General Math Exercises

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Exercise 1. There exists a positive real number x for which $x^2 < \sqrt{x}$.

Proof. Observe that when $x = \frac{1}{2}$ then $(\frac{1}{2})^2 = \frac{1}{4}$ which is less than $\sqrt{\frac{1}{2}} = \frac{1}{\sqrt{2}}$ $\square$

Exercise 2. Let $x, y \in \mathbb{R}$.
If $x^2 + 5y = y^2 + 5x$, then $x = y$ or $x + y = 5$.

Proof. Suppose $x^2 + 5y = y^2 + 5x$.
Then $x^2 - y^2 = 5x - 5y$, and factoring gives $(x - y)(x + y) = 5(x - y)$.
There are two cases to consider.
Case 1. Suppose $x - y = 0$.
Then adding y to both sides gives $x = y$.
Case 2. Suppose $x - y \neq 0$.
We can divide both sides of the equation $(x - y)(x + y) = 5(x - y)$ by the nonzero number $x - y$ to get $x + y = 5$.

Thus either $x = y$ or $x + y = 5$. $\square$

Exercise 3. Let $x \in \mathbb{R}$. If $x^5 + 7x^3 + 5x \geq x^4 + x^2 + 8$, then $x \geq 0$.

Solution. Using proof by contrapositive is more useful than using direct proof in this specific instance. $\square$

Proof. Suppose it is not the case that $x \geq 0$, so $x < 0$.
Then x^5, $7x^3$, and $5x$ are negative and x^4, x^2, and 8 are positive.
Thus the sum $x^5 + 7x^3 + 5x$ must be less than the sum $x^4 + x^2 + 8$, so $x^5 + 7x^3 + 5x < x^4 + x^2 + 8$.
Consequently it is not the case that $x^5 + 7x^3 + 5x \geq x^4 + x^2 + 8$. $\square$

Exercise 4. Let $x \in \mathbb{R}$. If $x^3 - x > 0$ then $x > -1$.

Proof. Suppose it is not the case that $x > -1$.
Then $x \leq -1$ which implies $x + 1 \leq 0$.
Since $x < 0$ and $x - 1 < 0$ then we have $x(x - 1) > 0$.
Multiply the inequality by $x + 1$ to get $x(x - 1)(x + 1) \leq 0$.
Simplifying gives $x^3 - x \leq 0$.
Thus it is not the case that $x^3 - x > 0$. $\square$

Exercise 5. Let $x \in \mathbb{R}$. If $x^2 + 5x < 0$, then $x < 0$.

Proof. We use proof by contrapositive.
Suppose it is not the case that $x < 0$.
Then $x \geq 0$.
Neither x^2 nor $5x$ is negative, so $x^2 + 5x \geq 0$.
Thus it is not true that $x^2 + 5x < 0$. $\square$

Exercise 6. Let $x, y \in \mathbb{R}$ such that $xy = 6$ and $x > 2$.
Then $y < 3$.

Proof. Since $x > 2$, then $3x > 3 \cdot 2 = 6 = xy = yx$, so $3x > yx$.
Since $x > 2$ and $2 > 0$, then $x > 0$.
Hence, $3 > y$, so $y < 3$. $\square$

Exercise 7. Let $x, y \in \mathbb{R}$. If $y^3 + yx^2 \leq x^3 + xy^2$, then $y \leq x$.

Proof. We use proof by contrapositive.
Suppose it is not true that $y \leq x$. Then $y > x$ which implies that $y - x > 0$.
Multiply both sides of $y - x > 0$ by the positive value $x^2 + y^2$.

$$\begin{aligned} (y-x)(x^2+y^2) &> 0 \\ yx^2 + y^3 - x^3 - xy^2 &> 0 \\ y^3 + yx^2 &> x^3 + xy^2 \end{aligned}$$

Therefore $y^3 + yx^2 > x^3 + xy^2$, so it is not true that $y^3 + yx^2 \leq x^3 + xy^2$. $\square$

Exercise 8. Let $x, y \in \mathbb{R}^+$. If $x < y$, then $x^2 < y^2$.

Proof. Let $x, y \in \mathbb{R}^+$ and $x < y$. Then $x - y < 0$.
Since $x > 0$ and $y > 0$, then $x + y > 0$.
Since the product of a positive and negative real number is negative, then $(x - y)(x + y) < 0$.
Multiplying gives $x^2 - y^2 < 0$.
Adding y^2 to both sides gives $x^2 < y^2$.
Therefore $x^2 < y^2$. $\square$

Exercise 9. Prove or disprove the conjecture: If $x, y \in \mathbb{R}$, then $|x+y| = |x|+|y|$.

Solution. We try various values of x and y.
If we have $x = 5$ and $y = -9$, then $|5 + (-9)| = 4$,but $|5| + |-4| = 14$.
One counterexample suffices to disprove the conjecture since this is a universal quantification, so the proof below works. $\square$

Proof. The conjecture is false.
Note that if $x = 5$ and $y = -9$, then $|5 + (-9)| = 4$ but $|5| + |-9| = 14$. $\square$

Exercise 10. For every real number $\theta \in [0, \pi/2]$, $\sin\theta + \cos\theta \geq 1$.

Proof. Suppose for the sake of contradiction that this is not true. Then there exists $\theta \in [0, \pi/2]$ for which $\sin\theta + \cos\theta < 1$. Since $\theta \in [0, \pi/2]$, neither $\sin\theta$ nor $\cos\theta$ is negative, so $0 \le \sin\theta + \cos\theta < 1$. Thus $0^2 \le (\sin\theta + \cos\theta)^2 < 1^2$ using the previously proved proposition, which gives $0^2 \le \sin^2\theta + 2\sin\theta\cos\theta + \cos^2\theta < 1^2$. As $\sin^2\theta + \cos^2\theta = 1$, this becomes $0 \le 1 + 2\sin\theta\cos\theta < 1$, so $1 + 2\sin\theta\cos\theta < 1$. Subtracting 1 and dividing by 2 from both sides gives $\sin\theta\cos\theta < 0$. Therefore either $\sin\theta$ is negative or $\cos\theta$ is negative (but not both). But this contradicts the fact that neither $\sin\theta$ nor $\cos\theta$ is negative. $\square$

Exercise 11. Let $S = \{x \in \mathbb{R} : x^2 - 7x + 10 < 0\}$. Then $S = \{x \in \mathbb{R} : 2 < x < 5\}$.

Proof. Let $p(x) = x^2 - 7x + 10$. Then $p(x) = (x-2)(x-5)$. We know $p(x)$ is negative iff either $x - 2 < 0$ and $x - 5 > 0$ or $x - 2 > 0$ and $x - 5 < 0$. Thus we consider two cases.

Case 1: Suppose $x - 2 < 0$ and $x - 5 > 0$.
Then $x < 2$ and $x > 5$ which is not possible.
Case 2: Suppose $x - 2 > 0$ and $x - 5 < 0$.
Then $x > 2$ and $x < 5$. Hence, $2 < x < 5$.
Therefore, $S = \{x \in \mathbb{R} : 2 < x < 5\}$. $\square$

Exercise 12. Let $x, y \in \mathbb{R}$. Then $(x+y)^2 = x^2 + y^2$ if and only if $x = 0$ or $y = 0$.

Proof. We first prove that if $(x+y)^2 = x^2 + y^2$, then $x = 0$ or $y = 0$. Suppose $(x+y)^2 = x^2 + y^2$. Then $x^2 + 2xy + y^2 = x^2 + y^2$, so $2xy = 0$, and hence $xy = 0$. Thus $x = 0$ or $y = 0$.

Conversely, we show that if $x = 0$ or $y = 0$, then $(x+y)^2 = x^2 + y^2$. Suppose $x = 0$ or $y = 0$. We consider two cases.
Case 1: Suppose $x = 0$.
Then $(x+y)^2 = (0+y)^2 = y^2 = 0^2 + y^2 = x^2 + y^2$.
Case 2: Suppose $y = 0$.
Then $(x+y)^2 = (x+0)^2 = x^2 + 0^2 = x^2 + y^2$.
Either way we have $(x+y)^2 = x^2 + y^2$. $\square$

Exercise 13. How can we remember $\sin(\frac{\pi}{6}) = \frac{1}{2}$?

Solution.

Recall the Pythagorean identity $x^2 + y^2 = r^2$ for a right triangle with hypotenuse r. Let $r = 1$ and one of the legs, say $y = 1/2$. Then $x^2 + (1/2)^2 = 1^2$, so $x = \frac{\sqrt{3}}{2}$.

We know function $\sin(x)$ is increasing on the interval $[0, \pi/2]$ and $1/2 < \sqrt{3}/2$, Therefore, the angle associated with $1/2$ must be less than the angle associated with $\sqrt{3}/2$. It is either $\pi/6$ or $\pi/3$. Since $\pi/6 < \pi/3$, it must be $\pi/6$. $\square$

Exercise 14. Is the following true or false?

For any natural number n the equation $4x^2 + x - n = 0$ has no rational root.

Solution. This statement is false. Let $n \in \mathbb{N}$. Then $4x^2 + x - n = 0$ implies $4x^2+x = n$, so $x^2+\frac{1}{4}x = \frac{n}{4}$. Hence, $(x+\frac{1}{8})^2 = \frac{n}{4}+(\frac{1}{8})^2$, so $(x+\frac{1}{8})^2 = \frac{n}{4}+\frac{1}{64} = \frac{16n+1}{64}$. Thus, $x + \frac{1}{8} = \pm\frac{\sqrt{16n+1}}{8}$, so $x = \frac{-1\pm\sqrt{16n+1}}{8}$. If the quantity $16n + 1$ is a perfect square, then we will have an integer. We try various values of n and find $n = 3$ produces $16 * 3 + 1 = 49$, a perfect square. □

Proof. The statement is false. Here is a counterexample.

Let $n = 3$. Then the equation $4x^2 + x - 3 = 0$ has a rational root $\frac{3}{4}$ since $4(\frac{3}{4})^2 + \frac{3}{4} - 3 = 0$. □

Exercise 15. Let $\mathbb{R}$ be the universal set(domain of discourse). Show that the set $\{x \in \mathbb{R} : 5x^2 + 3x + 2 < 0\}$ is empty.

Solution. Let $S = \{x \in \mathbb{R} : 5x^2 + 3x + 2 < 0\}$. We can factor the polynomial expression $5x^2+3x+2$ by completing the square(as if we were solving a quadratic equation). Thus we have $5x^2+3x+2 = 5(x^2+3x/5+2/5) = 5(x^2+(1/2)(3/5)x+9/100)+2-45/100 = 5(x+3/10)^2+31/20$. Since $(x+3/10)^2$ is greater than or equal to zero the entire expression $5(x+3/10)^2+31/20$ must be strictly greater than zero for every $x \in \mathbb{R}$. Hence $5x^2 + 3x + 2 > 0$ for each $x \in \mathbb{R}$, so there is no $x \in \mathbb{R}$ such that $5x^2 + 3x + 2 < 0$. Therefore S must be empty, so $S = \emptyset$.

We note that we could define function $f : \mathbb{R} \mapsto \mathbb{R}$ by $f(x) = 5x^2 + 3x + 2$ and graph f using calculus and realize that $f(x) > 0$ for all $x \in \mathbb{R}$. The graph provides concrete visualization that S is empty,but we're more interested in showing why S must be empty. □

Exercise 16. Prove by induction that $4^n > n^2 + 4n$ for each integer $n \geq 2$.

Proof. Let $p(n) : 4^n > n^2 + 4n$ be a predicate defined for each integer $n \geq 2$.

Basis:

Since $4^2 = 16 > 12 = 2^2 + 4 * 2$, then the statement $p(2)$ is true.

Induction:

Let $n \geq 2$ such that $p(n)$ is true. Then $4^n > n^2 + 4n$. To prove $p(n + 1)$ is true, we must prove $4^{n+1} > (n + 1)^2 + 4(n + 1)$. Thus, we must prove $4 > \frac{(n+1)^2+4(n+1)}{4^n}$.

Observe that $\frac{(n+1)^2+4(n+1)}{n^2+4n} = \frac{n^2+6n+5}{n^2+4n} = 1 + \frac{2n+5}{n^2+4n}$.

To prove $\frac{(n+1)^2+4(n+1)}{n^2+4n} < 4$, we prove $\frac{2n+5}{n^2+4n} < 1$.

