On deriving Mills constant for a prime-representing function

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Abstract

Mills showed the existence of a real number A such that $\lfloor A^{3^n} \rfloor$ is prime for every positive integer n, where $\lfloor X \rfloor$ denotes the floor function.

This paper reviews the proof details and calculates several prime values.

1 Introduction

Mills [1] showed in 1947 that there is a real number A such that $\lfloor A^{3^n} \rfloor$ is prime for every positive integer n.

This defines a prime-representing function $f : \mathbb{Z}^+ \to \mathbb{Z}^+$ defined by $f(n) = \lfloor A^{3^n} \rfloor$.

A prime-representing function is a function that generates primes.

The function f is known as Mills function.

The real number A is known as Mills' constant.

The primes generated by this function are known as Mills primes.

This paper reviews the proof details and calculates several Mills primes.

2 Proof Details

This paper begins with a proof of the existence of a greatest prime number less than any integer greater than 2.

Lemma 1. *Let $n \in \mathbb{Z}^+$ and $n > 2$.*

Let S be the set of all prime numbers less than n.

Then S has a greatest element.

Proof. Observe that $S = \{p \in \mathbb{Z}^+ : p \text{ is prime and } p < n\}$.

Since $n \in \mathbb{Z}$ and $n > 2$, then there is a prime p such that $p < n$.

Since $p \in \mathbb{Z}^+$ and p is prime and $p < n$, then $p \in S$, so S is non-empty.

Since $(\mathbb{Z}^+, \leq)$ is a totally ordered set and $S \subset \mathbb{Z}^+$, then $(S, \leq)$ is a totally ordered set.

Let T be the set of all positive integers less than n.

Then $T = \{k \in \mathbb{Z}^+ : k < n\} = \{1, ..., n-1\}$ is a finite set of cardinality $n-1$.

Let $x \in S$.

Then $x \in \mathbb{Z}^+$ and x is prime and $x < n$.

Since $x \in \mathbb{Z}^+$ and $x < n$, then $x \in T$.

Hence, $x \in S$ implies $x \in T$, so $S \subseteq T$.

A subset of a finite set is finite.

Since $S \subseteq T$ and T is finite, then S is finite.

A totally ordered non-empty finite set has a greatest element.

Since $(S, \leq)$ is a totally ordered non-empty finite set, then S has a greatest element. □

Ingham [2] established the result below on the difference of consecutive primes.

Proposition 2. *Let K be a fixed positive integer.*

Let p_n be the n^{th} prime number in the sequence of primes arranged in ascending order.

Then $p_{n+1} - p_n < K p_n^{\frac{5}{8}}$.

Lemma 3. *Let K be a fixed positive integer.*

If N is an integer greater than K^8, then there exists a prime p such that $N^3 < p < (N+1)^3 - 1$.

Proof. Let N be an integer greater than K^8.

Then $N \in \mathbb{Z}$ and $N > K^8$.

Since $K \in \mathbb{Z}^+$, then $K \geq 1$, so $K^8 \geq 1$.

Since $N > K^8$ and $K^8 \geq 1$, then $N > 1$.

Let S be the set of all prime numbers less than N^3.

Then $S = \{p \in \mathbb{Z}^+ : p \text{ is prime and } p < N^3\}$.

Since $N \in \mathbb{Z}$, then $N^3 \in \mathbb{Z}$.

Since $N \in \mathbb{Z}$ and $N > 1$, then $N \geq 2$, so $N^3 \geq 8 > 2$.

Hence, $N^3 > 2$.

Since $N^3 \in \mathbb{Z}$ and $N^3 > 2$ and S is the set of all prime numbers less than N^3, then S has a greatest element, by lemma 1.

Let p_n be the greatest element of S.

Then $p_n \in \mathbb{Z}^+$ and p_n is prime and $p_n < N^3$.

We prove N^3 is composite.

Since $N^3 = N \cdot N \cdot N$, then N divides N^3.

Since $N > 1$, then $N^2 > 1$.

