Transformational Geometry Exercises

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Geometric Transformations

Exercise 1. If $T = (4, 3)$ and $C = \{(x, y) : x^2 - 2x + y^2 - 4y - 5 = 0\}$, write an equation for line s such that $C \cap s = T$.

Solution. We must write an equation for line s. We can complete the square for the equation $x^2 - 2x + y^2 - 4y - 5 = 0$ to get $x^2 - 2x + 1 + y^2 - 4y + 4 = 5 + 1 + 4$. This implies $(x - 1)^2 + (y - 2)^2 = 10$ which is the standard form of the equation of a circle. Hence, C is a circle with center $(1, 2)$ and radius $\sqrt{10}$, so $C = \{(x, y) : (x - 1)^2 + (y - 2)^2 = 10\}$. We note that $T \in C$ since $(4 - 1)^2 + (3 - 2)^2 = 9 + 1 = 10$. The intersection of a circle and a line tangent to a point on the circle is a line that is perpendicular to the line segment joining the center and the point.

Let l be the line joining the center $(1, 2)$ and point $T = (4, 3)$. Then s is the tangent line perpendicular to l that contains the point $T = (4, 3)$. Two non-vertical lines are perpendicular if and only if the slopes of the lines are negative reciprocals. The slope of l is $m_l = \frac{\Delta y}{\Delta x} = \frac{3-2}{4-1} = 1/3$. Thus the slope of line s is $m_s = -3$. Since we know the slope of line s and a point $T = (4, 3)$ on s, we have $m_s = \frac{\Delta y}{\Delta x} = \frac{y-3}{x-4} = -3$. Hence, $3x + y - 15 = 0$. Therefore, $s = \{(x, y) : 3x + y - 15 = 0\}$. $\square$

Exercise 2. Let G be the function which assigns to each non-horizontal line the point where the line intersects the x-axis.

If $t = \{(x, y) : 3x + 2y = 12\}$, find $G(t)$. What is the range of G ? Is G one to one?

If $(2, 7) \in s$ and s is a pre-image of $(-3, 0)$, write an equation for s.

Solution. Let us precisely characterize function G. Let A be the set of all non-horizontal lines in the xy Cartesian plane. Let l be any line in A. Then function G maps l to its x intercept. Thus, G maps A onto the entire x axis. Hence, the range of G is the x axis, so G is an onto function. In other words, the codomain of G is the x axis. Let $B = \{(x, y) : y = 0\}$. Then G is a function from A onto B.

Is G one to one? Well, there do exist distinct lines $l_1, l_2 \in A$ such that $G(l_1) = G(l_2)$. For example, the line $l_1 = \{(x, y) : y = x\}$ and $l_2 = \{(x, y) :$

$y = 2x\}$ map to the same point, namely the origin. Thus, this counter-example shows that G is not one to one.

To find the function value $G(t)$, we must find the x intercept of the line t. Hence, we must find point $P = (p_1, p_2) \in t$ such that $P \in B$. Thus, $\{P\} = t \cap B$. This means find P such that $3p_1 + 2p_2 = 12$ and $p_2 = 0$. This implies $p_1 = 4$, so $P = (4, 0)$. Hence, $G(t) = (4, 0)$.

Let $s \in A$. We are given $(2, 7) \in s$ and $G(s) = (-3, 0)$, so we need to find the equation for line s. Since $(-3, 0)$ is the x intercept of s, then $(-3, 0) \in s$. So, we know 2 points on line s. Therefore, the slope of $s = m_s = \frac{\Delta y}{\Delta x} = \frac{0-7}{-3-2} = 7/5$. Hence, $\frac{y-0}{x-(-3)} = 7/5$, so $-7x+5y = 21$. Thus, $s = \{(x, y) : -7x+5y = 21\}$. $\square$

Exercise 3. Let f be a mapping from the plane to the plane defined for any point $P(x, y)$ by $f(P) = (|x|, |y|)$.

If f a function? If $A = (-3, 6)$, what is $f(A)$? What are the pre-images of $B(4, 2)$? What is the range of f? Is f one to one? Is f a geometric transformation?

Solution. We know $f : R^2 \mapsto R^2$ is a relation or mapping from the plane to the plane. So, more specifically, is f a function?

To decide if f is a function we need to answer 2 questions.
1) Does every point in the plane have some image?
To answer this we must prove every point P in the plane has an image.
In logic symbols: Prove: $\forall P \in \text{plane}, \exists f(P)$.
2) If points A and B in the plane are identical, is $f(A) = f(B)$?
To answer this we must prove if A and B are identical points, then $f(A) = f(B)$.
In logic symbols: Prove: $A = B \Rightarrow f(A) = f(B)$.