Since $n \geq 2$, then $2n \geq 4$ and $n > 0$. Since $n > 0$ and $n \geq 2$, then $n^2 \geq 2n$. Thus, $n^2 \geq 2n$ and $2n \geq 4$, so $n^2 \geq 4$. Adding $n^2 \geq 4$ and $2n \geq 4$, we obtain $n^2 + 2n \geq 8$. Hence, $n^2 + 2n > 5$, so $n^2 + 4n > 2n + 5$. Since $n > 0$, then $n^2 + 4n > 0$, so dividing we obtain $1 > \frac{2n+5}{n^2+4n}$. Therefore, $\frac{2n+5}{n^2+4n} < 1$.

Thus, we have

$$\begin{aligned}\frac{(n+1)^2+4(n+1)}{n^2+4n} &= 1+\frac{2n+5}{n^2+4n}\\ &< 2\\ &< 4.\end{aligned}$$

Hence, $\frac{(n+1)^2+4(n+1)}{n^2+4n} < 4$. Since $n^2+4n > 0$, then $(n+1)^2+4(n+1) < 4(n^2+4n)$.

By hypothesis, $n^2+4n < 4^n$, so $4(n^2+4n) < 4^{n+1}$. Thus, we have $(n+1)^2 + 4(n+1) < 4(n^2+4n)$ and $4(n^2+4n) < 4^{n+1}$, so $(n+1)^2+4(n+1) < 4^{n+1}$. Therefore, $4^{n+1} > (n+1)^2+4(n+1)$, so $p(n+1)$ is true. Hence, $p(n)$ implies $p(n+1)$ for all $n \geq 2$.

Since $p(2)$ is true and $p(n)$ implies $p(n+1)$ for all $n \geq 2$, then by induction, $p(n)$ is true for all $n \geq 2$. Therefore, $4^n > n^2+4n$ for all integers $n \geq 2$. □

Exercise 17. Let $x \in \mathbb{R}^*$ with $x > -1$. Then $(1+x)^n > 1$ for every integer $n \geq 2$.

Solution. We must prove the proposition $\forall (n \in N, n \geq 2), S_n$ where the statement S_n is $(1+x)^n > 1+nx$.

Since S_n is a statement about the natural numbers, we use proof by induction(weak).

Our basis is $n_0 = 2$ and we must prove S_2.

For induction we must prove $S_k \to S_{k+1}$ for any $k \geq 2$.

Thus we must prove $(1+x)^k > 1+kx \to (1+x)^{k+1} > 1+(k+1)x$ for $k \geq 2$. We use direct proof to assume $(1+x)^k > 1+kx$ for any $k \geq 2$. This is our induction hypothesis. □

Proof. Let $x \in \mathbb{R}^*, x > -1$. Let $n \in \mathbb{N}$ and let S_n be the statement $(1+x)^n > 1+nx$. We prove using mathematical induction(weak).

Basis: For $n = 2$ the statement S_2 is $(1+x)^2 > 1+2x$.

Since $x \neq 0$, then $x^2 > 0$. Thus, $(1+2x)+x^2 > (1+2x)+0$. Hence, $(1+x)^2 > 1+2x$, so S_2 is true.

Induction: Let $k \in \mathbb{N}$. Suppose $(1+x)^k > 1+kx$ for any $k \geq 2$.

Since $x > -1$, then $1+x > 0$. Hence, $(1+x)^k(1+x) > (1+kx)(1+x)$. Thus, $(1+x)^{k+1} > (1+x+kx)+kx^2$.

Since $k \geq 2$, then $k > 0$. Since $x \neq 0$, then $x^2 > 0$. Thus, $kx^2 > 0$.

Since $(1+x)^{k+1} > (1+x+kx)+kx^2$ and $kx^2 > 0$, then lemma implies $(1+x)^{k+1} > 1+x+kx$. Thus, $(1+x)^{k+1} > 1+(k+1)x$.

Hence, $(1+x)^k > 1+kx$ implies $(1+x)^{k+1} > 1+(k+1)x$ for any $k \geq 2$.

Since S_2 is true and S_k implies S_{k+1} for any integer $k \geq 2$, then by induction S_n is true for every integer $n \geq 2$. □

Exercise 18. Conjecture: There is a real number x for which $x^4 < x < x^2$.

Solution. The conjecture in logic symbols is:
$\exists (x \in \mathbb{R})(x^4 < x < x^2)$.
If we assume there is $x \in \mathbb{R}$ for which $x^4 < x < x^2$ is true, then we have

$$\begin{aligned}
(x^4 < x) \wedge (x < x^2) &\to (x^4 - x < 0) \wedge (0 < x^2 - x) \\
&\to x(x^3-1) < 0 \wedge (x^2 - x > 0) \\
&\to x(x-1)(x^2+x+1) < 0 \wedge (x(x-1) > 0).
\end{aligned}$$

Let the function $f(x) = x(x-1)(x^2+x+1)$ and $g(x) = x(x-1)$. Using graphing techniques and/or calculus (to find the derivative of f and g) we realize that the quartic function f is strictly negative when $x \in (0,1)$ and the quadratic function g is strictly positive when $x < 0$ or $x > 1$. Thus we conclude that there can be no real number x for which $x^4 < x < x^2$. Now we will prove this result below. $\square$

Proof. We disprove the conjecture $\exists(x \in \mathbb{R})(x^4 < x < x^2)$ by proving its logical negation is true. Thus we must prove $\forall(x \in \mathbb{R})\neg(x^4 < x < x^2)$. We use proof by contradiction.

Suppose it is not the case that $\forall(x \in \mathbb{R})\neg(x^4 < x < x^2)$. Then $\exists(x \in \mathbb{R})(x^4 < x < x^2)$. Since $x > x^4$ and $x^4 \geq 0$, then $x > 0$. We can therefore divide the inequality by positive x to get

$$x^3 < 1 < x.$$

We subtract 1 from the above inequality to get

$$x^3 - 1 < 0 < x - 1.$$

We factor to get

$$(x-1)(x^2+x+1) < 0 < (x-1). \tag{1}$$

Since $x^4 < x < x^2$ then $x \neq 1$, for if $x = 1$, then $1^4 < 1$, which is false. Thus $x - 1 \neq 0$, so either $x - 1 < 0$ or $x - 1 > 0$. We consider these cases separately.
Case 1: Suppose $x - 1 < 0$.
Then we can divide inequality 1 by $x - 1$ to get

$$(x^2+x+1) > 0 > 1.$$

This implies $0 > 1$, which is false.
Case 2: Suppose $x - 1 > 0$.
Then we can divide inequality 1 by $x - 1$ to get

$$(x^2+x+1) < 0 < 1.$$

Since $x > 0$, then $(x^2+x+1) > 0$. But, the above inequality implies $x^2+x+1 < 0$, a contradiction.

Both cases show that a contradiction results when we assume $\forall(x \in \mathbb{R})\neg(x^4 < x < x^2)$ is false. Therefore the statement $\forall(x \in \mathbb{R})\neg(x^4 < x < x^2)$ is true. Thus we've proved that the logical negation of the conjecture is true, and it follows that the conjecture is false. $\square$

Proof. Suppose for the sake of contradiction that this conjecture is true. Let x be a real number such that $x^4 < x < x^2$. Then x is positive, since it is greater than the positive number x^4. Also, $x \neq 1$ for $1^4 \not< 1^2$. Dividing all parts of the inequality by positive x produces $x^3 < 1 < x$. Subtracting 1 from all parts of the inequality we obtain $x^3 - 1 < 0 < x - 1$ and reason as follows:

$$\begin{aligned} x^3 - 1 &< 0 < x - 1 \\ (x-1)(x^2+x+1) &< 0 < x - 1 \\ x^2+x+1 &< 0 < 1 \end{aligned}$$

Now we have $x^2 + x + 1 < 0$, which is a contradiction because x is positive. Thus the conjecture must be false. $\square$

Exercise 19. Demonstrate that $\cos(\pi/2 - \theta) = \sin\theta$.

Solution.
Let $\triangle ABC$ be a right triangle with hypotenuse 1 and right angle at C.
Let θ be the angle at A.
Then the complement of θ is $\pi/2 - \theta$ which equals the angle at B.
Note: We draw a picture of this to visualize the relationships in the triangle. We observe that hypotenuse $AB = 1$.
Observe that $\sin(\theta) = BC/AB = BC/1 = BC$ and $\cos(\theta) = AC/AB = AC/1 = AC$. Thus, $\sin(\pi/2 - \theta) = AC/AB = \cos(\theta)/1 = \cos(\theta)$ and $\cos(\pi/2 - \theta) = BC/AB = \sin(\theta)/1 = \sin(\theta)$. $\square$

Exercise 20. Prove $\cos(\theta) = \sin(\pi/2 - \theta)$ given that $\sin(x) = \cos(\pi/2 - x)$ for all $x \in \mathbb{R}$.

Solution.
Let $\theta \in \mathbb{R}$.
Since $\sin(x) = \cos(\pi/2 - x)$ for every real x, then in particular, if we let $x = \pi/2 - \theta$, then

$$\begin{aligned} \sin(\pi/2 - \theta) &= \cos[\pi/2 - (\pi/2 - \theta)] \\ &= \cos(\pi/2 - \pi/2 + \theta) \\ &= \cos(\theta) \end{aligned}$$

$\square$

Exercise 21. Given the fact that $\cos(x - y) = \cos x \cos y + \sin x \sin y$ for all real numbers x and y, show that

a. $\cos[\pi/2 - y] = \sin y$ for all $y \in \mathbb{R}$.
b. $\cos(-y) = \cos y$ for all $y \in \mathbb{R}$. (Thus, cosine is an even function).
c. $\cos(x + \pi/2) = -\sin x$ for all $x \in \mathbb{R}$.
d. $\sin(-x) = -\sin(x)$ for all $x \in \mathbb{R}$. (Thus, sine is an odd function).
e. $\sin(x + \pi/2) = \cos x$ for all $x \in \mathbb{R}$.
f. $\cos(x + y) = \cos x \cos y - \sin x \sin y$ for all $x, y \in \mathbb{R}$.
g. $\sin(x + y) = \sin x \cos y + \cos x \sin y$ for all $x, y \in \mathbb{R}$.
h. $\sin(x - y) = \sin x \cos y - \cos x \sin y$ for all $x, y \in \mathbb{R}$.
i. $\sin(2x) = 2 \sin x \cos x$ for all $x \in \mathbb{R}$.
j. $\sin x - \sin y = 2\cos(\frac{x+y}{2})\sin(\frac{x-y}{2})$ for all $x, y \in \mathbb{R}$.

Solution. We can define a predicate for each statement.
For example, in f,
define predicate $p(x, y) : \cos(x + y) = \cos x \cos y - \sin x \sin y$.
Then we must prove $(\forall x, y \in \mathbb{R})[p(x, y)]$. $\square$

Proof. To prove a:

Let y be an arbitrary real number. Then

$$\begin{aligned} \cos(\pi/2 - y) &= \cos(\pi/2)\cos y + \sin(\pi/2)\sin y \\ &= 0 * \cos y + 1 * \sin y \\ &= 0 + \sin y \\ &= \sin y. \end{aligned}$$

To prove b:

Let y be an arbitrary real number. Then

$$\begin{aligned} \cos(-y) &= \cos(0 - y) \\ &= \cos 0 \cos y + \sin 0 \sin y \\ &= 1 * \cos y + 0 * \sin y \\ &= \cos y + 0 \\ &= \cos y. \end{aligned}$$

To prove c:

Let x be an arbitrary real number. Then

$$\begin{aligned} \cos(x + \pi/2) &= \cos(x - (-\pi/2)) \\ &= \cos x \cos(-\pi/2) + \sin x \sin(-\pi/2) \\ &= \cos x \cos(\pi/2) + \sin x \sin(= \pi/2) \\ &= \cos x * 0 + \sin x * (-1) \\ &= 0 - \sin x \\ &= -\sin x. \end{aligned}$$

To prove d:

Let x be an arbitrary real number. Then

$$\begin{aligned} \sin(-x) &= \cos(\pi/2 - (-x)) \\ &= \cos(\pi/2 + x) \\ &= \cos(x + \pi/2) \\ &= -\sin x. \end{aligned}$$

To prove e:

Let x be an arbitrary real number. Then

$$\begin{aligned} \sin(x + \pi/2) &= \cos(\pi/2 - (x + \pi/2)) \\ &= \cos(-x) \\ &= \cos(x). \end{aligned}$$