Since $N > 1 > 0$, then $N > 0$.

Since $N^2 > 1$ and $N > 0$, then $N^3 > N$.

Since $N^3 > N$ and $N > 1$, then $N^3 > N > 1$.

Since $N \in \mathbb{Z}$ and $1 < N < N^3$ and N divides N^3, then N^3 is composite, so N^3 is not prime.

By Euclid's theorem, there are infinitely many primes, so there exists a prime $p_{n+1} > p_n$.

Either $p_{n+1} < N^3$ or $p_{n+1} = N^3$ or $p_{n+1} > N^3$.

Since p_{n+1} is prime and N^3 is not prime, then $p_{n+1} \neq N^3$.

Since p_n is the largest prime less than N^3 and $p_{n+1} > p_n$, then p_{n+1} cannot be less than N^3.

Since p_{n+1} cannot be less than N^3 and $p_{n+1} \neq N^3$, then $p_{n+1} > N^3$.

Therefore, $N^3 < p_{n+1}$.

Since K is a positive integer, then $p_{n+1} - p_n < Kp_n^{\frac{5}{8}}$, by proposition 2.

Therefore, $p_{n+1} < p_n + Kp_n^{\frac{5}{8}}$.

We prove $p_n + Kp_n^{\frac{5}{8}} < N^3 + KN^{\frac{15}{8}}$.

Since $1 < p_n < N^3$, then $p_n^5 < (N^3)^5 = N^{15}$, so $p_n^{\frac{5}{8}} < N^{\frac{15}{8}}$.

Since $p_n^{\frac{5}{8}} < N^{\frac{15}{8}}$ and $K > 0$, then $Kp_n^{\frac{5}{8}} < KN^{\frac{15}{8}}$.

Since $p_n < N^3$ and $Kp_n^{\frac{5}{8}} < KN^{\frac{15}{8}}$, then $p_n + Kp_n^{\frac{5}{8}} < N^3 + KN^{\frac{15}{8}}$.

We prove $N^3 + KN^{\frac{15}{8}} < N^3 + N^2$.

Since $N > 1$, then $N^{15} > 1 > 0$, so $N^{15} > 0$.

Since $N > K^8$ and $N^{15} > 0$, then $N^{16} > K^8N^{15}$, so $N^2 > KN^{\frac{15}{8}}$.

Therefore, $N^3 + N^2 > N^3 + KN^{\frac{15}{8}}$, so $N^3 + KN^{\frac{15}{8}} < N^3 + N^2$.

We prove $N^3 + N^2 < (N+1)^3 - 1$.

Since $N > 1$, then $3N > 3$ and $N^2 > 1$, so $2N^2 > 2$.

Since $2N^2 > 2$ and $3N > 3$, then $2N^2 + 3N > 5 > 0$, so $2N^2 + 3N > 0$.

Hence, $0 < 2N^2 + 3N$.

Observe that

$$\begin{aligned} N^3 + N^2 + 1 &< (N^3 + N^2 + 1) + (2N^2 + 3N) \\ &= N^3 + 3N^2 + 3N + 1 \\ &= (N+1)^3. \end{aligned}$$

Therefore, $N^3 + N^2 + 1 < (N+1)^3$, so $N^3 + N^2 < (N+1)^3 - 1$.

Since $p_n < N^3$ and $N^3 < p_{n+1}$ and $p_{n+1} < p_n + Kp_n^{\frac{5}{8}}$ and $p_n + Kp_n^{\frac{5}{8}} < N^3 + KN^{\frac{15}{8}}$ and $N^3 + KN^{\frac{15}{8}} < N^3 + N^2$ and $N^3 + N^2 < (N+1)^3 - 1$, then $p_n < N^3 < p_{n+1} < p_n + Kp_n^{\frac{5}{8}} < N^3 + KN^{\frac{15}{8}} < N^3 + N^2 < (N+1)^3 - 1$, so $N^3 < p_{n+1} < (N+1)^3 - 1$.

