

General Math Notes

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Mathematics is the **science of quantity**.

Mathematics is the abstract science of measure and number.

Mathematics is the science which investigates the means of measuring quantity.

Measure

A magnitude is a primitive concept that specifies amount, size, dimension, or extent.

Definition 1. magnitude

A **magnitude** is whatever is capable of increase or diminution.

The word **quantity** is a synonym for magnitude.

A magnitude is not a number.

We use ratios to compare sizes and choose a unit as our measuring tool.

A ratio is a way to compare one thing to another.

A ratio is a comparison of two magnitudes—one being measured and the other serving as a reference.

Definition 2. ratio

A **ratio** is a qualitative comparison between two magnitudes, one being measured and the other serving as a reference.

The **antecedent of a ratio** indicates what is being measured.

The **consequent of a ratio** is the reference or common measure.

We may denote a ratio with antecedent a and consequent c by $a : c$.

A ratio shows relative extent, but not measured extent.

A ratio allows us to express how one magnitude relates to another, such as in size, proportion, or scale.

Definition 3. unit

A **unit** is a ratio of equal magnitudes.

A unit is arbitrary and serves as the standard of measure.

A unit introduces the possibility of algebraic measurement.

Definition 4. quotientness

A ratio has the property of **quotientness** iff both its parts can be measured without remainder using the same magnitude(unit).

If a given magnitude measures both parts of a ratio without remainder, then that ratio has the property of quotientness.

Quotientness property is the critical test for number concept.

Quotientness is purely geometric: match lengths/distances against a unit.

Quotientness does not require counting or number knowledge. Its focus is on relational fit, not numerical value.

Definition 5. natural number

A **natural number** is a ratio whose antecedent is measured by the unit.

A natural number arises from measure of ratios of magnitudes whose consequents are all the unit.

A natural number is a special kind of ratio in which the antecedent(numerator) is a multiple of the abstract unit, and the abstract unit is the consequent (denominator).

The natural number 1 is a ratio in which the unit measures the antecedent once.

The natural number 2 is a ratio in which the unit measures the antecedent twice.

The natural number 3 is a ratio in which the unit measures the antecedent three times.

The natural number 4 is a ratio in which the unit measures the antecedent four times.

The natural number 5 is a ratio in which the unit measures the antecedent five times.

Only successful geometric measure yields a natural number.

Natural numbers introduce counting numbers.

Definition 6. number

A **number** is a measure of a ratio of natural numbers.

Number arises from the measure of ratios of numbers whose denominators are all the abstract unit.

A number is a name given to a measure that describes a ratio of magnitudes.

A number is a rational number.

Example 7. number

Since $\frac{3}{4}$ represents 3 units measured out of 4, then $\frac{3}{4}$ is a number.

Observe that $\frac{3}{4} = \text{measure}(3 : 4)$.

The number concept is the basis for algebra.

A number is a name assigned to a measure of a ratio of magnitudes.

Only successful algebraic measure using abstract unit yields number.

Definition 8. ratio without measure

A **ratio without measure** is a ratio in which quotientness fails.

A ratio without measure cannot be measured.

If quotientness fails, then a ratio of magnitudes or numbers cannot be measured.

No quotientness means no natural number.

Example 9. Diagonal to side ratio of a square is a ratio without measure.

Let s be the length of the side of a square.

Let d be the length of the diagonal of the square.

Then $d^2 = s^2 + s^2 = 2s^2$, so $\frac{d}{s} = \sqrt{2}$.

Therefore, the ratio ($d : s$) is a ratio without measure.

The expression $\sqrt{2}$ is just an attempted measure, but not a number at all.

We regard $\sqrt{2}$ as just a constant and we can only approximate its value.

Example 10. Circumference of a circle to its diameter ratio is a ratio that cannot be measured using any unit.

Let C be the circumference of a circle with diameter d .

Then $C = \pi d$, so $\frac{C}{d} = \pi$.

Let u be a unit that measures the diameter.

Then u measures the diameter with no remainder.

If we try to measure the circumference by the diameter, then there is always a remainder of the circumference left that cannot be measured.

Hence, if we use u to measure the circumference, then there is always a remainder of the circumference that cannot be measured.

Therefore, the ratio $C : d$ cannot be measured by any unit u , so the ratio $C : d$ is a ratio without measure.