To prove f:

Let x and y be arbitrary real numbers. Then

$$\begin{aligned}
\cos(x+y) &= \cos(x-(-y)) \\
&= \cos x \cos(-y) + \sin x \sin(-y) \\
&= \cos x \cos y + \sin x(-\sin y) \\
&= \cos x \cos y - \sin x \sin y.
\end{aligned}$$

To prove g:
Let x and y be arbitrary real numbers. Then

$$\begin{aligned}
\sin(x+y) &= \cos(\pi/2-(x+y)) \\
&= \cos(\pi/2-x-y) \\
&= \cos((\pi/2-x)-y) \\
&= \cos(\pi/2-x)\cos y + \sin(\pi/2-x)\sin y \\
&= \sin x \cos y + \cos x \sin y.
\end{aligned}$$

To prove h:
Let y be an arbitrary real number. Then

$$\begin{aligned}
\sin(x-y) &= \sin(x+(-y)) \\
&= \sin x \cos(-y) + \cos x \sin(-y) \\
&= \sin x \cos y + \cos x \sin(-y) \\
&= \sin x \cos y - \cos x \sin y.
\end{aligned}$$

To prove i:
Let x be an arbitrary real number. Then

$$\begin{aligned}
\sin(2x) &= \sin(x+x) \\
&= \sin x \cos x + \cos x \sin x \\
&= \sin x \cos x + \sin x \cos x \\
&= 2 \sin x \cos x.
\end{aligned}$$

To prove j:
Let x and y be arbitrary real numbers.
Let $\theta_1 = \frac{x+y}{2}$.
Let $\theta_2 = \frac{x-y}{2}$.
Then

$$\begin{aligned}
\sin x - \sin y &= \sin(\theta_1+\theta_2) - \sin(\theta_1-\theta_2) \\
&= (\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2) - (\sin\theta_1\cos\theta_2 - \cos\theta_1\sin\theta_2) \\
&= \cos\theta_1\sin\theta_2 + \cos\theta_1\sin\theta_2 \\
&= 2\cos\theta_1\sin\theta_2 \\
&= 2\cos\frac{x+y}{2}\sin\frac{x-y}{2}.
\end{aligned}$$

$\square$

Exercise 22. For every $x \in [\pi/2, \pi]$, $\sin x - \cos x \geq 1$.

Proof. Suppose for the sake of contradiction that there exists $x \in [\pi/2, \pi]$ such that $\sin x - \cos x < 1$.

Since $x \in [\pi/2, \pi]$, then $\cos x \leq 0$ and $\sin x \geq 0$, so $\sin x \cos x \leq 0$ and $\cos x \leq 0 \leq \sin x$.

Since $\cos x \leq 0 \leq \sin x$, then $\cos x \leq \sin x$, so $0 \leq \sin x - \cos x$.

Since $0 \leq \sin x - \cos x$ and $\sin x - \cos x < 1$, then $0 \leq \sin x - \cos x < 1$.

Hence, $0 \leq (\sin x - \cos x)^2 < 1$, so $(\sin x - \cos x)^2 < 1$.

Thus, $\sin^2(x) - 2\sin x \cos x + \cos^2(x) < 1$, so $1 - 2\sin x \cos x < 1$.

Therefore, $0 < 2\sin x \cos x$, so $0 < \sin x \cos x$.

But, we now have a contradiction $\sin x \cos x > 0$ and $\sin x \cos x \leq 0$. ☐

Exercise 23. Show that $\arccos(x) = \frac{\pi}{2} - \arcsin(x)$.

Solution. Let $x \in \mathbb{R}$.

We must prove $\arccos(x) = \frac{\pi}{2} - \arcsin(x)$.

Define predicate $p(x) : \arccos(x) = \frac{\pi}{2} - \arcsin(x)$.

We must prove $(\forall x \in \mathbb{R})[p(x)]$. ☐

Proof. Let $a \in \mathbb{R}$ be arbitrary.

Since $\sin(\pi/2 - x) = \cos(x)$ for every $x \in \mathbb{R}$, then in particular, $\sin(\pi/2 - a) = \cos a$.

Let $b = \cos a$. Then $\sin(\pi/2 - a) = b$, so $\arcsin b = \pi/2 - a$.

Hence, $a = \pi/2 - \arcsin b$.

Since $b = \cos a$, then $\arccos b = a$.

Therefore, $\arccos b = \pi/2 - \arcsin b$.

Since a is arbitrary, then $\arccos x = \pi/2 - \arcsin x$ for every $x \in \mathbb{R}$. ☐

Exercise 24. Given that arcsin is an odd function on domain $[-1, 1]$ and the fact that $\arccos x = \pi/2 - \arcsin x$, prove that $\arccos(-x) = \pi - \arccos x$ for all x such that $-1 \leq x \leq 1$.

Solution. Let $x \in [-1, 1]$ be arbitrary.

We must prove $\arccos(-x) = \pi - \arccos x$. We note that x and $-x$ must be in the domain of arccos function. Since domain of arccos is the interval $[-1, 1]$, then x is in the domain of arccos. Since $x \in [-1, 1]$, then $-1 \leq x \leq 1$, so $-1 \leq x$ and $x \leq 1$. Hence, $1 \geq -x$ and $-x \geq -1$, so $-x \leq 1$ and $-x \geq -1$. Thus, $-1 \leq -x \leq 1$, so $-x \in [-1, 1]$, too.

We can work backwards to find the relationships. ☐

Proof. Let x be an arbitrary real number in the interval $[-1, 1]$.

Then

$$\begin{aligned}
\pi - \arccos x &= \pi - (\pi/2 - \arcsin x) \\
&= \pi/2 + \arcsin x \\
&= \pi/2 - (-\arcsin x) \\
&= \pi/2 - (\arcsin(-x)) \\
&= \arccos(-x).
\end{aligned}$$

$\square$

Exercise 25. Given the fact $\arcsin(-x) = -\arcsin(x)$ and $\sec^{-1} x = \arccos(1/x)$, prove that $\sec^{-1}(-x) = \pi - \sec^{-1}(x)$.

Solution. The conclusion has the form $\forall x.p(x)$ where predicate $p(x) : \sec^{-1}(-x) = \pi - \sec^{-1}(x)$. The domain of discourse is the domain of the arsecant function, namely, $(-\infty, -1] \cup [1, \infty)$.

Let x be an arbitrary real number in the domain of arcsecant function. We must prove $\sec^{-1}(-x) = \pi - \sec^{-1}(x)$. $\square$

Proof. Let x be an arbitrary real number such that $x \in (-\infty, -1] \cup [1, \infty)$.

Then

$$\begin{aligned} \sec^{-1}(-x) &= \cos^{-1}(-1/x) \\ &= \pi - \cos^{-1}(1/x) \\ &= \pi - \sec^{-1}(x). \end{aligned}$$

$\square$

Exercise 26. Given the fact that $\cot^{-1}(x) = \pi/2 - \tan^{-1}(x)$ and that $\tan^{-1}$ is an odd function, show that $\cot^{-1}(-x) = \pi - \cot^{-1}(x)$ for all $x \in \mathbb{R}$.

Solution.

Define predicate $p(x) : \cot^{-1}(-x) = \pi - \cot^{-1}(x)$.

We must prove $(\forall x \in \mathbb{R})p(x)$.

Note that the domain of discourse is the same as the domain of function $\cot^{-1}$, namely $\mathbb{R}$. $\square$

Proof. Let x be an arbitrary real number.

Then

$$\begin{aligned} \cot^{-1}(-x) &= \pi/2 - \tan^{-1}(-x) \\ &= \pi/2 - [-\tan^{-1}(x)] \\ &= \pi - \pi/2 - [-\tan^{-1}(x)] \\ &= \pi - [\pi/2 - \tan^{-1}(x)] \\ &= \pi - \cot^{-1}(x). \end{aligned}$$

$\square$

Exercise 27. Prove that if C is the graph of a function $y = f(x)$, then C is symmetric with respect to the point (h, k) if and only if $f(-x+h) = 2k - f(x+h)$ for every x such that $x + h$ is in the domain of f.

Solution.

Our hypothesis is:

1. $y = f(x)$ is a real valued function in $\mathbb{R}^2$.
2. The graph of f is $C = \{(x, y) \in \mathbb{R}^2 : y = f(x)\}$.

3. The point (h, k) is some fixed point in $\mathbb{R}^2$.

Our conclusion is:

$P \leftrightarrow Q$ where

P is the statement C is symmetric with respect to (h, k).

Q is the statement $f(-x + h) = 2k - f(x + h)$ for every x such that $x + h$ is in the domain of f.

By definition of symmetry with respect to a point, P is translated as:

$P : (\forall x \in \mathbb{R})(\forall y \in \mathbb{R})[(x + h, y + k) \in C \to (-x + h, -y + k) \in C]$.

Observe that P is a quantified statement with variables x, y bound and constants h, k, C.

Define predicate $p_1(x, y) : (x + h, y + k) \in C$.

Define predicate $p_2(x, y) : (-x + h, -y + k) \in C$.

Then $P : (\forall x \in \mathbb{R})(\forall y \in \mathbb{R})[p_1(x, y) \to p_2(x, y)]$.

We analyze statement Q.

$Q : (\forall x, x + h \in dom(f))[f(-x + h) = 2k - f(x + h)]$.

Observe that Q is a quantified statement with variable x and constants f, k.

Thus, define predicate $q(x) : f(-x + h) = 2k - f(x + h)$.

Then $Q : (\forall x, x + h \in dom(f))[q(x)]$.

To prove the conclusion $P \leftrightarrow Q$, we must prove:

1. $P \to Q$.
2. $Q \to P$.

To prove 1: To prove Q, we assume P. To prove Q, we let $a \in \mathbb{R}$ be arbitrary such that $a + h \in dom(f)$. We must prove $q(a)$, that is prove $f(-a + h) = 2k - f(a + h)$.

To prove 2: To prove P, we assume Q. To prove P, we let $a, b \in \mathbb{R}$ be arbitrary such that $p_1(a, b)$ is true. We must prove $p_2(a, b)$ is true. Thus, we assume $(a + h, b + k) \in C$. We must prove $(-a + h, -b + k) \in C$.

$\square$

Proof. Let $y = f(x)$ be function in $\mathbb{R}^2$ with graph $C = \{(x, y) \in \mathbb{R} \times \mathbb{R} : y = f(x)\}$. Let (h, k) be some fixed point in $\mathbb{R}^2$.

$\Rightarrow$ Let a be an arbitrary real number such that $a + h$ is in the domain of f.

To prove $f(-a+h) = 2k - f(a+h)$, we assume C is symmetric with respect to the point (h, k). Since $a + h$ is in the domain of f, then $f(a + h)$ exists, so the point $(a + h, f(a + h))$ is on the graph C. Thus, $(a + h, f(a + h)) \in C$.

Every point on C can be expressed in terms of h, k and some point $(x, y) \in \mathbb{R}^2$.

Therefore, let $a + h = x + h$ and $f(a + h) = y + k$ for some point $(x, y) \in \mathbb{R}^2$. Since $a + h = x + h$, then $x = a$. Hence, $(a + h, y + k) \in C$.

Since C is symmetric with respect to (h, k) and $(a + h, y + k) \in C$, then the point $(-a+h, -y+k)$ is on C, so $(-a+h, -y+k) \in C$. Thus, by definition of C, $-y + k = f(-a + h)$. Hence, $y + k = f(a + h)$ and $-y + k = f(-a + h)$. Adding these equations we obtain $2k = f(a + h) + f(-a + h)$, so that $f(-a + h) = 2k - f(a + h)$, as desired.

$\Leftarrow$ Conversely, to prove C is symmetric with respect to (h,k), let a be an arbitrary real number such that $a+h$ is in the domain of f and $f(-a+h) = 2k - f(a+h)$. Let b be an arbitrary real number such that $(a+h, b+k) \in C$. To prove $(-a+h, -b+k) \in C$, we must prove $-b+k = f(-a+h)$.

Since $f(-a+h) = 2k - f(a+h)$, then $2k = f(a+h) + f(-a+h)$. Since $(a+h, b+k) \in C$, then by definition of C, $b+k = f(a+h)$. Thus, we have $2k = f(a+h) + f(-a+h)$ and $b+k = f(a+h)$. If we subtract these equations, we obtain $-b+k = f(-a+h)$, as desired. $\square$

Exercise 28. If $M > 0$, then the linear function $f(x) = Mx + B$ is increasing on $\mathbb{R}$.

