# Book Multivariable Math by Williamson Exercises

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## Chapter 1 Vectors

### Chapter 1.1 Coordinate Vectors

**Exercise 1.** Let $x, y, z \in \mathbb{R}^3$ such that $x = (3, -1, 0)$ and $y = (0, 1, 5)$ and $z = (2, 5, -1)$.

Compute $3x$ and $y + z$ and $4x - 2y + 3z$.

**Solution.** Observe that $3x = 3(3, -1, 0) = (9, -3, 0)$.

Observe that $y + z = (0, 1, 5) + (2, 5, -1) = (2, 6, 4)$.

Observe that

$$\begin{aligned} 4x - 2y + 3z &= 4(3, -1, 0) - 2(0, 1, 5) + 3(2, 5, -1) \\ &= (12, -4, 0) - (0, 2, 10) + (6, 15, -3) \\ &= (12, -6, -10) + (6, 15, -3) \\ &= (18, 9, -13). \end{aligned}$$

Therefore, $(18, 9, -13) = 4x - 2y + 3z$ is a linear combination of the vectors $x, y, z$. □

**Exercise 2.** Find numbers $a$ and $b$ such that $ax + by = (9, -1, 10)$, where $x = (3, -1, 0)$ and $y = (0, 1, 5)$.

Is there more than one solution?

**Solution.** Let $a, b \in \mathbb{R}$ such that $ax + by = (9, -1, 10)$.

Observe that

$$\begin{aligned} (9, -1, 10) &= ax + by \\ &= a(3, -1, 0) + b(0, 1, 5) \\ &= (3a, -a, 0) + (0, b, 5b) \\ &= (3a, -a + b, 5b). \end{aligned}$$

Thus, $(9, -1, 10) = (3a, -a + b, 5b)$, so $3a = 9$ and $-a + b = -1$ and $5b = 10$.

Hence, $a = 3$ and $-a + b = -1$ and $b = 2$, and $-1 = -a + b = -3 + 2 = -1$.
Therefore, $a = 3$ and $b = 2$.
There is only one solution, $a = 3$ and $b = 2$. $\square$

**Exercise 3.** Show that no choice of numbers $a$ and $b$ can make $ax+by = (3, 0, 0)$, where $x = (3, -1, 0)$ and $y = (0, 1, 5)$.

For what values of $c$, if any, can the equation $ax + by = (3, 0, c)$ be satisfied?

**Solution.** Suppose there exist real numbers $a$ and $b$ such that $ax+by = (3, 0, 0)$.

Observe that

$$\begin{aligned} (3,0,0) &= ax + by \\ &= a(3,-1,0) + b(0,1,5) \\ &= (3a,-a,0) + (0,b,5b) \\ &= (3a,-a+b,5b). \end{aligned}$$

Thus, $(3, 0, 0) = (3a, -a + b, 5b)$, so $3a = 3$ and $-a + b = 0$ and $5b = 0$.
Hence, $a = 1$ and $-a + b = 0$ and $b = 0$.
Consequently, $0 = -a + b = -1 + 0 = -1$, so $0 = -1$, a contradiction.
Therefore, there are no real numbers $a$ and $b$ such that $ax+by = (3, 0, 0)$. $\square$

**Solution.** Suppose there exist real numbers $a$ and $b$ such that $ax+by = (3, 0, c)$, where $c \in \mathbb{R}$.

Observe that

$$\begin{aligned} (3,0,c) &= ax + by \\ &= a(3,-1,0) + b(0,1,5) \\ &= (3a,-a,0) + (0,b,5b) \\ &= (3a,-a+b,5b). \end{aligned}$$

Thus, $(3, 0, c) = (3a, -a + b, 5b)$, so $3a = 3$ and $-a + b = 0$ and $5b = c$.
Hence, $a = 1$ and $-a + b = 0$ and $b = \frac{c}{5}$.
Consequently, $0 = -a + b = -1 + \frac{c}{5} = \frac{-5 + c}{5}$, so $0 = \frac{-5 + c}{5}$.
Therefore, $0 = -5 + c$, so $c = 5$.
If there are real numbers $a$ and $b$ such that $ax+by = (3, 0, c)$, then $c = 5$. $\square$

**Exercise 4.** Prove that the representation of a vector $x \in \mathbb{R}^n$ in terms of the natural basis is unique.

That is, show that if $x_1e_1 + x_2e_2 + ... + x_ne_n = y_1e_1 + y_2e_2 + ... + y_ne_n$, then $x_k = y_k$ for each $k = 1, 2, ..., n$.

