence, $s = \frac{2x}{3}$.

Thus, $\frac{ds}{dt} = \frac{2}{3}\frac{dx}{dt} = \frac{2}{3} \cdot 5 = \frac{10}{3}$.

Therefore, the rate at which the man's shadow length is changing is $\frac{10}{3}$ feet per second.

Let m be the distance between the light base and the man's shadow tip.

Then $m = x + s$.

To find the rate at which the man's shadow tip is moving, we must find $\frac{dm}{dt}$.

Since $m = x + s$ and $s = \frac{2x}{3}$, then $m = x + \frac{2x}{3} = \frac{5x}{3}$.

Thus, $\frac{dm}{dt} = \frac{5}{3}\frac{dx}{dt} = \frac{5}{3} \cdot 5 = \frac{25}{3}$.

Therefore, the rate at which the man's shadow tip is moving is $\frac{25}{3}$ feet per second. □

Exercise 22. As a spherical raindrop falls, it reaches a layer of dry air and begins to evaporate at a rate that is proportional to its surface area $S = 4\pi r^2$.

Show that the radius decreases at a constant rate.

Solution. Let r be the radius of the spherical raindrop.

The surface area is $S = 4\pi r^2$.

We must show that the rate of change of its radius $\frac{dr}{dt}$ is constant.

We are given that the evaporation rate is proportional to S.

This means that the change in volume of the raindrop is proportional to S.

Thus, there exists a constant k such that $\frac{dV}{dt} = kS$, where V is the volume of the raindrop.

Since the raindrop is modeled as a sphere, then $V = \frac{4\pi r^3}{3}$.

Since $\frac{dV}{dt} = \frac{dV}{dr}\frac{dr}{dt}$, then $\frac{dr}{dt} = \frac{dV}{dt} / \frac{dV}{dr}$.

Since $V = \frac{4\pi r^3}{3}$, then $\frac{dV}{dr} = 4\pi r^2 = S$.

Therefore, $\frac{dr}{dt} = \frac{kS}{S} = k$, as desired. $\square$

Exercise 23. A balloon rises at a rate of 10 feet per second from a point on the ground 100 feet from an observer. What is the rate of change of the angle of elevation of the balloon from the observer when the balloon is 100 feet above the ground?

Solution. Let θ be the angle of elevation of the balloon with respect to the observer.

Let y be the height of the balloon.

Then we have a right triangle with vertical side y and horizontal side 100.

Thus, $\tan(\theta) = \frac{y}{100}$.

We must find $\frac{d\theta}{dt}$ when $y = 100$.

We are given $\frac{dy}{dt} = 10$ feet per second.

Since $\tan(\theta) = \frac{y}{100}$, then $\sec^2(\theta)\frac{d\theta}{dt} = \frac{dy}{dt}/100 = \frac{10}{100} = \frac{1}{10}$, so $\frac{d\theta}{dt} = \frac{\cos^2(\theta)}{10}$.

If $y = 100$, then $\tan(\theta) = \frac{100}{100} = 1$, so $\theta = \frac{\pi}{4}$.

Thus, $\cos(\theta) = \frac{1}{\sqrt{2}}$, so $\frac{d\theta}{dt} = \frac{(\frac{1}{\sqrt{2}})^2}{10} = \frac{1}{20}$.

Therefore, the rate of change is $\frac{d\theta}{dt} = \frac{1}{20}$ rad/sec. $\square$

Exercise 24. What is the smallest initial velocity required to throw a stone to the top of a 49 foot silo?

Solution. We model this as one dimensional linear motion with constant acceleration due to gravity $g = -32ft/sec^2$.

Thus, the position function of the stone is given by $s(t) = s_0 + v_0 t + \frac{gt^2}{2}$.

Since $s_0 = 0$, then $s(t) = 0 + v_0 t - \frac{32t^2}{2} = v_0 t - 16t^2$.

We must find v_0.

When the stone reaches the silo top, its velocity will be zero, so $s'(t) = 0$ when $s(t) = 49$.

Since $s(t) = v_0 t - 16t^2$, then $s'(t) = v_0 - 32t$, so $0 = s'(t) = v_0 - 32t$ implies that $v_0 = 32t$.