Let $P(x, y)$ be any point in the xy plane. Then f specifies a well defined rule for obtaining the image of P. Hence, the domain of f is the entire plane, R^2. Therefore, every point in the domain of f has at least one image.

Let $A(a_1, a_2)$ and $B(b_1, b_2)$ be points in the xy plane such that $A = B$. Then $a_1 = b_1$ and $a_2 = b_2$. So, $f(A) = (|a_1|, |a_2|) = (|b_1|, |b_2|) = f(B)$. Hence, $A = B$ implies $f(A) = f(B)$.

We have shown that $A = B$ implies $f(A) = f(B)$. This means two identical points map to the same point(i.e., they don't map to two different points). Since we have shown that each point in the domain of f has at least one image, then this implies f maps each point in the domain to at most one image. Hence, each point in the domain is mapped to exactly one image. Therefore, f is a function.

Let $P(a, b)$ be any point in the domain of f. Then $f(P) = f(a, b) = (|a|, |b|)$, so $a \geq 0$ and $b \geq 0$. Hence, the range of f is any point in quadrant I or on the positive x or y axis.

We find $f(A) = f(-3, 6) = (|-3|, |6|) = (3, 6)$.

We must find the pre-images of point $B = (4, 2)$. Thus, we must find all points $X \in \text{domain} f$ such that $f(X) = (4, 2)$. Let $X = (x_1, x_2) \in R^2$. Then $f(X) = (|x_1|, |x_2|) = (4, 2)$, so $|x_1| = 4$ and $|x_2| = 2$. Hence, $x_1 = \pm 4$ and $x_2 = \pm 2$. Thus, there are 4 pre-images of B. The set of pre-images of $B(4, 2)$ is $\{(4, 2), (4, -2), (-4, 2), (-4, -2)\}$.

Is f a one to one function?
To answer this we must prove f satisfies the conditions for a one to one function.

We have multiple approaches to consider:
a) We could use direct proof and prove: if $A \neq B$, then $f(A) \neq f(B)$.
b) We could use proof by contrapositive and prove: if $f(A) = f(B)$, then $A = B$.
c) We could use proof by contradiction and prove: $A \neq B \wedge f(A) = f(B)$ leads to some contradiction. Or, we could disprove f is a one to one function by providing a counter example.

Since we just showed that $f(4, -2) = f(4, 2)$, then f is not a one to one function. Since f is not one to one, then f is not a geometric transformation. $\square$

Exercise 4. Let T be a function from the plane to the plane defined for any point $P(x, y)$ by $T(P) = (x + 2, 2y - 3)$.

If T a function? If $A = (1, -6)$, what is $T(A)$? What are the pre-images of $B(-2, 4)$? What is the range of T? Is T one to one? Is T a geometric transformation?

Solution. We know $T : R^2 \mapsto R^2$ is a relation or mapping from the plane to the plane. So, more specifically, is T truly a function?

To decide if T is a function we need to answer 2 questions.
1) Does every point in the plane have some image?
To answer this we must prove every point P in the plane has an image.
In logic symbols: Prove: $\forall P \in \text{plane}, \exists T(P)$.
2) If points A and B in the plane are identical, is $T(A) = T(B)$?
To answer this we must prove if A and B are identical points, then $T(A) = T(B)$.
In logic symbols: Prove: $A = B \Rightarrow T(A) = T(B)$.

To decide if T is a transformation(bijection) we need to answer 2 questions.
1) Is T an onto function?
To answer this we must prove every point P in the plane(codomain of T) is the image of some point in the domain of T.
In logic symbols: Prove: $\forall P \in \text{plane}, \exists Q \in \text{plane} \ni T(Q) = P$.
2) Is T a one to one function?
To answer this we must prove T satisfies the conditions for a one to one function.

We have multiple approaches to consider:
a) We could use direct proof and prove: if $A \neq B$, then $T(A) \neq T(B)$.
b) We could use proof by contrapositive and prove: if $T(A) = T(B)$, then $A = B$.
c) We could use proof by contradiction and prove: $A \neq B \wedge T(A) = T(B)$ leads to some contradiction.

To prove T is onto, we suppose $P(x, y)$ is any point in the codomain of T and find some pre-image $Q(a, b)$ such that $T(Q) = P$. This means find Q such that $T(a, b) = (x, y)$ which implies $(a + 2, 2b - 3) = (x, y)$. This means $a + 2 = x$ and $2b - 3 = y$, so $a = x - 2$ and $b = (y + 3)/2$. Thus, we can let $Q = (x - 2, (y + 3)/2)$.

