Math Notes Theory

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Laws of arithmetic operations

Proposition 1. *Let a be a homogeneous quantity.*
Then $a \div 1 = a$.

Proof. The measurement of a using 1 as a unit of measure means that it requires a of a unit to measure a.
Therefore, $a \div 1 = a$. $\square$

Proposition 2. *Let a be a homogeneous quantity.*
Then $a \div a = 1$.

Proof. The measurement of a using a as a unit of measure means that it requires exactly one unit to measure a.
Therefore, $a \div a = 1$. $\square$

Proposition 3. ***The reciprocal of the reciprocal of a is a.***
Let a be a nonzero homogeneous quantity.
Then $\frac{1}{\frac{1}{a}} = a$.

Proof. The division of 1 by $\frac{1}{a}$ as a unit of measure means 1 is measured by $\frac{1}{a}$, so it requires a units to measure 1.
Hence, $1 \div \frac{1}{a} = a$, so $\frac{1}{\frac{1}{a}} = 1 \div \frac{1}{a} = a$.
Therefore, $\frac{1}{\frac{1}{a}} = a$. $\square$

Proposition 4. ***The product of a homogeneous quantity and its reciprocal is the unit.***
Let a be a homogeneous quantity.
Then $\frac{1}{a} \cdot a = 1$.

Proof. Since $\frac{1}{a} \cdot a = \frac{1}{a} \div \frac{1}{a} = 1$, then $\frac{1}{a} \cdot a = 1$. $\square$

Lemma 5. *Let a and b be natural numbers.*

Then $a \div \frac{1}{b} = b \div \frac{1}{a}$.

Proof. When a is measured by $\frac{1}{b}$ as the unit of measure, then a is divided into equal $\frac{1}{b}$ parts.

This requires b units to measure 1, so a sum of a b's is required to measure a.

Hence, the sum $b + b + ... + b$, where there are a occurrences of b, measures a, so $a \div \frac{1}{b} = b + b + ... + b$.

When b is measured by $\frac{1}{a}$ as the unit of measure, then b is divided into equal $\frac{1}{a}$ parts.

This requires a units to measure 1, so a sum of b a's is required to measure b.

Hence, the sum $a + a + ... + a$, where there are b occurrences of a, measures b, so $b \div \frac{1}{a} = a + a + ... + a$.

TODO: How to finish this proof? □

Proposition 6. ***Multiplication of rational numbers is commutative.***

Let a and b be rational numbers.

Then $ab = ba$.

Proof. Observe that

$$\begin{aligned} ab &= a \div \frac{1}{b} \\ &= b \div \frac{1}{a} \\ &= ba. \end{aligned}$$

□