Thus, $49 = s(t) = v_0 t - 16t^2 = 32t^2 - 16t^2 = 16t^2$, so $t^2 = \frac{49}{16}$.
Hence, $t = \frac{7}{4}$, so $v_0 = 32 \cdot \frac{7}{4} = 56$.
Therefore, the smallest initial velocity is 56 feet per second. □

Exercise 25. Let x and $x + n$ be real numbers.
The geometric mean of x and $x + n$ is $g = \sqrt{x(x+n)}$ and the arithmetic mean is $a = \frac{x+(x+n)}{2}$.
Show that $\frac{dg}{dx} = \frac{a}{g}$.

Solution. Observe that

$$\begin{aligned} \frac{dg}{dx} &= \frac{d}{dx}\sqrt{x(x+n)} \\ &= \frac{d}{dx}(x^2+nx)^{\frac{1}{2}} \\ &= \frac{1}{2}(x^2+nx)^{\frac{-1}{2}}(2x+n) \\ &= \frac{2x+n}{2\sqrt{x^2+nx}} \\ &= \frac{x+(x+n)}{2\sqrt{x^2+nx}} \\ &= \frac{a}{\sqrt{x^2+nx}} \\ &= \frac{a}{\sqrt{x(x+n)}} \\ &= \frac{a}{g}. \end{aligned}$$

Therefore, $\frac{dg}{dx} = \frac{a}{g}$. □

Exercise 26. A cross section of a 5 foot trough is an isosceles trapezoid with a 2 foot lower base, a 3 foot upper base, and an altitude of 2 feet. Water runs into the trough at 1 cubic foot per minute. How fast is the water level rising when the water is 1 foot deep?

Solution. We draw a picture of the trapezoid cross section.
Let h be the water height in the trough.
We must find $\frac{dh}{dt}$ when $h = 1$.
We are given $\frac{dV}{dt} = 1$ cubic foot per minute, where V is the volume of the water in the trough.
The volume V of the water is the area of the trapezoid of height h multiplied by the length of the trough.
Let x be the horizontal top distance of the water at height h.
Then we have two similar right triangles, so $\frac{x}{h} = \frac{1}{2}/2 = \frac{1}{4}$.
Hence, $x = \frac{h}{4}$.

The volume is $V = [2h + 2(\frac{1}{2})xh]5 = 5(2h + xh) = 5(2h + \frac{h^2}{4}) = 10h + \frac{5h^2}{4} = \frac{5h^2}{4} + 10h$.

Since $\frac{dV}{dt} = \frac{dV}{dh} \cdot \frac{dh}{dt}$, then $1 = \frac{dV}{dh} \cdot \frac{dh}{dt}$, so $\frac{dh}{dt} = \frac{1}{\frac{dV}{dh}}$.

Since $V = \frac{5h^2}{4} + 10h$, then $\frac{dV}{dh} = \frac{5h}{2} + 10$, so $\frac{dV}{dh} = \frac{5*1}{2} + 10 = \frac{25}{2}$.

Hence, $\frac{dh}{dt} = \frac{1}{\frac{25}{2}} = \frac{2}{25}$.

Therefore, the water is rising at $\frac{2}{25}$ feet per minute when the water is 1 foot deep. $\square$

Exercise 27. The formula for the output P of a battery is $P = VI - RI^2$, where V is voltage in volts, R is resistance in ohms, and I is current in amperes.

Find the current that corresponds to a maximum value of P in which $V = 12$ volts and $R = 0.5$ ohms. Assume that a 15 ampere fuse bounds the output in the interval $[0, 15]$.

Solution. Since $V = 12$ and $R = 0.5$, then $P = 12I - \frac{I^2}{2}$, so P is a function of I.

Thus, $P(I) = 12I - \frac{I^2}{2} = -\frac{I^2}{2} + 12I$.

We must find I such that P is maximum on the interval $[0, 15]$.

Since P is a continuous function on $[0, 15]$, then by EVT, P has a maximum on $[0, 15]$.

We observe that $P(0) = 0$ and $P(15) = 67.5$.

Since P is a polynomial function of I, then P is differentiable and $\frac{dP}{dI} = -I + 12$.

We have $\frac{dP}{dI} = 0$ iff $-I + 12 = 0$ iff $I = 12$ and $P(12) = 72$.

Since $P(12) = 72 > 67.5 = P(15) > 0 = P(0)$, then we conclude P has a maximum value when $I = 12$ amperes. $\square$

Exercise 28. The height of an object t seconds after it is dropped from a height of 500 feet is given by $s(t) = -16t^2 + 500$.