Solution. Our hypothesis is $M > 0$ and a real valued function $f(x) = Mx + B$ defined on domain $\mathbb{R}$. To prove the conclusion f is increasing on $\mathbb{R}$, we must prove $(\forall x_1, x_2 \in \mathbb{R})[x_1 < x_2 \to f(x_1) < f(x_2)]$. Thus, we let x_1, x_2 be arbitrary real numbers. To prove $x_1 < x_2 \to f(x_1) < f(x_2)$ is true, we assume $x_1 < x_2$. We must prove $f(x_1) < f(x_2)$. $\square$

Proof. Let x_1 and x_2 be arbitrary real numbers. To prove f is increasing on $\mathbb{R}$, we assume $x_1 < x_2$. We must prove $f(x_1) < f(x_2)$.

Since $x_1 < x_2$ and $M > 0$, then $Mx_1 < Mx_2$. Thus, $Mx_1 + B < Mx_2 + B$. Hence, $f(x_1) < f(x_2)$, as desired. $\square$

Exercise 29. If the linear function $f(x) = Mx + B$ is increasing on $\mathbb{R}$, then $M > 0$.

Solution. This is the converse of the previous exercise.

Hypothesis is: $f(x) = Mx + B$ is increasing.

Conclusion is: $M > 0$.

The hypothesis is: $(\forall x_1 \in \mathbb{R})(\forall x_2 \in \mathbb{R})(x_1 < x_2 \to f(x_1) < f(x_2))$.

Since the conclusion is a simple statement and the hypothesis is complicated, let's try proving by contrapositive.

We assume the negation of the conclusion:

Thus, we assume $M \leq 0$.

We must prove the negation of the hypothesis:

Thus, we must prove $(\exists x_1 \in \mathbb{R})(\exists x_2 \in \mathbb{R})(x_1 < x_2 \land f(x_1) \geq f(x_2))$.

Hence, we must find concrete real numbers x_1 and x_2 such that $x_1 < x_2$ and $f(x_1) \geq f(x_2)$.

Let $x_1 = 3$ and $x_2 = 5$. Since $3 < 5$, then $x_1 < x_2$.

We must prove $f(3) \geq f(5)$.

Since $M \leq 0$, then $M(3) \geq M(5)$. Hence, $M(3) + B \geq M(5) + B$, so $f(3) \geq f(5)$. Therefore, $f(x_1) \geq f(x_2)$. $\square$

Proof. We prove by contrapositive. Assume $M \leq 0$. We must prove there exist real numbers x_1 and x_2 such that $x_1 < x_2$ and $f(x_1) \geq f(x_2)$.

Let $x_1 = 3$ and $x_2 = 5$. Since $3 < 5$, then $x_1 < x_2$. Since $M \leq 0$, then $Mx_1 = 3M \geq Mx_2 = 5M$. Hence, $3M + B \geq 5M + B$, so $f(3) \geq f(5)$. Therefore, $f(x_1) \geq f(x_2)$. $\square$

Exercise 30. If the linear function $f(x) = Mx + B$ is one to one on $\mathbb{R}$, then $M \neq 0$.

Solution. Hypothesis is: f is one to one.

Conclusion is: $M \neq 0$.

Let $H : f$ is one to one.

Let $C : M \neq 0$.

We must prove $H \to C$.

Note $H : (\forall a, b)(a \neq b \to f(a) \neq f(b))$.

Thus, we must prove $(\forall a, b)(a \neq b \to f(a) \neq f(b)) \Rightarrow M \neq 0$.

If we try direct proof, we get nowhere because we can't find a way to deduce $M \neq 0$. Thus, we try indirect proof. We can try proof by contrapositive or proof by contradiction. In this case, since the hypothesis is complicated and the conclusion is simple, let's try proof by contrapositive.

Thus, since $H \to C \Leftrightarrow \neg C \to \neg H$, we assume $M = 0$. We must prove $\neg H \Leftrightarrow \neg(\forall a, b)(a \neq b \to f(a) \neq f(b)) \Leftrightarrow (\exists a, b)(a \neq b \wedge f(a) = f(b))$. $\square$

Proof. We prove by contrapositive. Suppose $M = 0$. We must find real numbers a and b such that $a \neq b$ and $f(a) = f(b)$.

Since $M = 0$, then $f(x) = Mx + B = 0x + B = B$.

Let $a = 3$ and $b = 5$. Since $3 \neq 5$, then $a \neq b$. Observe that $f(a) = f(3) = B = f(5) = f(b)$. $\square$

Exercise 31. Let $g : [1, 4] \to \mathbb{R}$ be a function defined by $g(x) = 2x^2 + \sqrt{x}$.

Then g is increasing and the inverse image of 0 is the empty set.

Proof. We first prove g is increasing on the interval $[1, 4]$.

Let a and b be arbitrary real numbers such that $a, b \in [1, 4]$ and $a < b$.

Then $1 \leq a < b \leq 4$.

To prove g is increasing we must prove $g(a) < g(b)$.

Since $1 \leq a < b \leq 4$, then $0 < 1 \leq a < b \leq 4$, so $1 \leq a$ and $1 < b$ and $0 < a < b$.

Since $a \geq 1$ and $b > 1$, then $a + b > 2$.

Since $a \geq 1$, then $\sqrt{a} \geq 1$.

Since $b > 1$, then $\sqrt{b} > 1$.

Thus, $\sqrt{a} + \sqrt{b} > 2$.

Since $a + b > 2$ and $\sqrt{a} + \sqrt{b} > 2$, then $(a + b)(\sqrt{a} + \sqrt{b}) > 4 > \frac{-1}{2}$, so $(a + b)(\sqrt{a} + \sqrt{b}) > \frac{-1}{2}$.

Hence, $-2(a + b)(\sqrt{a} + \sqrt{b}) < 1$.

Since $0 < a < b$, then $0 < \sqrt{a} < \sqrt{b}$, so $\sqrt{a} < \sqrt{b}$.

Thus, $\sqrt{b} - \sqrt{a} > 0$.

Since $-2(a + b)(\sqrt{a} + \sqrt{b}) < 1$ and $\sqrt{b} - \sqrt{a} > 0$, then $-2(a + b)(\sqrt{a} + \sqrt{b})(\sqrt{b} - \sqrt{a}) < \sqrt{b} - \sqrt{a}$.

Consequently, $2(a+b)(\sqrt{a}+\sqrt{b})(\sqrt{a}-\sqrt{b}) < \sqrt{b}-\sqrt{a}$, so $2(a+b)(a-b) < \sqrt{b} - \sqrt{a}$.

Hence, $2(a^2 - b^2) < \sqrt{b} - \sqrt{a}$, so $2a^2 - 2b^2 < \sqrt{b} - \sqrt{a}$.

Thus, $2a^2 + \sqrt{a} < 2b^2 + \sqrt{b}$, so $g(a) < g(b)$, as desired. $\square$

Proof. We next prove the inverse image of 0 is the empty set.

Let $S = \{x \in [1,4] : g(x) = 0\}$.

To prove the inverse image of 0 is the empty set, we must prove $S = \emptyset$.

To prove $S = \emptyset$, let $x \in [1,4]$.

We must prove $g(x) \neq 0$.

Since $x \in [1,4]$, then $1 \leq x \leq 4$, so $1 \leq x$.

Hence, either $x > 1$ or $x = 1$.

We consider these cases separately.

Case 1: Suppose $x = 1$.

Then $g(x) = g(1) = 3 \neq 0$, so $g(x) \neq 0$.

Case 2: Suppose $x > 1$.

Then $1 < x$.

Since g is increasing, then $g(1) < g(x)$, so $3 < g(x)$.

Hence, $g(x) > 3$, so $g(x) \neq 0$.

Therefore, in all cases, $g(x) \neq 0$, as desired. □

Exercise 32. Let $h : [1,3] \to \mathbb{R}$ be a function defined by $h(x) = x + \frac{6}{x}$.

Then h is not decreasing and h is not one to one.

Proof. We first prove h is not decreasing.

Since $2 \in [1,3]$ and $3 \in [1,3]$ and $2 < 3$ and $h(2) = 5 = h(3)$, then h is not decreasing. □

Proof. We next prove h is not one to one.

Since $2 \in [1,3]$ and $3 \in [1,3]$ and $2 \neq 3$ and $h(2) = 5 = h(3)$, then h is not one to one. □

Exercise 33. If function f is increasing on an interval I, then f is one to one on I.

Solution. Let f be an arbitrary function on an arbitrary interval I.

The hypothesis is: f is increasing on I.

The conclusion is: f is one to one on I.

The hypothesis means: $(\forall x_1, x_2 \in \mathbb{R})(x_1 < x_2 \to f(x_1) < f(x_2))$.

The conclusion means:

$(\forall x_1, x_2 \in \mathbb{R})[x_1 \neq x_2 \to f(x_1) \neq f(x_2)]$.

Thus, let $x_1, x_2 \in \mathbb{R}$ such that $x_1 \neq x_2$. We must prove $f(x_1) \neq f(x_2)$

Suppose $(\forall x_1, x_2 \in \mathbb{R})(x_1 < x_2 \to f(x_1) < f(x_2))$.

Since $x_1 \neq x_2$, then either $x_1 < x_2$ or $x_1 > x_2$.

Case 1: Suppose $x_1 < x_2$. Since $a < b \Rightarrow f(a) < f(b)$ for all real a, b, then in particular, if we let $a = x_1$ and $b = x_2$, then $f(x_1) < f(x_2)$. Thus, $f(x_1) \neq f(x_2)$.

Case 2: Suppose $x_1 > x_2$. Then $x_2 < x_1$. Since $a < b \Rightarrow f(a) < f(b)$ for all real a, b, then in particular, if we let $a = x_2$ and $b = x_1$, then $f(x_2) < f(x_1)$. Thus, $f(x_1) \neq f(x_2)$.

Hence, in all cases, $x_1 \neq x_2$ implies $f(x_1) \neq f(x_2)$, as desired. □

Proof. To prove f is one to one, let x_1 and x_2 be arbitrary distinct real numbers.

We must prove $f(x_1) \neq f(x_2)$.

Since x_1 and x_2 are distinct, then $x_1 \neq x_2$. Hence, either $x_1 < x_2$ or $x_1 > x_2$. We consider these cases separately.

Case 1: Suppose $x_1 < x_2$.

Since f is increasing on I, then, for every real number a and b, $a < b$ implies $f(a) < f(b)$. Hence, in particular, if we let $a = x_1$ and $b = x_2$, then $x_1 < x_2$ implies $f(x_1) < f(x_2)$. Thus, since $x_1 < x_2$, then $f(x_1) < f(x_2)$. Therefore, $f(x_1) \neq f(x_2)$.

Case 2: Suppose $x_1 > x_2$.

Then $x_2 < x_1$. Since f is increasing on I, then, for every real number a and b, $a < b$ implies $f(a) < f(b)$. Hence, in particular, if we let $a = x_2$ and $b = x_1$, then $x_2 < x_1$ implies $f(x_2) < f(x_1)$. Thus, since $x_2 < x_1$, then $f(x_2) < f(x_1)$. Therefore, $f(x_2) \neq f(x_1)$.

Hence, in all cases, $x_1 \neq x_2$ implies $f(x_1) \neq f(x_2)$, as desired. □

Exercise 34. For each $n \in \mathbb{N}, (1+x)^n \geq 1 + nx$ for all $x \in \mathbb{R}$ with $x > -1$.

Solution. Let $x \in \mathbb{R}, x > -1$. We must prove the proposition $\forall (n \in N), S_n$ where the statement S_n is $(1+x)^n \geq 1 + nx$.

Since S_n is a statement about the natural numbers, we use proof by induction(weak).

Our basis is $n_0 = 1$ and we must prove S_1.

For induction we must prove $S_k \to S_{k+1}$ for any $k \geq 1$.

Thus we must prove $(1+x)^k \geq 1 + kx \to (1+x)^{k+1} \geq 1 + (k+1)x$ for $k \geq 1$. We use direct proof to assume $(1+x)^k > 1 + kx$ for any $k \geq 1$. This is our induction hypothesis. □

Proof. Let $x \in \mathbb{R}, x > -1$.

Let $n \in \mathbb{N}$ and let S_n be the statement $(1+x)^n \geq 1 + nx$. We prove using mathematical induction(weak).