*Proof.* Let $x \in \mathbb{R}^n$.
Then there exist $x_1, x_2, ..., x_n \in \mathbb{R}$ such that $x = (x_1, x_2, ..., x_n)$.
Let $x_1e_1 + x_2e_2 + ... + x_ne_n$ be a representation of $x$.
Then $x = x_1e_1 + x_2e_2 + ...x_ne_n$.
Suppose $y_1e_1 + y_2e_2 + ... + y_ne_n$ is another representation of $x$ for some $y_1, y_2, ..., y_n \in \mathbb{R}$.

Then $x = y_1e_1 + y_2e_2 + ... + y_ne_n$.

Thus, $x_1e_1 + x_2e_2 + ... + x_ne_n = x = y_1e_1 + y_2e_2 + ... + y_ne_n$, so $x_1e_1 + x_2e_2 + ... + x_ne_n = y_1e_1 + y_2e_2 + ... + y_ne_n$.

Observe that

$$\begin{aligned} x_1e_1 + x_2 + e_2 + ... + x_ne_n &= y_1e_1 + y_2e_2 + ... + y_ne_n \\ x_1(1,0,...,0) + x_2(0,1,...,0) + ... + x_n(0,0,...,1) &= y_1(1,0,...,0) + y_2(0,1,...,0) + ... + y_n(0,0,...,1) \\ (x_1,0,...,0) + (0,x_2,...,0) + ... + (0,0,...,x_n) &= (y_1,0,...,0) + (0,y_2,...,0) + ... + (0,0,...,y_n) \\ (x_1,x_2,...,x_n) &= (y_1,y_2,...,y_n). \end{aligned}$$

Therefore, $x_1 = y_1$ and $x_2 = y_2$ and ... and $x_n = y_n$, so $x_k = y_k$ for each $k = 1,2,...,n$. □

**Exercise 5.** Represent the first vector as a linear combination of the remaining vectors, either by inspection or by solving an appropriate system of equations.

a. $(2,3,4); (1,1,1), (1,2,1), (-1,1,2)$

b. $(2,-7); (1,1), (1,-1)$

c. $(-2,3); e_1, e_2$

**Solution.** a. Let $a,b,c \in \mathbb{R}$ such that $(2,3,4) = a(1,1,1)+b(1,2,1)+c(-1,1,2)$.

Observe that

$$\begin{aligned} (2,3,4) &= a(1,1,1) + b(1,2,1) + c(-1,1,2) \\ &= (a,a,a) + (b,2b,b) + (-c,c,2c) \\ &= (a+b, a+2b, a+b) + (-c,c,2c) \\ &= (a+b-c, a+2b+c, a+b+2c). \end{aligned}$$

Thus, $(2,3,4) = (a+b-c, a+2b+c, a+b+2c)$, so $a+b-c = 2$ and $a+2b+c = 3$ and $a+b+2c = 4$.

Hence, $2a + 3b = 5$ and $-a - 3b = -2$, so $a = 3$.

Consequently, $3b = 5 - 2a = 5 - 2(3) = -1$, so $b = -\frac{1}{3}$, and $c = a + b - 2 = 3 - \frac{1}{3} - 2 = \frac{2}{3}$.

Therefore, $a = 3$ and $b = -\frac{1}{3}$ and $c = \frac{2}{3}$, so $(2,3,4) = 3(1,1,1) - \frac{1}{3}(1,2,1) + \frac{2}{3}(-1,1,2)$.

b. Let $a,b \in \mathbb{R}$ such that $(2,-7) = a(1,1) + b(1,-1)$.

Observe that

$$\begin{aligned} (2,-7) &= a(1,1) + b(1,-1) \\ &= (a,a) + (b,-b) \\ &= (a+b, a-b). \end{aligned}$$

Thus, $(2,-7) = (a+b, a-b)$, so $a+b = 2$ and $a-b = -7$, so $2a = -5$.

Hence, $a = -\frac{5}{2}$ and $b = 2 - a = 2 - (-\frac{5}{2}) = \frac{9}{2}$, so $a = -\frac{5}{2}$ and $b = \frac{9}{2}$.
Therefore, $(2, -7) = -\frac{5}{2}(1, 1) + \frac{9}{2}(1, -1)$.

c. Let $a, b \in \mathbb{R}$ such that $(-2, 3) = ae_1 + be_2$.
Then $a = -2$ and $b = 3$, so $(-2, 3) = -2e_1 + 3e_2$. □

**Exercise 6.** Let $x = (5, 500, 10)$ represent the amount of ink, paper, and binding material needed to produce a single copy of some book, and let $y = (4, 800, 90)$ be the same vector for some other book.

Interpret $100x + 50y$.

What about $100x - 50y$?

**Solution.** The expression $100x + 50y$ represents the total amount of ink, paper, and binding material required to produce 100 copies of the first book and 50 copies of the second book, whereas the expression $100x - 50y$ is the difference in the amount of ink, paper, and binding material required to produce 100 copies of the first book once 50 copies of the second book have been produced. □

## Chapter 1.2 Geometric Vectors