We observe that π is a constant that can only be approximated. In fact, π is just a magnitude, not a number.

No common divisor means no fraction(number)??

Geometric measure uses a physical unit and/or parts of a physical unit.

Geometric measure uses only magnitudes.

Algebraic measure uses an abstract unit and/or equal parts of an abstract unit.

Algebraic measure uses numbers.

By using the abstract unit, through natural numbers, we can give any ratio that has a quotient, a name or measure.

Example 11. ratio with a name or measure

The ratio of two eggs to three bowls of rice is $2 : 3$.

The ratio of 5 black boxes to 2 balls is $5 : 2$.

Definition 12. fraction

A **fraction** is the measure of a ratio of numbers.

Any well-formed number can be expressed as a fraction of ratios, where the ratio antecedent is a multiple of the ratio consequent.

Definition 13. placeholder

A placeholder is a symbol that can be replaced by a number name.

A placeholder is incorrectly called a variable in mainstream math.

Definition 14. zero

Zero, denoted 0, is a placeholder that represents the absence of magnitude.

Zero is not a number because it does not represent a measure of a ratio of magnitudes.

Hence, zero cannot act as a part of any ratio or be used to measure anything.

Since zero represents a lack of magnitude, zero contradicts the idea that numbers are intended to quantify and measure.

However, zero is useful as a placeholder and in communicating numbers in a given radix system, such as the base ten decimal system.

Since zero represents the absence of magnitude, then zero cannot act as a dividend, divisor or valid part of a ratio.

This is because the division process inherently involves measuring one magnitude by another magnitude.

But, with zero, there is no magnitude to measure.

The expression $0 : 0$ is not a valid ratio at all.

Arithmetic Operations

There are 4 arithmetic operations: subtraction, addition, division, multiplication.

These arithmetic operations are not mere human constructs; they exist independently of the human mind, as objective truths.

The arithmetic operation of division is about the divisor measuring the dividend.

Division operation defines the concept of measure.

Definition 15. division

Let a and b be homogeneous quantities.

The **division** of a by b , denoted $a \div b$, is the measurement of a using b as a unit of measure.

The **dividend** is a and the **divisor** is b .

The result of dividing a by b is a quantity q called the **quotient** and a quantity r called the **remainder** such that $a \div b = q$ with remainder r .

Long division algorithm is a technique that involves measuring a dividend or partial dividend with a divisor or partial divisor, resulting in a quotient or partial quotients whose sum is the final quotient.

Division by zero is not defined.

Remainder and factor- a remainder represents an incomplete measure or a non-whole measure.

The absence of a remainder leads to the identification of a factor.

Division is a complete measure; when there is a remainder, it involves parts of the divisor and when there is no remainder, the divisor is a factor of the dividend.

Polynomial division- this process directly emulates the actions of long division.

In polynomial division the division is exactly the same as long division, except there is the use of placeholders instead of numerical values.

multiplication- this operation is fundamentally derived from reciprocal division

Multiplication is the inverse of division.

In geometry, measurement is conducted using a **physical unit** and, when necessary, parts of that physical unit.

In algebra, measurement is performed using an **abstract unit** and, when needed, equal parts of that same abstract unit.

Definition 16. reciprocal of a quantity

Let a be a homogeneous quantity.

The **reciprocal** of a , denoted $\frac{1}{a}$, is the division of 1 by a .

Therefore, $\frac{1}{a} = 1 \div a$.

Should we include definition of multiplicative inverse of a as $a^{-1} = \frac{1}{a}$?

Definition 17. division of a reciprocal

Let p and q be natural numbers.

Then $p \div \frac{1}{q} = \frac{p + p + \dots + p}{\frac{1}{q} + \frac{1}{q} + \dots + \frac{1}{q}}$, where the numerator is the sum of q p 's and

the denominator is the sum of q $\frac{1}{q}$'s.

Definition 18. multiplication

Let a and b be homogeneous quantities.

The **multiplication** of a by b , denoted ab , is the division of a by the reciprocal of b .

Therefore, $ab = a \div \frac{1}{b}$ and ab is the **product** of a and b .

Factors arise from division.

The concept of a factor is inherently linked to the process of division.