Therefore, p_{n+1} is a prime such that $N^3 < p_{n+1} < (N+1)^3 - 1$, so there exists a prime p such that $N^3 < p < (N+1)^3 - 1$. □

Theorem 4. ***Mills theorem***

There is a prime-representing function $f : \mathbb{Z}^+ \to \mathbb{Z}^+$ defined by $f(n) = \lfloor A^{3^n} \rfloor$ for some real number A.

Proof. Let K be a fixed positive integer.

Then $K \in \mathbb{Z}^+$, so $K^8 \in \mathbb{Z}^+$.

For every positive integer n, there is a prime p greater than n.

Since K^8 is a positive integer, then there is a prime greater than K^8.

Let P_0 be a prime greater than K^8.

Then P_0 is prime and $P_0 > K^8$.

Since $P_0 > K^8$, then there exists a prime P_1 such that $P_0^3 < P_1 < (P_0 + 1)^3 - 1$, by lemma 3.

We construct a sequence of primes (P_n) such that $P_n < P_{n+1}$ and $P_n^3 < P_{n+1} < (P_n + 1)^3 - 1$ for all $n \in \mathbb{Z}^+$.

We prove by induction on n.

Define predicate $q(n)$ over $\mathbb{Z}^+$ by 'P_n is prime and P_{n+1} is prime and $P_n > K^8$ and $P_{n-1} < P_n$ and $P_n^3 < P_{n+1} < (P_n + 1)^3 - 1$'.

Basis:

Let $n = 1$.

Since $P_0^3 < P_1 < (P_0 + 1)^3 - 1$, then $P_0^3 < P_1$, so $P_1 > P_0^3$.

Since P_0 is prime, then $P_0 > 1$, so $P_0 > 0$.

Since $P_0 > 1$ and $P_0 > 1$, then $P_0^2 > 1$.

Since $P_0^2 > 1$ and $P_0 > 0$, then $P_0^3 > P_0$.

Since $P_1 > P_0^3$ and $P_0^3 > P_0$, then $P_1 > P_0$.

Since $P_1 > P_0$ and $P_0 > K^8$, then $P_1 > K^8$.

Since $P_1 > K^8$, then there exists a prime P_2 such that $P_1^3 < P_2 < (P_1 + 1)^3 - 1$, by lemma 3.

Since P_1 is prime and P_2 is prime and $P_1 > K^8$ and $P_0 < P_1$ and $P_1^3 < P_2 < (P_1 + 1)^3 - 1$, then $q(1)$ is true.

Induction:

Let $k \in \mathbb{Z}^+$ such that $q(k)$ is true.

Then P_k is prime and P_{k+1} is prime and $P_k > K^8$ and $P_{k-1} < P_k$ and $P_k^3 < P_{k+1} < (P_k + 1)^3 - 1$.

Since $P_k^3 < P_{k+1} < (P_k + 1)^3 - 1$, then $P_k^3 < P_{k+1}$, so $P_{k+1} > P_k^3$.

Since P_k is prime, then $P_k > 1$, so $P_k > 0$.

Since $P_k > 1$ and $P_k > 1$, then $P_k^2 > 1$.

Since $P_k^2 > 1$ and $P_k > 0$, then $P_k^3 > P_k$.

Since $P_{k+1} > P_k^3$ and $P_k^3 > P_k$, then $P_{k+1} > P_k$.

Since $P_{k+1} > P_k$ and $P_k > K^8$, then $P_{k+1} > K^8$.

Since $P_{k+1} > K^8$, then there exists a prime P_{k+2} such that $P_{k+1}^3 < P_{k+2} < (P_{k+1}+1)^3 - 1$.

Since P_{k+1} is prime and P_{k+2} is prime and $P_{k+1} > K^8$ and $P_k < P_{k+1}$ and $P_{k+1}^3 < P_{k+2} < (P_{k+1}+1)^3 - 1$, then $q(k+1)$ is true.

Therefore, $q(k)$ implies $q(k+1)$ for all $k \in \mathbb{Z}^+$.