Basis: If $n = 1$ then the statement S_1 is $(1+x)^1 \geq 1 + 1x$, which is true since both sides equal $1 + x$.

Induction: Let $k \in \mathbb{N}$. Suppose $(1+x)^k \geq 1+kx$ for any $k \geq 1$. Since $x > -1$, then $x + 1 > 0$, so we can multiply, preserving the inequality: $(1+x)(1+x)^k \geq (1+x)(1+kx)$. This implies $(1+x)^{k+1} \geq 1 + kx + x + kx^2$ which implies $(1+x)^{k+1} \geq 1 + (k+1)x + kx^2$. Since $x^2 > 0$ and $k > 0$, then $kx^2 > 0$. Thus $(1+x)^{k+1} \geq 1 + (k+1)x$. □

Exercise 35. Let f be a real valued function with domain $\mathbb{R}$. Then f is even if and only if the curve $C = \{(x, f(x)) : x \in \mathbb{R}\}$ is symmetric with respect to the y axis.

Solution. Our hypothesis is $f : \mathbb{R} \mapsto \mathbb{R}$ is a function and curve $C = \{(x, f(x)) : x \in \mathbb{R}\}$. Our conclusion is f is even iff C is symmetric with respect to the y axis. To prove this conclusion we must prove both:

1. if f is even, then C is symmetric with respect to the y axis.

2. if C is symmetric with respect to the y axis, then f is even.

To prove 1: we assume f is even. We must prove $(\forall x, y \in \mathbb{R})[(x, y) \in C \to (-x, y) \in C]$. Therefore, we let $x, y \in \mathbb{R}$ be arbitrary. To prove $(x, y) \in C \to (-x, y) \in C$, we assume $(x, y) \in C$. We must prove $(-x, y) \in C$.

To prove 2: We assume C is symmetric with respect to the y axis. We must prove $(\forall x \in \mathbb{R})(f(-x) = f(x))$. Therefore, we let $x \in \mathbb{R}$ be arbitrary. We must prove $f(-x) = f(x)$.

We note that $C = \{(x, y) : x \in \mathbb{R}, y = f(x)\} = \{(x, y) \in \mathbb{R} \times \mathbb{R} : y = f(x)\}$. Therefore, $C \subseteq \mathbb{R}^2$.

□

Proof. Let $f : \mathbb{R} \mapsto \mathbb{R}$ be a function and let $C = \{(x, f(x)) : x \in \mathbb{R}\}$.

$\Rightarrow$ Suppose f is even. To prove C is symmetric with respect to the y axis, let $(x, y) \in C$ be arbitrary. We must prove $(-x, y) \in C$, that is, prove $y = f(-x)$.

Since $(x, y) \in C$, then by definition of C, $y = f(x)$. Since f is even, then $f(-x) = f(x)$. Hence, $y = f(-x)$, as desired.

$\Leftarrow$ Conversely, suppose C is symmetric with respect to the y axis. To prove f is even, let x be an arbitrary real number. We must prove $f(-x) = f(x)$.

Since $x \in \mathbb{R}$, then $f(x) \in \mathbb{R}$, by definition of f. Hence, $(x, f(x)) \in C$. Since $x \in \mathbb{R}$, then $-x \in \mathbb{R}$. Hence, $f(-x) \in \mathbb{R}$, so $(-x, f(-x)) \in C$. Since C is symmetric with respect to the y axis and $(x, f(x)) \in C$, then $(-x, f(x)) \in C$. Since $(-x, f(-x)) \in C$, then by definition of C, $f(-x) = f(-x)$. Since $(-x, f(x)) \in C$, then by definition of C, $f(-x) = f(x)$. Since f is a function, then $f(-x)$ is unique. Therefore, $f(-x) = f(x)$, as desired. □

Exercise 36. Let C be a subset of $\mathbb{R} \times \mathbb{R}$ such that C is symmetric with respect to the y axis, but not with the origin. Assume the fact that if a curve C is symmetric with respect to the x and y axes, then it is symmetric with respect to the origin. Show that C cannot be symmetric with respect to the x axis.

Solution. We define statements.

Let $P : C$ is symmetric with respect to the x axis.

Let $Q : C$ is symmetric with respect to the y axis.

Let $R : C$ is symmetric with respect to the origin.

The hypothesis is: $Q \wedge \neg R$.

The conclusion is: $\neg P$.

We are given the theorem: $P \wedge Q \to R$.

We use direct proof, so we must prove $Q \wedge \neg R \to \neg P$.

Since we have negative statements, we'd like to get rid of the negatives.

We can use logical equivalence involving the contrapositive:

We know $Q \wedge \neg R \to \neg P \Leftrightarrow Q \wedge P \to R$.

Thus, we prove the equivalent statement $Q \wedge P \to R$.

Hence, we assume P and Q and show that R is true.

We can apply theorem $P \wedge Q \to R$ to conclude R.

Suppose C is symmetric with respect to the x axis. By hypothesis C is symmetric with respect to the y axis. By the theorem, C is symmetric with respect to the x and y axes imply C is symmetric with respect to the origin. Hence, C is symmetric with respect to the origin, as desired. $\square$

Proof. We prove C is symmetric with respect to the x and y axes imply C is symmetric with respect to the origin. Assume C is symmetric with respect to the x axis. By hypothesis, C is symmetric with respect to the y axis. By a theorem, if C is symmetric with respect to the x and y axes, then C is symmetric with respect to the origin. Since C is symmetric with respect to the x and y axes, then we conclude C is symmetric with respect to the origin. $\square$

Exercise 37. Let x, y, z be any real numbers. Let $\vee$ be the max function. Then $x \vee (y \vee z) = (x \vee y) \vee z$.

Solution.

We must prove that the max function is associative.

Observe that by definition of max function, for any two real numbers x and y, either $x \leq y$ or $y \leq x$ are considered. (This is because of the trichotomy law of real numbers-either $x < y$ or $x = y$ or $x > y$, so either $x \leq y$ or $x > y$, so either $x \leq y$ or $y < x$, so either $x \leq y$ or $y \leq x$). Hence, for each pair of numbers, we have two cases. Since we're doing two max operations, then we have 4 cases to consider:

1. $x \leq y$ and $y \leq z$.
2. $x \leq y$ and $z \leq y$.
3. $y \leq x$ and $y \leq z$.
4. $y \leq x$ and $z \leq y$. $\square$

Proof. Let x, y, and z be arbitrary real numbers.

Either
$x \leq y$ and $y \leq z$, or
$x \leq y$ and $z \leq y$, or
$y \leq x$ and $y \leq z$, or
$y \leq x$ and $z \leq y$.

We consider these cases separately.

Case 1: Suppose $x \leq y$ and $y \leq z$.

Then $x \leq z$, by transitivity of $\leq$.

Observe that

$$\begin{aligned} x \vee (y \vee z) &= x \vee z \\ &= z \\ &= y \vee z \\ &= (x \vee y) \vee z \end{aligned}$$

Case 2: Suppose $x \leq y$ and $z \leq y$.

Observe that

$$\begin{aligned} x \vee (y \vee z) &= x \vee y \\ &= y \\ &= y \vee z \\ &= (x \vee y) \vee z \end{aligned}$$

Case 3: Suppose $y \leq x$ and $y \leq z$.
Observe that

$$\begin{aligned} x \vee (y \vee z) &= x \vee z \\ &= (x \vee y) \vee z \end{aligned}$$

Case 4: Suppose $y \leq x$ and $z \leq y$.
Since $z \leq y$ and $y \leq x$, then $z \leq x$.
Observe that

$$\begin{aligned} x \vee (y \vee z) &= x \vee y \\ &= x \\ &= x \vee z \\ &= (x \vee y) \vee z \end{aligned}$$

In all cases, $x \vee (y \vee z) = (x \vee y) \vee z$, as desired, so *max* is associative. $\square$

Exercise 38. Prove $|\sum_{k=1}^{n} x_k| \leq \sum_{k=1}^{n} |x_k|$ for every positive integer n and every real number $x_1, x_2, ..., x_n$.

Solution.
The statement to prove is:
$(\forall x_1, x_2, ..., x_n \in \mathbb{R})(\forall n \in \mathbb{N})[|\sum_{k=1}^{n} x_k| \leq \sum_{k=1}^{n} |x_k|]$.
Let $x_1, x_2, ..., x_n \in \mathbb{R}$ be arbitrary.
Define predicate $p(n) : |\sum_{k=1}^{n} x_k| \leq \sum_{k=1}^{n} |x_k|$ over $\mathbb{N}$.
Let S be the truth set of $p(n)$.
To prove $(\forall n \in \mathbb{N})[p(n)]$, we must prove $S = \mathbb{N}$.
We prove by induction.
Thus, we must prove:
1. $1 \in S$. Thus,we must prove $p(1)$ is true.
2. $(\forall m \in \mathbb{N})(m \in S \to m + 1 \in S)$. To prove, we assume $m \in \mathbb{N}$ is arbitrary such that $m \in S$. To prove $m + 1 \in S$, we must prove $p(m + 1)$ is true.
We note that this is the generalized triangle inequality. $\square$

Proof. Let $x_1, x_2, ..., x_n$ be arbitrary real numbers.
Let S be the truth set of $p(n) : |\sum_{k=1}^{n} x_k| \leq \sum_{k=1}^{n} |x_k|$ over $\mathbb{N}$.
We prove $S = \mathbb{N}$ by induction.
Basis:
Clearly, $1 \in S$, since $|\sum_{k=1}^{1} x_k| = |x_1| = \sum_{k=1}^{1} |x_k|$.

Induction:
Suppose $m \in S$.
To prove $m+1 \in S$, we must prove $|\sum_{k=1}^{m+1} x_k| \leq \sum_{k=1}^{m+1} |x_k|$.
Since $m \in S$, then $|\sum_{k=1}^{m} x_k| \leq \sum_{k=1}^{m} |x_k|$.
Observe that

$$\begin{aligned}
|\sum_{k=1}^{m+1} x_k| &= |\sum_{k=1}^{m} x_k + x_{m+1}| \\
&\leq |\sum_{k=1}^{m} x_k| + |x_{m+1}| \\
&\leq \sum_{k=1}^{m} |x_k| + |x_{m+1}| \\
&\leq \sum_{k=1}^{m+1} |x_k| \text{ , as desired.}
\end{aligned}$$

□

Exercise 39. Let T be a real number. Let $f : \mathbb{R} \mapsto \mathbb{R}$ be a function such that $f(x+T) = f(x)$ for all $x \in \mathbb{R}$. Then $f(x+nT) = f(x)$ for all $x \in \mathbb{R}$ and $n \in \mathbb{N}$.

Solution.
The hypothesis is:
T is an arbitrary real number.
$f : \mathbb{R} \mapsto \mathbb{R}$ is a function.
$(\forall x \in \mathbb{R})[f(x+T) = f(x)]$.
The conclusion is:
$(\forall x \in \mathbb{R})(\forall n \in \mathbb{N})[f(x+nT) = f(x)]$.
Let $x \in \mathbb{R}$.
To prove $(\forall n \in \mathbb{N})[f(x+nT) = f(x)]$, we use induction since this is a statement about $\mathbb{N}$
Define predicate $p(n) : f(x+nT) = f(x)$.
Let S be the truth set of $p(n)$.
To prove $(\forall x \in \mathbb{N})[p(n)]$, we must prove $S = \mathbb{N}$.
Therefore, we must prove:
1. $1 \in S$.
2. S is inductive. That is, prove $(\forall m \in \mathbb{N})(m \in S \rightarrow m+1 \in S)$.

□

Proof. Let x be an arbitrary real number.
Let S be the truth set of $p(n) : f(x+nT) = f(x)$.
We prove $S = \mathbb{N}$ by induction.
Basis:
Clearly, $1 \in S$, since $f(x+1T) = f(x+T) = f(x)$.
Induction:

Suppose $m \in S$.
To prove $m+1 \in S$, we must prove $f(x+(m+1)T) = f(x)$.
Since $m \in S$, then $f(x+mT) = f(x)$.
Observe that

$$\begin{aligned} f(x+(m+1)T) &= f(x+mT+T) \\ &= f((x+mT)+T) \\ &= f(x+mT) \\ &= f(x) \text{ , as desired.} \end{aligned}$$

□

Exercise 40. For every natural number $n, \cos(n\pi) = (-1)^n$.