Since we prove T is actually a transformation, then we know we can find $T(A)$. Thus, $T(A) = T(1, -6) = (1 + 2, 2 * -6 - 3) = (3, -15)$.

We find the pre-images of $B = (-2, 4)$. Thus, we must find all $X \in$ domain of T such that $T(X) = (-2, 4)$. Let $X = (x_1, x_2) \in R^2$. Then $T(X) = T(x_1, x_2) =$

$(x_1+2, 2x_2-3) = (-2, 4)$. This implies $x_1 + 2 = -2$ and $2x_2 - 3 = 4$, so $x_1 = -4$ and $x_2 = 7/2$. Thus, the set of pre-images of B is $\{(-4, 7/2)\}$. This is consistent with the fact that T is one to one and onto(ie, bijection). There is exactly one pre-image of B.

Since T is onto function, then we know the range of T is simply the entire plane, R^2. □

Proof. We prove $T : R^2 \mapsto R^2$ is a function.

Let $P(x, y)$ be any point in the xy plane. Then T specifies a well defined rule for obtaining the image of P. Hence, the domain of T is the entire plane, R^2. Therefore, every point in the domain of T has at least one image.

Suppose points $A(a_1, a_2)$ and $B(b_1, b_2)$ are in the domain of T such that $A = B$. Then $a_1 = b_1$ and $a_2 = b_2$. Thus, $T(A) = T(a_1, a_2) = (a_1+2, 2a_2-3) = (b_1 + 2, 2b_2 - 3) = T(B)$. Hence, $A = B$ implies $T(A) = T(B)$.

We have shown that $A = B$ implies $T(A) = T(B)$. This means two identical points map to the same point(i.e., they don't map to two different points). Since we have shown that each point in the domain of T has at least one image, then this implies T maps each point in the domain to at most one image. Hence, each point in the domain is mapped to exactly one image. Therefore, T is a function. □

Proof. We prove $T : R^2 \mapsto R^2$ is a geometric transformation.

We prove T is onto. Let $P(x, y)$ be any point in the plane(codomain of T). Let $Q = (x - 2, (y + 3)/2)$ be in the domain of T. Then $T(Q) = ((x - 2) + 2, 2 * (y + 3)/2 - 3) = (x, y) = P$. Hence, T is an onto function.

We prove T is one to one. We use proof by contrapositive. Suppose points $A(a_1, a_2)$ and $B(b_1, b_2)$ exist in the domain of T such that $T(A) = T(B)$. Then $T(A) = (a_1 + 2, 2a_2 - 3)$ and $T(B) = (b_1 + 2, 2b_2 - 3)$. Thus, $a_1 + 2 = b_1 + 2$ and $2a_2 - 3 = 2b_2 - 3$. Hence, $a_1 = b_1$ and $a_2 = b_2$, so $A = B$. Therefore, $T(A) = T(B)$ implies $A = B$, so T is one to one.

Since T is both one to one and onto function, then T is a geometric transformation of the plane onto the plane. □

Exercise 5. Given $s = \{(x, y) : x = -3\}$, determine $M_s(P)$ if $P(x, y)$ is any point.

Solution. We must determine a formula for the line reflection M_s. We know M_s is a geometric transformation from the plane onto the plane.

Let $P(x, y)$ be any point in the domain of M_s. Either P is on s or not.

If $P \in s$, then $x = -3$, so $M_s(P) = M_s(x, y) = (x, y) = (-3, y)$.

If $P \notin s$, then s is the perpendicular bisector of $\overline{PP'}$ where $P' = M_s(P)$. Let $M = (m_1, m_2) = s \cap \overline{PP'}$ be the midpoint of $\overline{PP'}$. Let $P' = (p_1, p_2)$. We must find the coordinates of P'.

Since $M \in s \cap \overline{PP'}$, then $M \in s$ and $M \in \overline{PP'}$. Thus, $m_1 = -3$ and $m_2 = y$ (since $\overline{PP'}$ is horizontal), so $M = (-3, y)$. We use the midpoint formula to obtain: $-3 = \frac{p_1+x}{2}$ and $y = \frac{p_2+y}{2}$. Thus, $p_1 = -x - 6$ and $p_2 = y$. Hence, $M_s(P) = M_s(x, y) = P' = (-x - 6, y)$. □

Exercise 6. Let $s = \{(x, y) : y = x\}$. Let T be the transformation of the plane defined for any point $P(x, y)$ by $T(P) = P' = (y, x)$. Is T a line reflection in s?