Since $q(1)$ is true, and $q(k)$ implies $q(k+1)$ for all $k \in \mathbb{Z}^+$, then by induction, $q(k)$ is true for all $k \in \mathbb{Z}^+$, so $q(n)$ is true for all $n \in \mathbb{Z}^+$.

Therefore, $P_n > K^8$ and $P_{n-1} < P_n$ and $P_n^3 < P_{n+1} < (P_n+1)^3 - 1$ for all $n \in \mathbb{Z}^+$.

Hence, there is a sequence of primes $P_0, P_1, P_2, ...$ such that $K^8 < P_0 < P_1 < P_2 < ... < P_n < P_{n+1} < ...$ and $P_n^3 < P_{n+1} < (P_n+1)^3 - 1$ for all non-negative integers n.

Let (u_n) be the sequence defined by $u_n = P_n^{3^{-n}}$ for all $n \in \mathbb{Z}^+$.

Let (v_n) be the sequence defined by $v_n = (P_n+1)^{3^{-n}}$ for all $n \in \mathbb{Z}^+$.

We prove $3^{-n} > 0$ for all $n \in \mathbb{Z}^+$.

Let $n \in \mathbb{Z}^+$.

Then $3^n \in \mathbb{Z}^+$, so $3^n > 0$.

Hence, $\frac{1}{3^n} > 0$, so $3^{-n} > 0$.

Therefore, $3^{-n} > 0$ for all $n \in \mathbb{Z}^+$.

We prove $v_n > u_n$ for all $n \in \mathbb{Z}^+$.

Let $n \in \mathbb{Z}^+$.

Then $u_n = P_n^{3^{-n}}$ and $v_n = (P_n+1)^{3^{-n}}$ and $n > 0$ and $3^{-n} > 0$.

Since P_n is prime, then $P_n > 0$.

Since $P_n + 1 > P_n > 0$ and $3^{-n} > 0$, then $(P_n+1)^{3^{-n}} > P_n^{3^{-n}}$, so $v_n > u_n$.

Therefore, $v_n > u_n$ for all $n \in \mathbb{Z}^+$.

We prove the sequence (u_n) is strictly increasing.

Let $n \in \mathbb{Z}^+$.

Then $n+1 \in \mathbb{Z}^+$ and $u_n = (P_n)^{3^{-n}}$ and $u_{n+1} = (P_{n+1})^{3^{-n-1}}$ and $P_n^3 < P_{n+1} < (P_n+1)^3 - 1$.

Since $P_n^3 < P_{n+1} < (P_n+1)^3 - 1$, then $P_n^3 < P_{n+1}$.

Since P_n is prime, then $P_n > 0$, so $P_n^3 > 0$.

Since $P_{n+1} > P_n^3$ and $P_n^3 > 0$, then $P_{n+1} > P_n^3 > 0$.

Since $3^{-n} > 0$ for all $n \in \mathbb{Z}^+$ and $n+1 \in \mathbb{Z}^+$, then $3^{-(n+1)} = 3^{-n-1} > 0$.

Thus, $3^{-n-1} > 0$.

Observe that

$$\begin{aligned} 0 < P_n^3 < P_{n+1} &\Rightarrow (P_n^3)^{3^{-n-1}} < (P_{n+1})^{3^{-n-1}} \\ &\Rightarrow (P_n)^{3^{-n}} < (P_{n+1})^{3^{-n-1}} \\ &\Rightarrow u_n < u_{n+1}. \end{aligned}$$

Hence, $u_n < u_{n+1}$, so $u_n < u_{n+1}$ for all $n \in \mathbb{Z}^+$.
Therefore, the sequence (u_n) is strictly increasing.