Find the time at which the instantaneous velocity equals the average velocity in the first three seconds of fall.

Solution. The function $s : \mathbb{R} \to \mathbb{R}$ given by $s(t) = -16t^2 + 500$ is a polynomial function, so s is continuous and differentiable and $s'(t) = -32t$.

The average velocity on the time interval $[0, 3]$ is $\frac{\Delta s}{\Delta t} = \frac{s(3)-s(0)}{3-0} = -48$ feet per second.

Since s is continuous, then s is continuous on $[0, 3]$.

Since s is differentiable, then s is differentiable on $(0, 3)$.

Hence, by MVT, there exists $t \in (0, 3)$ such that $s'(t) = \frac{s(3)-s(0)}{3-0}$.

Since $-48 = \frac{s(3)-s(0)}{3-0} = s'(t) = -32t$, then $-48 = -32t$, so $t = \frac{3}{2}$.

Therefore, at time $t = \frac{3}{2} = 1.5$ seconds the instantaneous velocity equals the average velocity. $\square$

Exercise 29. The height (in feet) of a ball at time t (in seconds) is given by the position function $s(t) = 96t - 16t^2$.

Find the time interval when the ball is moving up and the interval on which it is moving down and the maximum height of the ball.

Solution. Since the position function s is a polynomial function, then s is continuous and differentiable.

If $s(t) = 0$, then $96t - 16t^2 = 0$, so $16t(6 - t) = 0$.

Hence, either $t = 0$ or $t = 6$, so s is defined on the closed interval $[0, 6]$.

The ball moves up when s is increasing and the ball moves down when s is decreasing.

Since $s(t) = 96t - 16t^2 = -16t^2 + 96t$, then $s'(t) = -32t + 96 = -32(t - 3)$.

Since $s'(t) = 0$ iff $-32t+96 = 0$ iff $32t = 96$ iff $t = 3$, then $s'(t) = 0$ iff $t = 3$.

If $t \in (0, 3)$, then $0 < t < 3$, so $t < 3$. Hence, $t-3 < 0$, so $s'(t) = -32(t-3) > 0$.

Hence, $s'(t) > 0$ for all $t \in (0, 3)$, so s is strictly increasing on the open interval $(0, 3)$.

If $t \in (3, 6)$, then $3 < t < 6$, so $3 < t$. Hence, $t > 3$, so $t - 3 > 0$.

Thus, $s'(t) = -32(t - 3) < 0$, so $s'(t) < 0$ for all $t \in (3, 6)$.

Therefore, s is strictly decreasing on the open interval $(3, 6)$.

The ball is moving up on the time interval $(0, 3)$ and is moving down on the time interval $(3, 6)$.

Since s is continuous on $(0, 6)$ and differentiable on $(0, 3)$ and differentiable on $(3, 6)$ and $s'(t) > 0$ for all $x \in (0, 3)$ and $s'(t) < 0$ for all $x \in (3, 6)$, then $s(3) = 144$ is a relative maximum, by FDT.

Therefore, the maximum height of the ball is 144 feet. $\square$

Exercise 30. Coughing forces the trachea(windpipe) to contract, which in turn affects the velocity v of the air through the trachea. Suppose the velocity of the air during coughing is $v = k(R - r)r^2$ where k is a constant, R is the normal radius of the trachea, and r is the radius during coughing. What radius will produce the maximum air velocity?

Solution. Since k and R are constants, then v is a function of r and $v(r) = k(R - r)r^2 = kr^2(R - r) = kRr^2 - kr^3$.

Thus, v is a polynomial function of r, so v is continuous and differentiable.

Since $v(r) = 0$ iff $k(R - r)r^2 = 0$ iff either $R - r = 0$ or $r^2 = 0$ iff either $r = R$ or $r = 0$, then v is defined on the interval $[0, R]$.

Since $v(r) = kRr^2 - kr^3$, then $\frac{dv}{dr} = kR(2r) - 3kr^2 = kr(2R - 3r)$.

Since k is a constant, then $k > 0$.

Since $\frac{dv}{dr} = 0$ iff $kr(2R - 3r) = 0$ iff either $kr = 0$ or $2R - 3r = 0$ iff either $r = 0$ or $3r = 2R$ iff either $r = 0$ or $r = \frac{2R}{3}$, then $\frac{dv}{dr} = 0$ iff either $r = 0$ or $r = \frac{2R}{3}$.