Solution.
The conclusion is:
$(\forall n \in \mathbb{N})[\cos(n\pi) = (-1)^n]$.
Let $x \in \mathbb{R}$.
To prove $(\forall n \in \mathbb{N})[\cos(n\pi) = (-1)^n]$, we use induction since this is a statement about $\mathbb{N}$
Define predicate $p(n) : \cos(n\pi) = (-1)^n$.
Let S be the truth set of $p(n)$.
To prove $(\forall x \in \mathbb{N})[p(n)]$, we must prove $S = \mathbb{N}$.
Therefore, we must prove:
1. $1 \in S$.
2. S is inductive. That is, prove $(\forall m \in \mathbb{N})(m \in S \to m+1 \in S)$.

□

Proof. Let S be the truth set of $p(n) : \cos(n\pi) = (-1)^n$.
We prove $S = \mathbb{N}$ by induction.
Basis:
Clearly, $1 \in S$, since $\cos(1\pi) = \cos(\pi) = -1 = (-1)^1$.
Induction:
Suppose $m \in S$.
To prove $m+1 \in S$, we must prove $\cos((m+1)\pi) = (-1)^{m+1}$.
Since $m \in S$, then $\cos(m\pi) = (-1)^m$.
Observe that

$$\begin{aligned} \cos((m+1)\pi) &= \cos(m\pi + \pi) \\ &= \cos(m\pi)\cos(\pi) - \sin(m\pi)\sin(\pi) \\ &= \cos(m\pi)(-1) - (\sin(m\pi))(0) \\ &= \cos(m\pi)(-1) - 0 \\ &= \cos(m\pi)(-1) \\ &= (-1)^m(-1) \\ &= (-1)^{m+1}, \text{ as desired.} \end{aligned}$$

□

Exercise 41. If $x, a, b \in \mathbb{R}$ and $xa = xb$, then either $x = 0$ or $a = b$.

Solution.
Let $x, a, b \in \mathbb{R}$.
Define:
$h(x) : xa = xb$.
$c_1(x) : x = 0$.
$c_2 : a = b$.
We must prove $h(x) \Rightarrow (c_1(x) \vee c_2)$. Observe that $h(x) \Rightarrow (c_1(x) \vee c_2) \Leftrightarrow (\forall x)(h(x) \to (c_1(x) \vee c_2)$.
Thus, we prove $(\forall x)(h(x) \to (c_1(x) \vee c_2)$.
Let $x \in \mathbb{R}$ such that $h(x)$ is true. We must prove either $x = 0$ or $a = b$. $\square$

Proof. Let real numbers a and b be given. Let x be an arbitrary real number such that $xa = xb$.

We must prove either $x = 0$ or $a = b$.

Since $xa = xb$, then $xa - xb = 0$, so $x(a - b) = 0$. Hence, either $x = 0$ or $a - b = 0$. Therefore, either $x = 0$ or $a = b$, as desired. $\square$

Exercise 42. Let f, g, h be real valued functions of a real variable. If $A = \{x \in \mathbb{R} : f(x) \neq g(x)\}$ and $B = \{x \in \mathbb{R} : g(x) \neq h(x)\}$ and $C = \{x \in \mathbb{R} : f(x) \neq h(x)\}$, then $C \subset A \cup B$.

Solution. To prove $C \subset A \cup B$, let $x \in C$. We must prove either $x \in A$ or $x \in B$.
One approach is to consider:
$x \in C \to (x \in A \vee x \in B) \Leftrightarrow (x \in C \wedge x \notin A) \to x \in B$.
Thus, we may assume $x \notin A$ and prove $x \in B$.

$\square$

Proof. To prove $C \subset A \cup B$, let $x \in C$. We must prove either $x \in A$ or $x \in B$.

Suppose $x \notin A$. We must prove $x \in B$.

Since $x \in C$, then $x \in \mathbb{R}$ and $f(x) \neq h(x)$. Since $x \in A$ iff $f(x) \neq g(x)$, then $x \notin A$ iff $f(x) = g(x)$. Since $x \notin A$, then we conclude $f(x) = g(x)$.

Since f, g and h are real valued functions, then $f(x), g(x)$, and $h(x)$ are real numbers.

We know equality of real numbers is transitive. Thus, for any real numbers a, b, c, if $a = b$ and $b = c$, then $a = c$. Hence, if $a \neq c$ then either $a \neq b$ or $b \neq c$.

Thus, since $f(x) \neq h(x)$, then either $f(x) \neq g(x)$ or $g(x) \neq h(x)$. Since $f(x) = g(x)$, then we conclude $g(x) \neq h(x)$. Since $x \in \mathbb{R}$ and $g(x) \neq h(x)$, then $x \in B$, as desired. $\square$

Exercise 43. Let S be a subset of $\mathbb{R}$ that is closed under addition and has the property that $-x \in S$ whenever $x \in S$. Then if $x \in S$ and $y \notin S$, then $x + y \notin S$.

Solution.
The hypothesis is:
1. $S \subset \mathbb{R}$.

2. S is closed under addition.
3. $(\forall x)(x \in S \rightarrow -x \in S)$.
The conclusion is:
$x \in S$ and $y \notin S \Rightarrow x + y \notin S$.
We must prove: $x \in S \wedge y \notin S \Rightarrow (x + y) \notin S$ which is equivalent to $(\forall x, y)(x \in S \wedge y \notin S \rightarrow (x + y) \notin S)$.
Direct proof is not fruitful, so let's try indirect proof such as using contrapositive.
Observe that $x \in S \wedge y \notin S \Rightarrow (x + y) \notin S \Leftrightarrow x \in S \wedge (x + y) \in S \Rightarrow y \in S$.
Let $x, y \in \mathbb{R}$ be arbitrary. To prove $x \in S$ and $y \notin S \Rightarrow (x + y) \notin S$, we prove $x \in S$ and $x + y \in S \Rightarrow y \in S$.
Suppose $x \in S$ and $x + y \in S$. We must prove $y \in S$.
We note the properties of S specified in the hypothesis:
S is closed under addition means $a \in S$ and $b \in S \Rightarrow a + b \in S$.
The property $-x \in S$ whenever $x \in S$ means $x \in S \Rightarrow -x \in S$.
How can we show $y \in S$? We use the properties of S(hypothesis). We know $x + y \in S$, so if we could add $-x$ we would get $(-x + x) + y = 0 + y$.
Thus, we would need to establish that $-x \in S$ and $0 \in S$.
How can we deduce $-x \in S$? We use the property regarding negatives.
How can we deduce $0 \in S$? Use additive inverses: $x + (-x) = 0$. But, we must show $-x \in S$. How can we deduce $-x \in S$? We use the property regarding negatives.
Thus, since $x \in S$, then $-x \in S$. Hence, $x + (-x) = 0 \in S$. Since $x + y \in S$ and $-x \in S$, then $-x+(x+y) \in S$. Since $-x+(x+y) = (-x+x)+y = 0+y = y$, then this implies $y \in S$.
We also observe that $\mathbb{Z}$ and $\mathbb{Q}$ satisfy the hypotheses of this theorem since $(\mathbb{Z}, +)$ and $(\mathbb{Q}, +)$ are groups. □

Proof. Let x and y be arbitrary real numbers.
To prove $x \in S$ and $y \notin S$ imply $(x + y) \notin S$, we prove $x \in S$ and $x + y \in S$ imply $y \in S$.
Suppose $x \in S$ and $x + y \in S$. We must prove $y \in S$.
Since $x \in S$, then $-x \in S$. Since $x+y \in S$ and $-x \in S$, then $-x+(x+y) \in S$. Since $-x + (x + y) = (-x + x) + y = 0 + y = y$, then this implies $y \in S$. □

Exercise 44. Let S be a subset of $\mathbb{R}$ that is closed under multiplication and has the property that $1/x \in S$ whenever $x \in S$ and $x \neq 0$. Then if $x \in S, x \neq 0$ and $y \notin S$, then $xy \notin S$.

Solution.
The hypothesis is:
1. $S \subset \mathbb{R}$.
2. S is closed under multiplication.
3. $(\forall x)(x \in S \wedge x \neq 0 \rightarrow 1/x \in S)$.
The conclusion is:
$x \in S, x \neq 0$ and $y \notin S \Rightarrow xy \notin S$.

We must prove: $x \in S \wedge x \neq 0 \wedge y \notin S \Rightarrow xy \notin S$ which is equivalent to $(\forall x, y)(x \in S \wedge x \neq 0 \wedge y \notin S \rightarrow xy \notin S)$.

Direct proof is not fruitful, so let's try indirect proof such as using contrapositive.

Observe that $x \in S \wedge x \neq 0 \wedge y \notin S \Rightarrow xy \notin S \Leftrightarrow x \in S \wedge x \neq 0 \wedge xy \in S \Rightarrow y \in S$.

Let $x, y \in \mathbb{R}$ be arbitrary. To prove $x \in S, x \neq 0$ and $y \notin S \Rightarrow xy \notin S$, we prove $x \in S, x \neq 0$ and $xy \in S \Rightarrow y \in S$.

Suppose $x \in S, x \neq 0$ and $xy \in S$.

We must prove $y \in S$.

We note the properties of S specified in the hypothesis:

S is closed under multiplication means $a \in S$ and $b \in S \Rightarrow ab \in S$.

The inverse property $1/x \in S$ whenever $x \in S, x \neq 0$ means $x \in S, x \neq 0 \Rightarrow 1/x \in S$.

How can we show $y \in S$? We use the properties of S(hypothesis). We know $xy \in S$, so if we could multiply by $1/x$ we would get $(1/x \cdot x)y = 1y = y$.

Thus, we would need to establish that $1/x \in S$ and $1 \in S$.

How can we deduce $1/x \in S$? We use the property regarding inverses.

Thus, since $x \in S$ and $x \neq 0$, then $1/x \in S$. Since $xy \in S$ and $1/x \in S$, then $1/x \cdot (xy) \in S$. Since $1/x \cdot (xy) = (1/x \cdot x)y = 1y = y$, then this implies $y \in S$.

We also observe that $\mathbb{Q}$ satisfy the hypotheses of this theorem. □

Proof. Let x and y be arbitrary real numbers.

To prove $x \in S, x \neq 0$ and $y \notin S \Rightarrow xy \notin S$, we prove $x \in S, x \neq 0$ and $xy \in S \Rightarrow y \in S$.

Suppose $x \in S, x \neq 0$ and $xy \in S$.

We must prove $y \in S$.

Since $x \in S$ and $x \neq 0$, then $1/x \in S$. Since $xy \in S$ and $1/x \in S$, then $1/x \cdot (xy) \in S$. Since $1/x \cdot (xy) = (1/x \cdot x)y = 1y = y$, then this implies $y \in S$. □

Exercise 45. Prove there is a unique solution to the equation $7x - 5 = 0$.

Solution.

We define predicate $p(x) : 7x - 5 = 0$ over $\mathbb{R}$.

To prove $(\exists! x \in \mathbb{R})(p(x))$, we must prove:

1. at least one such x exists $(\exists x \in \mathbb{R})(p(x))$ and
2. at most one such x exists.

We solve for x.

Observe that $7x - 5 = 0 \Rightarrow x = 5/7$. Therefore, at most one solution exists.

We now prove at least one solution exists.

Let $x = 5/7$. Then $7x - 5 = 7(5/7) - 5 = 5 - 5 = 0$, as desired. □

Proof. We prove at most one solution to the equation $7x - 5 = 0$ exists. Suppose $7x - 5 = 0$. Then $7x = 5$, so $x = 5/7$. Therefore, at most one solution exists.

We now prove at least one solution exists. Let $x = 5/7$. Then $7x - 5 = 7(5/7) - 5 = 5 - 5 = 0$, as desired.

Since at least one solution exists and at most one solution exists, then exactly one solution exists. Therefore, there is a unique solution to the equation. $\square$

Exercise 46. Prove there is a unique solution to the equation $x - 5 = \sqrt{x+7}$.

Solution.

We define predicate $p(x) : x - 5 = \sqrt{x+7}$ over $\mathbb{R}$.

To prove $(\exists! x \in \mathbb{R})(p(x))$, we must prove:

1. at least one such x exists $(\exists x \in \mathbb{R})(p(x))$ and
2. at most one such x exists.

We solve for x.

Suppose $x - 5 = \sqrt{x+7}$. Then $x^2 - 10x + 25 = x + 7$, so $x^2 - 11x + 18 = 0$. Thus, $(x-9)(x-2) = 0$, so either $x = 2$ or $x = 9$.