Solution. To decide if transformation T is a line reflection in s we need to answer 2 questions.
1) If $P \in s$, is $T(P) = P$? To prove this we must show that $P \in s$ implies $T(P) = P$.
2) If $P \notin s$, is s the perpendicular bisector of line segment $\overline{PP'}$? To prove this we must show that $P \notin s$ implies 2 things:
a) $s \perp \overline{PP'}$ (s is perpendicular to the line segment $\overline{PP'}$) and
b) if $M = s \cap \overline{PP'}$, then M is the midpoint of $\overline{PP'}$. $\square$

Proof. We prove T is a line reflection in s.

Let $P = (x, y)$ be any point in the domain of T. Then $P' = T(P) = T(x, y) = (y, x)$.

Either P is on s or not. We consider these cases separately. There are two cases to consider.

Case 1: Suppose $P \in s$.
Then $y = x$, so $T(P) = (y, x) = (x, x) = (x, y) = P$. Hence, $T(P) = P$. Therefore, if $P \in s$, then $T(P) = P$.

Case 2: Suppose $P \notin s$.
Then the slope of $\overline{PP'}$ is $\frac{x-y}{y-x} = -1$. The slope of s is 1,so the slope of $\overline{PP'}$ is the negative reciprocal of the slope of s. Therefore, $s \perp \overline{PP'}$.

Let $M = (m, n) = s \cap \overline{PP'}$. Then $M \in s$, so $n = m$. Therefore, $M = (m, m)$. We compute the distance MP. Observe that $MP = \sqrt{(m-x)^2 + (m-y)^2} = \sqrt{(m-y)^2 + (m-x)^2} = MP'$. Since M is between P and P' and $MP = MP'$, then M is the midpoint of line segment $\overline{PP'}$.

Since $s \perp \overline{PP'}$ and $s \cap \overline{PP'}$ is the midpoint of $\overline{PP'}$, then s is the perpendicular bisector of $\overline{PP'}$.

Thus, both cases show that T is a line reflection in s. $\square$

Exercise 7. Prove or disprove that the transformation T defined for all points $P(x, y)$ by $T(P) = (2x, y - 1)$ is an isometry.

Solution. Let T be the transformation from the plane onto the plane defined by $T(P) = (2x, y - 1)$ where $P(x, y)$ is any point in the domain of T. Is T an isometry?

To decide if T is an isometry, we need to answer 1 question.
1) Does $A'B' = AB$? That is, if A and B are any points in the domain of T, does $A'B' = AB$ where $A' = T(A)$ and $B' = T(B)$?

After playing with some examples, we realize that T is not an isometry. Therefore, we devise a counter example. $\square$

Proof. The transformation T defined for all points $P(x, y)$ by $T(P) = (2x, y-1)$ is not an isometry. Let A and B be points in the domain of T such that $A = (1, 2)$ and $B = (2, 5)$. Then $A' = T(A) = T(1, 2) = (2, 1)$ and $B' = T(B) = T(2, 5) = (4, 4)$. We compute the distance $A'B' = \sqrt{(2-4)^2 + (1-4)^2} = \sqrt{13}$

and $AB = \sqrt{(1-2)^2 + (2-5)^2} = \sqrt{10}$. Hence $A'B' \neq AB$, so T is not an isometry. $\square$

Exercise 8. Let $u = \{(x, y) : y = 3x\}$. If $A = (4, 3)$, find the coordinates of $A' = M_u(A)$.

Solution. Let M_u be a line reflection in line u. We must find the coordinates of point $A' = M_u(A)$ where $A = (4, 3)$. Is $A \in u$? We compute $3 * 4 = 12 \neq 3$. Hence, $A \notin u$.

By definition of M_u, if $A \notin u$, then $M_u(A) = A'$ such that u is the perpendicular bisector of line segment $\overline{AA'}$. Thus, u is the perpendicular bisector of $\overline{AA'}$ where $M_u(A) = A'$. Let line $l = \overleftrightarrow{AA'}$. Then, $u \perp l$, so $m_u * m_l = -1$. Since $m_u = 3$, then $m_l = \frac{-1}{3}$. We compute the equation of a line l with slope $m_l = -1/3$ through the point $A = (4, 3) \in l$ to be $\frac{y-3}{x-4} = -1/3$, or equivalently, $x + 3y = 13$. Thus, $l = \{(x, y) : x + 3y = 13\}$.