Observe that v is continuous on $(0, R)$ and differentiable on $(0, \frac{2R}{3})$ and differentiable on $(\frac{2R}{3}, R)$.

Let $r \in (0, \frac{2R}{3})$.

Then $0 < r < \frac{2R}{3}$, so $0 < r$ and $r < \frac{2R}{3}$.

Since $r < \frac{2R}{3}$, then $3r < 2R$, so $2R - 3r > 0$.

Since $k > 0$ and $r > 0$ and $2R - 3r > 0$, then $\frac{dv}{dr} > 0$.

Thus, $\frac{dv}{dr} > 0$ for all $r \in (0, \frac{2R}{3})$, so v is increasing on the interval $(0, \frac{2R}{3})$.

Let $r \in (\frac{2R}{3}, R)$.

Then $\frac{2R}{3} < r < R$, so $\frac{2R}{3} < r$ and $r < R$.

Since $R > 0$, then $\frac{2R}{3} > 0$.

Since $0 < \frac{2R}{3} < r$, then $0 < r$.

Since $\frac{2R}{3} < r$, then $2R < 3r$, so $2R - 3r < 0$.

Since $k > 0$ and $r > 0$ and $2R - 3r < 0$, then $\frac{dv}{dr} < 0$.

Thus, $\frac{dv}{dr} < 0$ for all $r \in (\frac{2R}{3}, R)$, so v is decreasing on the interval $(\frac{2R}{3}, R)$.

Therefore, by FDT, $v(\frac{2R}{3})$ is a relative maximum, so $r = \frac{2R}{3}$ will produce maximum air velocity. $\square$

Exercise 31. After a drug is administered to a patient, the drug concentration in the patient's bloodstream over a two hour period is given by $C = 0.29483t + 0.04253t^2 - 0.00035t^3$, where C is measured in milligrams and t is the time in minutes. On what interval is C increasing or decreasing?

Solution. Since C is a function of t over the interval $[0, 120]$ and $C(t) = 0.29483t + 0.04253t^2 - 0.00035t^3$, then C is a polynomial function. Hence, C is continuous and differentiable and $\frac{dC}{dt} = 0.29483 + 0.08506t - 0.00105t^2$.

Since $\frac{dC}{dt} = 0$ iff $0.29483 + 0.08506t - 0.00105t^2 = 0$ if $t = 84.3388$, then $\frac{dC}{dt} = 0$ when $t = 84.3388$ minutes.

Observe that C is continuous on $(0, 120)$ and differentiable on $(0, 84.3388)$ and differentiable on $(84.3388, 120)$.

Let $t \in (0, 84.3388)$ be $t = 10$.

Then $\frac{dC}{dt} = 0.29483 + 0.08506(10) - 0.00105(10^2) > 0$, so we assume $\frac{dC}{dt} > 0$ for all $t \in (0, 84.3388)$.

Hence, C is increasing on the interval $(0, 84.3388)$.

Let $t \in (84.3388, 120)$ be $t = 100$.

Then $\frac{dC}{dt} = 0.29483 + 0.08506(100) - 0.00105(100^2) < 0$, so we assume $\frac{dC}{dt} < 0$ for all $t \in (84.3388, 120)$.

Hence, C is decreasing on the interval $(84.3388, 120)$.

Therefore, C is increasing when t is between 0 and 84.3388 minutes and C is decreasing when t is between 84.3388 and 120 minutes. $\square$

Exercise 32. The resistance R of a certain type of resistor is given by $R = \sqrt{0.001T^4 - 4T + 100}$, where R is measured in ohms and the temperature T is measured in degrees Celsius. What temperature produces a minimum resistance for this type of resistor?

Solution. Observe that R is a function of T given by $R(T) = \sqrt{0.001T^4 - 4T + 100}$.

Thus, $\frac{dR}{dT} = \frac{1}{2}(0.001T^4 - 4T + 100)^{\frac{-1}{2}}[4(0.001)T^3 - 4] = \frac{2(0.001T^3 - 1)}{\sqrt{0.001T^4 - 4T + 100}}$.

We see that $0.001T^4 - 4T + 100 > 0$ for all T, so $\frac{dR}{dT} = 0$ iff $0.001T^3 - 1 = 0$ iff $T = 10$.