Suppose $x = 2$. Then $x - 5 = 2 - 5 = -3 \neq 3 = \sqrt{9} = \sqrt{2+7} = \sqrt{x+7}$. Therefore, 2 is not a solution. Hence, at most one solution exists.

We prove at least one solution exists. Let $x = 9$.

Then $x - 5 = 9 - 5 = 4 = \sqrt{16} = \sqrt{9+7} = \sqrt{x+7}$, as desired. $\square$

Proof. **Uniqueness:**

We prove at most one solution exists.

Suppose $x - 5 = \sqrt{x+7}$ for some real number x. Then $x^2 - 10x + 25 = x + 7$, so $x^2 - 11x + 18 = 0$. Thus, $(x-9)(x-2) = 0$, so either $x = 2$ or $x = 9$.

Suppose $x = 2$. Then $x - 5 = 2 - 5 = -3 \neq 3 = \sqrt{9} = \sqrt{2+7} = \sqrt{x+7}$. Therefore, 2 is not a solution. Hence, at most one solution exists.

Existence: We prove at least one solution exists.

Let $x = 9$.

Then $x - 5 = 9 - 5 = 4 = \sqrt{16} = \sqrt{9+7} = \sqrt{x+7}$, as desired.

Since at least one solution exists and at most one solution exists, then exactly one solution exists. Therefore, there is a unique solution to the equation $x - 5 = \sqrt{x+7}$. $\square$

Exercise 47. Let A be the set of integers which can be expressed as $2k - 1$ for some $k \in \mathbb{Z}$.

Let B be the set of integers which can be expressed as $2n+1$ for some $n \in \mathbb{Z}$.

Then $A = B$.

Proof. We first prove $A \subset B$.

Let $a \in A$ be arbitrary.

Then $a = 2k - 1$ for some $k \in \mathbb{Z}$.

Let $n = k - 1$.

Since $k \in \mathbb{Z}$ and $\mathbb{Z}$ is closed under subtraction, then $n \in \mathbb{Z}$.

Thus,

$$\begin{aligned} a &= 2k - 1 \\ &= 2(n+1) - 1 \\ &= 2n + 2 - 1 \\ &= 2n + 1. \end{aligned}$$

Since $a = 2n + 1$ for some integer n, then $a \in B$.
Hence, if $a \in A$, then $a \in B$, so $A \subset B$.

We next prove $B \subset A$.
Let $b \in B$ be arbitrary.
Then $b = 2n + 1$ for some $n \in \mathbb{Z}$.
Let $k = n + 1$.
Since $n \in \mathbb{Z}$ and $\mathbb{Z}$ is closed under addition, then $k \in \mathbb{Z}$.
Thus,

$$\begin{aligned} b &= 2n + 1 \\ &= 2(k-1) + 1 \\ &= 2k - 2 + 1 \\ &= 2k - 1. \end{aligned}$$

Since $b = 2k - 1$ for some integer k, then $b \in A$.
Hence, if $b \in B$, then $b \in A$, so $B \subset A$.

Since $A \subset B$ and $B \subset A$, then $A = B$, as desired. ☐

Exercise 48. The number $\log_2 3$ is irrational.

Proof. We prove by contradiction.
Suppose $\log_2 3$ is rational.
Let $x = \log_2 3$.
Then $x \in \mathbb{Q}$ and $2^x = 3$.
Since $2^0 = 1 \neq 3$, then $x \neq 0$.
Suppose x is negative.
Then $x < 0$, so $-x > 0$.
Since $x \in \mathbb{R}$ and $-x > 0$, then $2^{-x} > 1$.
Hence, $\frac{1}{2^x} > 1$, so $\frac{1}{3} > 1$, a contradiction.
Therefore, x cannot be negative.
Since x cannot be negative and x cannot be zero, then by trichotomy, x must be positive.
Thus, $x > 0$, so there exist positive integers m and n such that $x = \frac{m}{n}$.
Hence, $2^{\frac{m}{n}} = 3$, so $2^m = 3^n$.
Since $m \in \mathbb{Z}$ and $m > 0$, then $m \geq 1$, so $m - 1 \geq 0$.
Since $m \in \mathbb{Z}$ and $\mathbb{Z}$ is closed under subtraction, then $m - 1 \in \mathbb{Z}$.
Since $m - 1 \in \mathbb{Z}$ and $m - 1 \geq 0$, then $2^{m-1} \in \mathbb{Z}$.
Since $2^m = 2 \cdot 2^{m-1}$, then this implies $2|2^m$, so 2 is the only prime factor of 2^m.
Since $n \in \mathbb{Z}$ and $n > 0$, then $n \geq 1$, so $n - 1 \geq 0$.
Since $n \in \mathbb{Z}$ and $\mathbb{Z}$ is closed under subtraction, then $n - 1 \in \mathbb{Z}$.
Since $n - 1 \in \mathbb{Z}$ and $n - 1 \geq 0$, then $3^{n-1} \in \mathbb{Z}$.
Since $3^n = 3 \cdot 3^{n-1}$, then this implies $3|3^n$, so 3 is the only prime factor of 3^n.

Hence, 2 cannot be a prime factor of 3^n.

Since 2 is a prime factor of 2^m, but 2 is not a prime factor of 3^n, then $2^m \neq 3^n$.

But, this contradicts the fact that $2^m = 3^n$.

Therefore, $\log_2 3$ is irrational. $\square$

Exercise 49. Let $S = \{n \in \mathbb{N} : n^2 - 3n + 2 = 0\}$.

Then $S = \{1, 2\}$.

Proof. We first prove $\{1, 2\} \subset S$.

Since $1 \in \mathbb{N}$ and $1^2 - 3 * 1 + 2 = 1 - 3 + 2 = 0$, then $1 \in S$.

Since $2 \in \mathbb{N}$ and $2^2 - 3 * 2 + 2 = 4 - 6 + 2 = 0$, then $2 \in S$.

Since $1 \in S$ and $2 \in S$, then $\{1, 2\} \subset S$.

We next prove $S \subset \{1, 2\}$.

Suppose $n \in S$.

Then $n \in \mathbb{N}$ and $n^2 - 3n + 2 = 0$.

Since $n \in \mathbb{N}$ and $\mathbb{N} \subset \mathbb{Z}$, then $n \in \mathbb{Z}$.

Since $n^2 - 3n + 2 = 0$, then $(n-2)(n-1) = 0$, so either $n - 2 = 0$ or $n - 1 = 0$.

Hence, either $n = 2$ or $n = 1$, so $n \in \{1, 2\}$.

Thus, if $n \in S$, then $n \in \{1, 2\}$, so $S \subset \{1, 2\}$.

Since $S \subset \{1, 2\}$ and $\{1, 2\} \subset S$, then $S = \{1, 2\}$, as desired. $\square$

Exercise 50. Let $S_n = \{(n+1)k : k \in \mathbb{N}\}$.

Compute

1. $S_1 \cap S_2$.
2. $\cup_{n=1}^{\infty} S_n$.
3. $\cap_{n=1}^{\infty} S_n$.

Proof. 1. We prove $S_1 \cap S_2 = S_5$.

(Thus, $S_1 \cap S_2$ is the set of multiples of 6).

Observe that $S_1 = \{2k : k \in \mathbb{N}\}$ and $S_2 = \{3k : k \in \mathbb{N}\}$ and $S_5 = \{6k : k \in \mathbb{N}\}$.

We first prove $S_n \subset \mathbb{N}$ for each $n \in \mathbb{N}$.

Let $n \in \mathbb{N}$ be given.

Then $S_n = \{(n+1)k : k \in \mathbb{N}\}$.

Suppose $x \in S_n$.

Then there exists $k \in \mathbb{N}$ such that $x = (n+1)k$.

Since $n \in \mathbb{N}$, then $n + 1 \in \mathbb{N}$.

Since $n + 1, k \in \mathbb{N}$ and $\mathbb{N}$ is closed under multiplication, then $x \in \mathbb{N}$.

Therefore, $S_n \subset \mathbb{N}$, as desired.

We prove $S_5 \subset S_1 \cap S_2$.
Let $x \in S_5$.
Then $x = 6k$ for some $k \in \mathbb{N}$.
Since $x = 6k = (2 * 3)k = 2(3k)$ and $3k \in \mathbb{N}$, then $x \in S_1$.
Since $x = 6k = (3 * 2)k = 3(2k)$ and $2k \in \mathbb{N}$, then $x \in S_2$.
Thus, $x \in S_1$ and $x \in S_2$, so $x \in S_1 \cap S_2$.
Therefore, $S_5 \subset S_1 \cap S_2$.

We prove $S_1 \cap S_2 \subset S_5$.
Let $y \in S_1 \cap S_2$.
Then $y \in S_1$ and $y \in S_2$.
Since $y \in S_1$, then $y = 2k$ for some $k \in \mathbb{N}$, so $2|y$.
Since $y \in S_2$, then $y = 3m$ for some $m \in \mathbb{N}$, so $3|y$.
Since $2|y$ and $3|y$ and $\gcd(2, 3) = 1$, then $2 * 3|y$, so $6|y$.
Since $S_1 \cap S_2 \subset S_1$ and $S_1 \subset \mathbb{N}$, then $S_1 \cap S_2 \subset \mathbb{N}$.
Since $y \in S_1 \cap S_2$, then $y \in \mathbb{N}$.
Since $6|y$ and $y \in \mathbb{N}$, then $y = 6t$ for some $t \in \mathbb{N}$, so $y \in S_5$.
Therefore, $S_1 \cap S_2 \subset S_5$.

Since $S_1 \cap S_2 \subset S_5$ and $S_5 \subset S_1 \cap S_2$, then $S_1 \cap S_2 = S_5$, as desired. □

Proof. 2. We prove $\cup_{n=1}^{\infty} S_n = \mathbb{N} - \{1\}$.

We first prove $\mathbb{N} - \{1\} \subset \cup_{n=1}^{\infty} S_n$.
Let $x \in \mathbb{N} - \{1\}$.
Then $x \in \mathbb{N}$ and $x \neq 1$.
Since $x \in \mathbb{N}$, then $x \geq 1$.
Since $x \neq 1$, then $x > 1$, so $x - 1 > 0$.
Since $x \in \mathbb{N}$ and $\mathbb{N} \subset \mathbb{Z}$, then $x \in \mathbb{Z}$.
Since $x \in \mathbb{Z}$, then $x - 1 \in \mathbb{Z}$.
Since $x - 1 \in \mathbb{Z}$ and $x - 1 > 0$, then $x - 1 \in \mathbb{N}$.
Thus, $S_{x-1} = \{xk : k \in \mathbb{N}\}$.
Since $x = x \cdot 1$ and $1 \in \mathbb{N}$, then $x \in S_{x-1}$.
Thus, there exists $x - 1 \in \mathbb{N}$ such that $x \in S_{x-1}$, so $x \in \cup_{n=1}^{\infty} S_n$.
Therefore, $\mathbb{N} - \{1\} \subset \cup_{n=1}^{\infty} S_n$.

We prove $\cup_{n=1}^{\infty} S_n \subset \mathbb{N} - \{1\}$.
Let $y \in \cup_{n=1}^{\infty} S_n$.
Then there exists $m \in \mathbb{N}$ such that $y \in S_m$.
Since $y \in S_m$ and $S_m = \{(m + 1)k : k \in \mathbb{N}\}$, then there exists $k \in \mathbb{N}$ such that $y = (m + 1)k$.
Since $m \in \mathbb{N}$, then $m + 1 \in \mathbb{N}$.
Since $k, m + 1 \in \mathbb{N}$ and $\mathbb{N}$ is closed under multiplication, then $y \in \mathbb{N}$, so $y \geq 1$.
Suppose for the sake of contradiction $y = 1$.

Then $1 \in \cup_{n=1}^{\infty} S_n$, so there exists $t \in \mathbb{N}$ such that $1 \in S_t$.

Since $1 \in S_t$ and $S_t = \{(t+1)k : k \in \mathbb{N}\}$, then there exists $r \in \mathbb{N}$ such that $1 = (t+1)r$.

Since $r, t \in \mathbb{N}$, then $r \geq 1$ and $t \geq 1$, so $t + 1 \geq 2$.

Thus, $(t+1)r \geq 2$, so $1 \geq 2$, a contradiction.

Hence, $y \neq 1$, so $y \notin \{1\}$.

Since $y \in \mathbb{N}$ and $y \notin \{1\}$, then $y \in \mathbb{N} - \{1\}$.