We prove the sequence (v_n) is strictly decreasing.
Let $n \in \mathbb{Z}^+$.
Then $v_n = (P_n + 1)^{3^{-n}}$ and $v_{n+1} = (P_{n+1} + 1)^{3^{-n-1}}$ and $P_n^3 < P_{n+1} < (P_n + 1)^3 - 1$.
Since $P_n^3 < P_{n+1} < (P_n + 1)^3 - 1$, then $P_{n+1} < (P_n + 1)^3 - 1$, so $P_{n+1} + 1 < (P_n + 1)^3$.
Since P_{n+1} is prime, then $P_{n+1} > 1$, so $P_{n+1} + 1 > 2 > 0$.
Hence, $P_{n+1} + 1 > 0$.
Since $0 < P_{n+1} + 1$ and $P_{n+1} + 1 < (P_n + 1)^3$, then $0 < P_{n+1} + 1 < (P_n + 1)^3$.
Thus, $(P_{n+1} + 1)^{3^{-1}} < P_n + 1$, so $((P_{n+1} + 1)^{3^{-1}})^{3^{-n}} < (P_n + 1)^{3^{-n}}$.
Hence, $(P_{n+1} + 1)^{3^{-n-1}} < (P_n + 1)^{3^{-n}}$, so $v_{n+1} < v_n$.
Consequently, $v_n > v_{n+1}$, so $v_n > v_{n+1}$ for all $n \in \mathbb{Z}^+$.
Therefore, the sequence (v_n) is strictly decreasing.

We prove the sequence (u_n) is bounded above by v_1.
Observe that $v_1 = (P_1 + 1)^{\frac{1}{3}}$ is a real number.
Since $v_n > u_n$ for all $n \in \mathbb{Z}^+$, then $v_1 > u_1$.
Let $k \in \mathbb{Z}^+$ and $k > 1$.
Since $v_n > u_n$ for all $n \in \mathbb{Z}^+$ and $k \in \mathbb{Z}^+$, then $v_k > u_k$.
Since the sequence (v_n) is strictly decreasing, then $v_1 > v_2 > v_3 > ...$, so $v_1 > v_n$ for all $n \in \mathbb{Z}^+$ with $n > 1$.
Since $k \in \mathbb{Z}^+$ and $k > 1$, then $v_1 > v_k$.
Since $v_1 > v_k$ and $v_k > u_k$, then $v_1 > u_k$.
Hence, $v_1 > u_k$ for all $k \in \mathbb{Z}^+$ with $k > 1$.
Since $v_1 > u_1$, and $v_1 > u_k$ for all $k \in \mathbb{Z}^+$ with $k > 1$, then $v_1 > u_n$ for all $n \in \mathbb{Z}^+$.
Since v_1 is a real number, and $v_1 > u_n$ for all $n \in \mathbb{Z}^+$, then the sequence (u_n) is bounded above by v_1.

Since the sequence (u_n) is strictly increasing and bounded above, then by the monotone convergence theorem, the sequence (u_n) converges.
Let A be the limit of (u_n).
Then $A = \lim_{n\to\infty} u_n = \sup(u_n)$ is a real number.

We prove $u_n > 1$ for all $n \in \mathbb{Z}^+$.

Since P_0 is prime, then $P_0 \geq 2$.

Since $P_1 > P_0$ and $P_0 \geq 2$, then $P_1 > 2$.

Since P_1 is prime and $P_1 > 2$, then $P_1 \geq 3$.

Since $P_n < P_{n+1}$ for all $n \in \mathbb{Z}^+$, then the sequence (P_n) is strictly increasing, so $P_1 < P_n$ for all positive integers $n > 1$.

Since $P_n > P_1$ and $P_1 \geq 3$, then $P_n > 3$ for all positive integers $n > 1$.

Since $P_1 \geq 3$ and $P_n > 3$ for all positive integers $n > 1$, then $P_n \geq 3$ for all $n \in \mathbb{Z}^+$.

Let $n \in \mathbb{Z}^+$.

Since $3^{-n} > 0$ for all $n \in \mathbb{Z}^+$ and $n \in \mathbb{Z}^+$, then $3^{-n} > 0$.

Since $3 > 1$ and $3^{-n} > 0$, then $3^{3^{-n}} > 1$.