Observe that R is continuous on $(-\infty, \infty)$ and differentiable on $(-\infty, 10)$ and differentiable on $(10, \infty)$.

Let $t \in (-\infty, 10)$ be $t = 0$.
Then $\frac{dR}{dT} < 0$, so we assume $\frac{dR}{dT} < 0$ for all $t \in (-\infty, 10)$.
Hence, R is strictly decreasing on the interval $(-\infty, 10)$.
Let $t \in (10, \infty)$ be $t = 1000$.
Then $\frac{dR}{dT} > 0$, so we assume $\frac{dR}{dT} > 0$ for all $t \in (10, \infty)$.
Hence, R is strictly increasing on the interval $(10, \infty)$.
Thus, by FDT, $R(10)$ is a relative minimum, so a temperature of 10 degrees Celsius produces a minimum resistance. □

Exercise 33. Find a, b, c, d so that the function defined by $f(x) = ax^3 + bx^2 + cx + d$ has a relative minimum $f(0) = 0$ and relative maximum $f(2) = 2$.

Solution. Since $0 = f(0) = d$, then $d = 0$.
Since $2 = f(2) = 8a + 4b + 2c + d = 8a + 4b + 2c$, then $8a + 4b + 2c = 2$, so $4a + 2b + c = 1$.
Since f is a polynomial function, then f is differentiable and $f'(x) = 3ax^2 + 2bx + c$.
A relative extremum occurs when $f' = 0$.
Since $f(0)$ is a relative minimum, then $f'(0) = 0$.
Hence, $0 = f'(0) = c$, so $c = 0$.
Thus, $4a + 2b = 1$.
Since $f(2)$ is a relative maximum, then $f'(2) = 0$.
Hence, $0 = f'(2) = 12a + 4b + c = 12a + 4b + 0 = 12a + 4b$, so $12a + 4b = 0$.
Thus, $12a = -4b$, so $-3a = b$.
Therefore, $1 = 4a + 2(-3a) = 4a - 6a = -2a$, so $a = \frac{-1}{2}$ and $b = \frac{3}{2}$.
Consequently, $a = -\frac{1}{2}$ and $b = \frac{3}{2}$ and $c = 0 = d$, so $f(x) = -\frac{x^3}{2} + \frac{3x^2}{2}$. □

Chapter 12 Vectors and the geometry of space

Chapter 12.1 Vectors in the plane

Exercise 34. Let P be the point $(10, 2)$.
Let Q be the point $(6, -1)$.
Let $\overrightarrow{PQ}$ be the directed line segment representing vector $v \in \mathbb{R}^2$.
Write v in component form.

Solution. Since $v \in \mathbb{R}^2$, then there exist $v_1, v_2 \in \mathbb{R}$ such that $v = (v_1, v_2)$.
Observe that

$$\begin{aligned} v &= (v_1, v_2) \\ &= (6 - 10, -1 - 2) \\ &= (-4, -3). \end{aligned}$$

Therefore, $v = (-4, -3)$. □

Exercise 35. Let P be the point $(\frac{3}{2}, \frac{4}{3})$.

Let Q be the point $(\frac{1}{2}, 3)$.

Let $\overrightarrow{PQ}$ be the directed line segment representing vector $v \in \mathbb{R}^2$.

Write v in component form.

Solution. Since $v \in \mathbb{R}^2$, then there exist $v_1, v_2 \in \mathbb{R}$ such that $v = (v_1, v_2)$.

Observe that

$$\begin{aligned} v &= (v_1, v_2) \\ &= (\frac{1}{2} - \frac{3}{2}, 3 - \frac{4}{3}) \\ &= (-1, \frac{5}{3}). \end{aligned}$$

Therefore, $v = (-1, \frac{5}{3})$. $\square$

Exercise 36. Let P be the point $(0.12, 0.60)$.

Let Q be the point $(0.84, 1.25)$.

Let $\overrightarrow{PQ}$ be the directed line segment representing vector $v \in \mathbb{R}^2$.

Write v in component form.

Solution. Since $v \in \mathbb{R}^2$, then there exist $v_1, v_2 \in \mathbb{R}$ such that $v = (v_1, v_2)$.