Therefore, $\cup_{n=1}^{\infty} S_n \subset \mathbb{N} - \{1\}$.

Since $\cup_{n=1}^{\infty} S_n \subset \mathbb{N} - \{1\}$ and $\mathbb{N} - \{1\} \subset \cup_{n=1}^{\infty} S_n$, then $\cup_{n=1}^{\infty} S_n = \mathbb{N} - \{1\}$, as desired. □

Proof. 3. We prove $\cap_{n=1}^{\infty} S_n = \emptyset$.

We prove by contradiction.

Suppose $\cap_{n=1}^{\infty} S_n \neq \emptyset$.

Then there exists $x \in \cap_{n=1}^{\infty} S_n$, so $x \in S_n$ for each $n \in \mathbb{N}$.

Let $n \in \mathbb{N}$.

Then $x \in S_n$.

Since $S_n = \{(n+1)k : k \in \mathbb{N}\}$, then $x = (n+1)k$ for some $k \in \mathbb{N}$.

Since $n \in \mathbb{N}$, then $n + 1 \in \mathbb{N}$.

Since $n+1 \in \mathbb{N}$ and $k \in \mathbb{N}$ and $\mathbb{N}$ is closed under multiplication, then $x \in \mathbb{N}$.

Since $k, n \in \mathbb{N}$, then $k \geq 1$ and $n \geq 1$, so $n + 1 \geq 2$.

Hence, $x \geq 2$.

Since $x \in \mathbb{N}$, then $S_x = \{(x+1)k : k \in \mathbb{N}\}$.

Suppose for the sake of contradiction $x \in S_x$.

Then there exists $m \in \mathbb{N}$ such that $x = (x+1)m$.

Since $x + 1 \geq 3 > 0$, then $x + 1 > 0$, so $x + 1 \neq 0$.

Hence, $\frac{x}{x+1} = m$.

Since $x < x + 1$ and $x + 1 > 0$, then $\frac{x}{x+1} < 1$, so $m < 1$.

Thus, $m \in \mathbb{N}$ and $m < 1$.

But, this contradicts the fact that every natural number is greater than or equal to one.

Therefore, $x \notin S_x$.

Thus, there exists $x \in \mathbb{N}$ such that $x \notin S_x$, so $x \notin \cap_{n=1}^{\infty} S_n$.

Hence, we have $x \in \cap_{n=1}^{\infty} S_n$ and $x \notin \cap_{n=1}^{\infty} S_n$, a contradiction.

Therefore, we conclude $\cap_{n=1}^{\infty} S_n = \emptyset$, as desired. □

Exercise 51. Let $S = \{x \in \mathbb{R} : x(x-1)(x-2)(x-3) < 0\}$.

Then $S = (0,1) \cup (2,3)$.

Proof. Since $\frac{1}{2}(\frac{1}{2} - 1)(\frac{1}{2} - 2)(\frac{1}{2} - 3) = \frac{1}{2}(\frac{-1}{2})(\frac{-3}{2})(\frac{-5}{2}) < 0$, then $\frac{1}{2} \in S$, so $S \neq \emptyset$.

We prove $S \subset (0,1) \cup (2,3)$.

Let $x \in \mathbb{R}$.

To prove $S \subset (0,1) \cup (2,3)$, we must prove if $x(x-1)(x-2)(x-3) < 0$, then $x \in (0,1) \cup (2,3)$.

Hence, we must prove if $x(x-1)(x-2)(x-3) < 0$, then either $0 < x < 1$ or $2 < x < 3$.

We prove by contrapositive.

Suppose it is not the case that either $0 < x < 1$ or $2 < x < 3$.

Then it is not the case that $0 < x < 1$ and it is not the case that $2 < x < 3$, so it is not the case that $0 < x$ and $x < 1$ and it is not the case that $2 < x$ and $x < 3$.

Hence, either $0 \geq x$ or $x \geq 1$ and either $2 \geq x$ or $x \geq 3$, so either $x \leq 0$ or $x \geq 1$ and either $x \leq 2$ or $x \geq 3$.

Thus, either $x \leq 0$ and $x \leq 2$ or $x \leq 0$ and $x \geq 3$ or $x \geq 1$ and $x \leq 2$ or $x \geq 1$ and $x \geq 3$.

Therefore, either $x \leq 0$ or $x \leq 0$ and $x \geq 3$ or $x \geq 1$ and $x \leq 2$ or $x \geq 3$.

Since the condition $x \leq 0$ and $x \geq 3$ is impossible, then either $x \leq 0$ or $x \geq 1$ and $x \leq 2$ or $x \geq 3$.

We consider these cases separately.

Case 1: Suppose $x \leq 0$.

Since $x-3 < x-2 < x-1 < x$, then $x-3 < x$ and $x-2 < x$ and $x-1 < x$.

Since $x-3 < x \leq 0$, then $x-3 < 0$.

Since $x-2 < x \leq 0$, then $x-2 < 0$.

Since $x-1 < x \leq 0$, then $x-1 < 0$.

Since $x \leq 0$ and $x-1 < 0$ and $x-2 < 0$ and $x-3 < 0$, then $x(x-1)(x-2)(x-3) \geq 0$.

Case 2: Suppose $x \geq 1$ and $x \leq 2$.

Then $x-1 \geq 0$ and $x-2 \leq 0$.

Since $x \geq 1 > 0$, then $x > 0$.

Since $x \leq 2 < 3$, then $x < 3$, so $x-3 < 0$.

Since $x > 0$ and $x-1 \geq 0$ and $x-2 \leq 0$ and $x-3 < 0$, then $x(x-1)(x-2)(x-3) \geq 0$.

Case 3: Suppose $x \geq 3$.

Then $x-3 \geq 0$.

Since $x > x-1 > x-2 > x-3$, then $x > x-3$ and $x-1 > x-3$ and $x-2 > x-3$.

Since $x > x-3 \geq 0$, then $x > 0$.

Since $x-1 > x-3 \geq 0$, then $x-1 > 0$.

Since $x-2 > x-3 \geq 0$, then $x-2 > 0$.

Since $x > 0$ and $x-1 > 0$ and $x-2 > 0$ and $x-3 \geq 0$, then $x(x-1)(x-2)(x-3) \geq 0$.

Thus, in all cases, $x(x-1)(x-2)(x-3) \geq 0$, as desired.

Therefore, if $x(x-1)(x-2)(x-3) < 0$, then $x \in (0,1) \cup (2,3)$, so $S \subset (0,1) \cup (2,3)$. $\square$

Proof. We prove $(0,1) \cup (2,3) \subset S$.

Let $x \in (0,1) \cup (2,3)$.

Then either $x \in (0,1)$ or $x \in (2,3)$, so either $0 < x < 1$ or $2 < x < 3$.

We consider these cases separately.

Case 1: Suppose $0 < x < 1$.

Then $0 < x$ and $x < 1$, so $x > 0$ and $x - 1 < 0$.
Since $x - 3 < x - 2 < x - 1 < 0$, then $x - 3 < 0$ and $x - 2 < 0$.
Since $x > 0$ and $x - 1 < 0$ and $x - 2 < 0$ and $x - 3 < 0$, then $x(x-1)(x-2)(x-3) < 0$.
Case 2: Suppose $2 < x < 3$.
Then $2 < x$ and $x < 3$, so $x > 2$ and $x - 3 < 0$.
Since $x > 2 > 0$, then $x > 0$.
Since $x > 2 > 1$, then $x > 1$, so $x - 1 > 0$.
Since $x > 2$, then $x - 2 > 0$.
Since $x > 0$ and $x - 1 > 0$ and $x - 2 > 0$ and $x - 3 < 0$, then $x(x-1)(x-2)(x-3) < 0$.
Thus, in all cases, $x(x-1)(x-2)(x-3) < 0$, so $x \in S$.
Therefore, $(0,1) \cup (2,3) \subset S$.
Since $S \subset (0,1) \cup (2,3)$ and $(0,1) \cup (2,3) \subset S$, then $S = (0,1) \cup (2,3)$. $\square$

Exercise 52. Let $A = \{(x,y) \in \mathbb{R} \times \mathbb{R} : x \neq 0, y \neq 0, \frac{x}{y} + \frac{y}{x} \geq 2\}$.
Then $A = [(0,\infty) \times (0,\infty)] \cup [(-\infty,0) \times (-\infty,0)]$.
(In other words, A is the union of quadrants I and III of the xy plane.)

Proof. Let $B = [(0,\infty) \times (0,\infty)] \cup [(-\infty,0) \times (-\infty,0)]$.
To prove $A = B$, we prove $A \subset B$ and $B \subset A$.
We first prove $B \subset A$.
Since $(1,1) \in (0,\infty) \times (0,\infty)$ and $(0,\infty) \times (0,\infty)$ is a subset of B, then $(1,1) \in B$, so $B \neq \emptyset$.
Let $(x,y) \in B$.
Then either $(x,y) \in (0,\infty) \times (0,\infty)$ or $(x,y) \in (-\infty,0) \times (-\infty,0)$, so either $x \in (0,\infty)$ and $y \in (0,\infty)$ or $x \in (-\infty,0)$ and $y \in (-\infty,0)$.
Hence, either $x > 0$ and $y > 0$ or $x < 0$ and $y < 0$, so either $xy > 0$ or $xy > 0$.
Thus, $xy > 0$.
Hence, $xy \neq 0$, so $x \neq 0$ and $y \neq 0$.
Since $0 \leq (x-y)^2 = x^2 - 2xy + y^2$, then $2xy \leq x^2 + y^2$.
Since $xy > 0$ and $x \neq 0$ and $y \neq 0$, then $2 \leq \frac{x^2+y^2}{xy} = \frac{x}{y} + \frac{y}{x}$.
Since $x \neq 0$ and $y \neq 0$ and $\frac{x}{y} + \frac{y}{x} \geq 2$, then $(x,y) \in A$.
Therefore, $B \subset A$. $\square$

Proof. We next prove $A \subset B$.
Since $1 \neq 0$ and $\frac{1}{1} + \frac{1}{1} = 1 + 1 = 2$, then $(1,1) \in A$, so $A \neq \emptyset$.
Let $(x,y) \in A$.
Then $(x,y) \in \mathbb{R}^2$ and $x \neq 0$ and $y \neq 0$ and $\frac{x}{y} + \frac{y}{x} \geq 2$.
Since $(x,y) \in \mathbb{R}^2$, then $x \in \mathbb{R}$ and $y \in \mathbb{R}$.
Since $x \neq 0$ and $y \neq 0$, then $xy \neq 0$, so either $xy > 0$ or $xy < 0$.
Suppose $xy < 0$.
Then either $x > 0$ and $y < 0$ or $x < 0$ and $y > 0$, so either $y < 0 < x$ or $x < 0 < y$.
Hence, either $y < x$ or $x < y$, so either $x > y$ or $x < y$.

Thus, $x \neq y$, so $x - y \neq 0$.
Therefore, $0 < (x-y)^2 = x^2 - 2xy + y^2$, so $2xy < x^2 + y^2$.
Since $xy < 0$ and $x \neq 0$ and $y \neq 0$, then $2 > \frac{x^2+y^2}{xy} = \frac{x}{y} + \frac{y}{x}$, so $\frac{x}{y} + \frac{y}{x} < 2$.
But, this contradicts the fact that $\frac{x}{y} + \frac{y}{x} \geq 2$.
Hence, xy cannot be less than zero, so $xy > 0$.
Thus, either $x > 0$ and $y > 0$, or $x < 0$ and $y < 0$.
We consider these cases separately.
Case 1: Suppose $x > 0$ and $y > 0$.
Since $x > 0$, then $x \in (0, \infty)$.
Since $y > 0$, then $y \in (0, \infty)$.
Thus, $(x, y) \in (0, \infty) \times (0, \infty)$.
Since $(0, \infty) \times (0, \infty)$ is a subset of B, then $(x, y) \in B$.
Case 2: Suppose $x < 0$ and $y < 0$.
Since $x < 0$, then $x \in (-\infty, 0)$.
Since $y < 0$, then $y \in (-\infty, 0)$.
Thus, $(x, y) \in (-\infty, 0) \times (-\infty, 0)$.
Since $(-\infty, 0) \times (-\infty, 0)$ is a subset of B, then $(x, y) \in B$.
Hence, in either case, $(x, y) \in B$, so $A \subset B$.
Since $A \subset B$ and $B \subset A$, then $A = B$, as desired. $\square$