Since $P_n \geq 3$ for all $n \in \mathbb{Z}^+$ and $n \in \mathbb{Z}^+$, then $P_n \geq 3$.

Since $P_n \geq 3 > 0$ and $3^{-n} > 0$, then $P_n^{3^{-n}} \geq 3^{3^{-n}}$.

Since $P_n^{3^{-n}} \geq 3^{3^{-n}}$ and $3^{3^{-n}} > 1$, then $P_n^{3^{-n}} > 1$, so $u_n > 1$.

Therefore, $u_n > 1$ for all $n \in \mathbb{Z}^+$.

We prove A is a lower bound of (v_n).

Suppose A is not a lower bound of (v_n).

Then there is $K \in \mathbb{Z}^+$ such that $v_K < A$.

Let $\epsilon = A - v_K > 0$.

Since $K \in \mathbb{Z}^+$, then $K + 1 \in \mathbb{Z}^+$ and $K + 1 > K$.

Since the sequence (v_n) is strictly decreasing, then $v_K > v_{K+1}$.

Since $v_n > u_n$ for all $n \in \mathbb{Z}^+$ and $K + 1 \in \mathbb{Z}^+$, then $v_{K+1} > u_{K+1}$.

Since $v_K > v_{K+1}$ and $v_{K+1} > u_{K+1}$, then $v_K > u_{K+1}$, so $v_K - u_{K+1} > 0$.

Since $u_n > 1$ for all $n \in \mathbb{Z}^+$ and $K + 1 \in \mathbb{Z}^+$, then $u_{K+1} > 1$.

Thus, $0 < 1 < u_{K+1} < v_K < A$.

Observe that

$$\begin{aligned}
|u_{K+1} - A| &= |u_{K+1} - v_K| + |v_k - A| \\
&= |v_K - u_{K+1}| + |A - v_k| \\
&= (v_K - u_{K+1}) + |\epsilon| \\
&= (v_K - u_{K+1}) + \epsilon.
\end{aligned}$$

Hence, $|u_{K+1} - A| = (v_K - u_{K+1}) + \epsilon$, so $|u_{K+1} - A| - \epsilon = v_K - u_{K+1} > 0$.

Thus, $|u_{K+1} - A| - \epsilon > 0$, so $|u_{K+1} - A| > \epsilon$.

Consequently, u_{K+1} is not in the ϵ neighborhood of A.

Therefore, there exists $\epsilon > 0$ such that for every $K \in \mathbb{Z}^+$ there is $K + 1 \in \mathbb{Z}^+$ and $K + 1 > K$ such that u_{K+1} is not in the ϵ neighborhood of A, so $A \neq \lim_{n\to\infty} u_n$.

But, this contradicts $A = \lim_{n\to\infty} u_n$.

Therefore, A is a lower bound of (v_n).

We prove $u_n < A$ for all $n \in \mathbb{Z}^+$.

Suppose for the sake of contradiction there is some $m \in \mathbb{Z}^+$ such that $u_m \geq A$.

Since $A = \sup(u_n)$, then A is the least upper bound of (u_n), so A is an upper bound of (u_n).

Hence, $u_m \leq A$.

Since $u_m \geq A$ and $u_m \leq A$, then $u_m = A$.

Since the sequence (u_n) is strictly increasing, then $u_m < u_{m+1}$.

Hence, $A < u_{m+1}$, so $u_{m+1} > A$.

But, this contradicts that A is an upper bound of (u_n).

Hence, there is no $m \in \mathbb{Z}^+$ such that $u_m \geq A$, so $u_m < A$ for all $m \in \mathbb{Z}^+$.

Therefore, $u_n < A$ for all $n \in \mathbb{Z}^+$.

We prove $v_n > A$ for all $n \in \mathbb{Z}^+$.

Suppose for the sake of contradiction there is some $t \in \mathbb{Z}^+$ such that $v_t \leq A$.

Since A is a lower bound of (v_n), then $A \leq v_t$.