Observe that

$$\begin{aligned} v &= (v_1, v_2) \\ &= (0.84 - 0.12, 1.25 - 0.60) \\ &= (0.72, 0.65). \end{aligned}$$

Therefore, $v = (0.72, 0.65)$. $\square$

Exercise 37. Let $u, v, w \in \mathbb{R}^2$ such that $u = (1, 2)$ and $v = (1, 1)$ and $w = (1, -1)$.

Express v as a linear combination of u and w, if possible.

Solution. Suppose v is a linear combination of u and w.

Then there exist scalars $a \in \mathbb{R}$ and $b \in \mathbb{R}$ such that $v = au + bw$.

Thus, $(1, 1) = a(1, 2) + b(1, -1) = (a, 2a) + (b, -b) = (a + b, 2a - b)$, so $a + b = 1$ and $2a - b = 1$.

Hence, $3a = 2$, so $a = \frac{2}{3}$ and $b = 1 - a = 1 - \frac{2}{3} = \frac{1}{3}$.

Therefore, $a = \frac{2}{3}$ and $b = \frac{1}{3}$, so $v = \frac{2}{3}u + \frac{1}{3}w$. $\square$

Exercise 38. Let P be the point $(4, 2)$.

Let $\overrightarrow{PQ}$ be the directed line segment representing vector $v \in \mathbb{R}^2$ given by $v = (-1, 3)$.

Find point Q.

Solution. Let $q_1, q_2 \in \mathbb{R}$ be the coordinates of Q.

Then $Q = (q_1, q_2)$.

Since $v = (-1, 3) = (q_1 - 4, q_2 - 2)$, then $q_1 - 4 = -1$ and $q_2 - 2 = 3$, so $q_1 = 3$ and $q_2 = 5$.

Therefore, $Q = (3, 5)$. $\square$

Exercise 39. Let $v \in \mathbb{R}^2$ be given by $v = 6e_1 - 5e_2$.

Compute the norm of vector v.

Solution. Since $v = 6e_1 - 5e_2 = (6, -5)$, then $||v|| = \sqrt{6^2 + (-5)^2} = \sqrt{61}$.

Therefore, the norm of v is $\sqrt{61}$. $\square$

Exercise 40. Let $u, v \in \mathbb{R}^2$ be given by $u = (1, \frac{1}{2})$ and $v = (2, 3)$.

Compute the following:

a. $||u||$

b. $||v||$

c. $||u + v||$

d. $||\frac{u}{||u||}||$

e. $||\frac{v}{||v||}||$

f. $||\frac{u+v}{||u+v||}||$

Solution. a. Observe that $||u|| = ||(1, \frac{1}{2})|| = \sqrt{1^2 + (\frac{1}{2})^2} = \frac{\sqrt{5}}{2}$.

b. Observe that $||v|| = ||(2, 3)|| = \sqrt{2^3 + 3^2} = \sqrt{13}$.

c. Observe that $||u+v|| = ||(1, \frac{1}{2}) + (2, 3)|| = ||(3, \frac{7}{2}|| = \sqrt{3^2 + (\frac{7}{2})^2} = \frac{\sqrt{85}}{2}$.

d. Observe that $||\frac{u}{||u||}|| = ||\frac{2(1, \frac{1}{2})}{\sqrt{5}}|| = ||(\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}|| = \sqrt{(\frac{2}{\sqrt{5}})^2 + (\frac{1}{\sqrt{5}})^2} = \sqrt{\frac{4}{5} + \frac{1}{5}} = 1$.

The vector $\frac{u}{||u||}$ is a unit vector in the direction of u, so its length is 1.

e. The vector $\frac{v}{||v||}$ is a unit vector in the direction of v, so its length is 1.

Therefore, $||\frac{v}{||v||}|| = 1$.

f. The vector $\frac{u+v}{||u+v||}$ is a unit vector in the direction of $u + v$, so its length is 1.

Therefore, $||\frac{u+v}{||u+v||}|| = 1$. $\square$

Exercise 41. Let $u, v \in \mathbb{R}^2$ be given by $u = (2, -4)$ and $v = (5, 5)$.