Number concept

Definition 19. factor

Let a and b be homogeneous quantities.

The product ab is a quantity, and we say that a is a **factor** of ab .

Similarly, we say b is a **factor** of ab .

Factors includes integers and fractions acting as units, and not solely just integers as factors.

So, factors can be understood as whole number measures as well as fractional units.

Thus, the factor concept emerges from the division operation as a measure, and not tied to being only in terms of integers.

This ensures a strong and coherent foundation for understanding and connecting to more complex mathematical ideas.

We cannot logically extend the factor concept to real numbers because the concept of a real number is flawed.

The concept of a real number is an ill-formed concept.

Infinite things do not exist.

There is no such thing as ‘all the positive integers’.

Laws of arithmetic operations

Proposition 20. *Let a be a homogeneous quantity.*

Then $a \div 1 = a$.

Proposition 21. *Let a be a homogeneous quantity.*

Then $a \div a = 1$.

Proposition 22. *The reciprocal of the reciprocal of a is a .*

Let a be a nonzero homogeneous quantity.

Then $\frac{1}{\frac{1}{a}} = a$.

Proposition 23. *The product of a homogeneous quantity and its reciprocal is the unit.*

Let a be a homogeneous quantity.

Then $\frac{1}{a} \cdot a = 1$.

Lemma 24. *Let a and b be natural numbers.*

Then $a \div \frac{1}{b} = b \div \frac{1}{a}$.

Proposition 25. *Multiplication of rational numbers is commutative.*

Let a and b be rational numbers.

Then $ab = ba$.

OLD Review carefully below and ensure everything has a proof that is logical and sound and has all details.

Trig Facts and Identities

θ	$\sin(\theta)$	$\cos(\theta)$
0	0	1
$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$
$\frac{\pi}{2}$	1	0

Let $\theta \in \mathbb{R}$.

Then $\cos(\frac{\pi}{2} - \theta) = \sin(\theta)$.

Let $a, b \in \mathbb{R}$.

$\sin(a + b) = \sin a \cos b + \cos a \sin b$.

$\cos(a + b) = \cos a \cos b - \sin a \sin b$

Definition 26. Max/Min binary operations

Let $x, y \in \mathbb{R}$.

Define binary operations $\max : \mathbb{R}^2 \rightarrow \mathbb{R}$ and $\min : \mathbb{R}^2 \rightarrow \mathbb{R}$ by

$$\max(x, y) = x \vee y = \begin{cases} x, & x \geq y \\ y, & y \geq x \end{cases}$$

$$\min(x, y) = x \wedge y = \begin{cases} x, & x \leq y \\ y, & y \leq x \end{cases}$$

Let $x, y, z \in \mathbb{R}$.

Then $x \wedge (y \wedge z) = (x \wedge y) \wedge z$ (max is associative)

Definition 27. Convex Set

Let V be a vector space over field $\mathbb{R}$.

Let $S \subseteq V$.

Then S is **convex** iff $(\forall \vec{v}, \vec{w} \in S)(\forall t \in \mathbb{R}, 0 \leq t \leq 1)[t\vec{v} + (1-t)\vec{w} \in S]$.

The closed interval $[0, 1] \in \mathbb{R}^1$ is convex.

Let S_1 and S_2 be arbitrary convex sets in vector space V .

Let $\vec{v}, \vec{w} \in S_1 \cap S_2$ be arbitrary.

Let $t \in \mathbb{R}$ such that $t \in [0, 1]$.

Since $\vec{v} \in S_1 \cap S_2$, then $\vec{v} \in S_1$ and $\vec{v} \in S_2$.

Since $\vec{w} \in S_1 \cap S_2$, then $\vec{w} \in S_1$ and $\vec{w} \in S_2$.

Since S_1 is convex, then $t\vec{v} + (1-t)\vec{w} \in S_1$.

Since S_2 is convex, then $t\vec{v} + (1-t)\vec{w} \in S_2$.

Hence, $t\vec{v} + (1-t)\vec{w} \in S_1 \cap S_2$.

Thus, the intersection of convex sets S_1 and S_2 is convex.

Since S_1 and S_2 are arbitrary, then the intersection of convex sets S_1 and S_2 is convex for every S_1 and every S_2 .

Therefore the intersection of any two convex sets in a vector space is convex.