Since $v_t \leq A$ and $v_t \geq A$, then $v_t = A$.

Since the sequence (v_n) is strictly decreasing, then $v_t > v_{t+1}$.

Hence, $A > v_{t+1}$, so $v_{t+1} < A$.

But, this contradicts that A is a lower bound of (v_n).

Hence, there is no $t \in \mathbb{Z}^+$ such that $v_t \leq A$, so $v_t > A$ for all $t \in \mathbb{Z}^+$.

Therefore, $v_n > A$ for all $n \in \mathbb{Z}^+$.

Since $u_n < A$ for all $n \in \mathbb{Z}^+$ and $A < v_n$ for all $n \in \mathbb{Z}^+$, then $u_n < A < v_n$ for all $n \in \mathbb{Z}^+$.

Let $n \in \mathbb{Z}^+$.

Then $u_n < A < v_n$.

Observe that

$$\begin{aligned} u_n < A < v_n &\Leftrightarrow P_n^{3^{-n}} < A < (P_n + 1)^{3^{-n}} \\ &\Rightarrow P_n < A^{3^n} < P_n + 1. \end{aligned}$$

Thus, $P_n < A^{3^n} < P_n + 1$, so $\lfloor A^{3^n} \rfloor = P_n$ is a prime number.

Hence, $\lfloor A^{3^n} \rfloor$ is a prime number for every $n \in \mathbb{Z}^+$.

Therefore, there is a prime-representing function $f : \mathbb{Z}^+ \to \mathbb{Z}^+$ defined by $f(n) = \lfloor A^{3^n} \rfloor$ for some real number A. $\square$

Let $f : \mathbb{Z}^+ \to \mathbb{Z}^+$ be the function defined by $f(n) = \lfloor A^{3^n} \rfloor$ for some real number A.

The real number A is the supremum of the strictly increasing convergent sequence (u_n) bounded above by $v_1 \in \mathbb{R}$, and $P_n < A^{3^n} < P_n + 1$.

Since P_n is prime, then $P_n \in \mathbb{Z}^+$, so $P_n + 1 \in \mathbb{Z}^+$.

There is no integer between consecutive integers.

Since P_n and $P_n + 1$ are consecutive integers and $P_n < A^{3^n} < P_n + 1$, then A^{3^n} is not an integer, so $A^{3^n} \notin \mathbb{Z}$.

Since $n \in \mathbb{Z}^+$, then $3^n \in \mathbb{Z}^+$.

Since $3^n \in \mathbb{Z}^+$ and $A^{3^n} \notin \mathbb{Z}$, then $A \notin \mathbb{Z}$, so A is not an integer.

Let $n \in \mathbb{Z}^+$.

Since $u_n > 1$ for all $n \in \mathbb{Z}^+$, then $u_n > 1$.

Since $u_n < A$ for all $n \in \mathbb{Z}^+$, then $u_n < A$.

Since $A > u_n$ and $u_n > 1$, then $A > 1$, so $A > 0$.

Hence, A is a positive real number that is not an integer and $A > 1$.

Therefore, the Mills' constant A is a positive real number that is not an integer and $A > 1$.

In fact, Mills' constant was proved irrational in 2025 [3].

Mills' constant is defined to be the smallest positive real number A such that $\lfloor A^{3^n} \rfloor$ generates prime numbers.

Numerical calculation of Mills primes

We use Sage to compute several Mills primes generated by Mills function $f : \mathbb{Z}^+ \to \mathbb{Z}^+$ defined by $f(n) = \lfloor A^{3^n} \rfloor$ for Mills' constant A.

If we assume the Riemann hypothesis is true, then $A \approx 1.306377883....$

We calculate several Mills primes based on this assumption.

$f(1) = \lfloor A^{3^1} \rfloor = 2$

$f(2) = \lfloor A^{3^2} \rfloor = 11$

$f(3) = \lfloor A^{3^3} \rfloor = 1361$

$f(4) = \lfloor A^{3^4} \rfloor = 2521008887$

References

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