Compute the following:

a. $||u||$

b. $||v||$
c. $||u+v||$

Solution. a. Observe that

$$\begin{aligned} ||u|| &= ||(2,-4)|| \\ &= ||2(1,-2)|| \\ &= |2| \cdot ||(1,-2)|| \\ &= 2\sqrt{1^2+(-2)^2} \\ &= 2\sqrt{5}. \end{aligned}$$

b. Observe that

$$\begin{aligned} ||v|| &= ||(5,5)|| \\ &= ||5(1,1)|| \\ &= |5| \cdot ||(1,1)|| \\ &= 5\sqrt{1^2+1^2} \\ &= 5\sqrt{2}. \end{aligned}$$

c. Observe that

$$\begin{aligned} ||u+v|| &= ||(2,-4)+(5,5)|| \\ &= ||(7,1)|| \\ &= \sqrt{7^2+1^2} \\ &= 5\sqrt{2}. \end{aligned}$$

□

Exercise 42. Let $u, v \in \mathbb{R}^2$ such that $||v|| = 4$ and $u = (1,1)$ and v has the same direction as u.

Find v.

Solution. Since v has length 4 and is in the same direction as u, then v is a scalar multiple of the unit vector in the direction of u.

Observe that

$$\begin{aligned} v &= ||v||\frac{u}{||u||} \\ &= 4\frac{(1,1)}{||(1,1)||} \\ &= \frac{4(1,1)}{\sqrt{1^2+1^2}} \\ &= \frac{4}{\sqrt{2}}(1,1) \\ &= 2\sqrt{2}(1,1) \\ &= (2\sqrt{2}, 2\sqrt{2}). \end{aligned}$$

Therefore, $v = (2\sqrt{2}, 2\sqrt{2})$. $\square$

Exercise 43. Let $f : \mathbb{R} \to \mathbb{R}$ be the function given by $f(x) = x^3$ and let P be the point $(1, 1)$.

a. Find a unit vector parallel to the graph of f at P.

b. Find a unit vector normal to the graph of f at P.

Solution. Part a.

A vector parallel to the graph of f at P has the same slope as the line tangent to the graph of f at P.

Let v be a vector parallel to the graph of f at P.

Let ℓ be the line tangent to the graph of f at P.

Since ℓ is a line tangent to f at P, then ℓ has slope equal to the slope of f at P.

Since $f(x) = x^3$, then $f'(x) = 3x^2$, so $f'(1) = 3$.

Thus, ℓ has slope 3 and contains the point P, so $\frac{y-1}{x-1} = 3$ is an equation of ℓ.

Therefore, $y = 3x - 2$ is another equation for line ℓ.

Since v has the same slope as line ℓ, then a representation of v can be chosen from any two points on ℓ.

Since $(0, -2) \in \ell$, let Q be the point $(0, -2)$.

Let $\overrightarrow{QP}$ be the directed line segment representing vector v.

Then $v = (1 - 0, 1 - (-2)) = (1, 3)$.

A unit vector parallel to the graph of f at P is $\frac{v}{||v||}$.

Observe that $\frac{v}{||v||} = \frac{(1,3)}{\sqrt{1^2+3^2}} = \frac{1}{\sqrt{10}}(1, 3)$.

Since v and $-v$ are both vectors parallel to the graph of f at P, then there are two such unit vectors: $\pm\frac{1}{\sqrt{10}}(1, 3)$. $\square$

Solution. Part b.

A vector normal to the graph of f at P has the same slope as the line normal to the graph of f at P.

Let w be a vector normal to the graph of f at P.

Let m be the line normal to the graph of f at P.

Since lines ℓ and m are perpendicular and the slope of ℓ is 3, then the slope of m is $-\frac{1}{3}$.

Since m has slope $-\frac{1}{3}$ and contains the point P, then $\frac{y-1}{x-1} = -\frac{1}{3}$ is an equation of m.

Therefore, $y = \frac{4-x}{3}$ is another equation for line m.

Since w has the same slope as line m, then a representation of w can be chosen from any two points on m.

Since $(4, 0) \in m$, let Q' be the point $(4, 0)$.

Let $\overrightarrow{PQ'}$ be the directed line segment representing vector w.

Then $w = (4 - 1, 0 - 1) = (3, -1)$.

A unit vector normal to the graph of f at P is $\frac{w}{||w||}$.

Observe that $\frac{w}{||w||} = \frac{(3,-1)}{\sqrt{3^2 + (-1)^2}} = \frac{(3,-1)}{\sqrt{10}} = \frac{1}{\sqrt{10}}(3, -1)$.

Since w and $-w$ are both vectors normal to the graph of f at P, then there are two such unit vectors: $\pm\frac{1}{\sqrt{10}}(3, -1)$. $\square$