Let $M = u \cap l$ be the midpoint of $\overline{AA'}$. Then $M = \{(m, n) : n = 3m \wedge m + 3n = 13\}$. Thus, we have the system of linear equations:

$$\begin{aligned} 3m - n &= 0 \\ m + 3n &= 13 \end{aligned}$$

Using linear algebra we find the solution to be $(m, n) = (1.3, 3.9)$. Thus $M = (1.3, 3.9)$.

Let $A' = (a_1, a_2)$. We must find a_1 and a_2. Since M is the midpoint of $\overline{AA'}$, we compute using the midpoint formula: $1.3 = \frac{4+a_1}{2}$ and $3.9 = \frac{3+a_2}{2}$. Thus, $a_1 = -1.4$ and $a_2 = 4.8$. Hence, $A' = (-1.4, 4.8)$. $\square$

Exercise 9. Define mapping T : plane $\mapsto$ plane as follows: $T(P) = P' = (x, -y)$ if $P = (x, y)$. Then T is a transformation and T is an isometry.

Solution. What facts or relationships can we deduce about T? Certainly, we know that T is a relation(mapping) from the plane to the plane. Is T a function? If T is a function, is T a transformation? If T is a transformation, is T an isometry?

To decide if M_s is a function we need to answer 2 questions.
1) Does every point in the plane have some image?
To answer this we must prove every point P in the plane has an image.
In logic symbols: Prove: $\forall P \in$ plane, $\exists M_s(P)$.
2) If points A and B in the plane are identical, is $M_s(A) = M_s(B)$?
To answer this we must prove if A and B are identical points, then $M_s(A) = M_s(B)$.
In logic symbols: Prove: $A = B \Rightarrow M_s(A) = M_s(B)$.

To decide if M_s is a transformation(bijection) we need to answer 2 questions.
1) Is M_s an onto function?
To answer this we must prove every point P in the plane(codomain of M_s) is the image of some point in the domain of M_s.
In logic symbols: Prove: $\forall P \in$ plane, $\exists Q \in$ plane $\ni M_s(Q) = P$.

2) Is M_s a one to one function?
To answer this we must prove M_s satisfies the conditions for a one to one function.

We have multiple approaches to consider:
a) We could use direct proof and prove: if $A \neq B$, then $M_s(A) \neq M_s(B)$.
b) We could use proof by contrapositive and prove: if $M_s(A) = M_s(B)$, then $A = B$.
c) We could use proof by contradiction and prove: $A \neq B \wedge M_s(A) = M_s(B)$ leads to some contradiction.

To decide if T is an isometry, we need to answer 1 question.
1) Does $P'Q' = PQ$? That is, if P and Q are any points in the domain of T, does $P'Q' = PQ$ where $P' = T(P)$ and $Q' = T(Q)$?

We must prove T is a function from the plane to the plane. Then we must prove T is a transformation(bijection). Lastly, we must prove T is an isometry.

Thus our strategy is to prove the following:
1) prove T is a function of the plane
2) prove T is onto
3) prove T is 1-1
4) prove T is an isometry

We observe that T is a line reflection in which the x-axis is the axis of reflection. □

Proof. Let T : plane $\mapsto$ plane be a mapping(relation).

We prove T is a function.

Let P be any point in the plane. The definition of T specifies a well defined rule for obtaining the image of P, so $T(P)$ exists. Hence, the domain of T is the entire plane. Thus, every point in the domain of T has at least one image.

Let $A = (a_1, a_2)$ and $B = (b_1, b_2)$ be points in the domain of T such that $A = B$. Then $a_1 = b_1$ and $a_2 = b_2$. We compute $T(A) = T(a_1, a_2) = (a_1, -a_2)$ and $T(B) = T(b_1, b_2) = (b_1, -b_2)$. Since $a_2 = b_2$, then $-a_2 = -b_2$. Thus, $a_1 = b_1$ and $-a_2 = -b_2$, so it follows that $T(A) = T(B)$.

We have shown that $A = B$ implies $T(A) = T(B)$. This means two identical points map to the same point(i.e., they don't map to two different points). Since we have shown that each point in the domain of T has at least one image, then this implies T maps each point in the domain to at most one image. Hence, each point in the domain is mapped to exactly one image. Therefore, T is a function. □

Proof. We prove T is a transformation of the plane.

We prove T is an onto function. Let $P = (x, y)$ be any point in the plane and point $Q = (x, -y)$. Then $T(Q) = T(x, -y) = (x, -(-y)) = (x, y) = P$. Hence, P is the image of some point Q in the domain of T. Thus, T is onto.

We prove T is a one to one function. We use proof by contrapositive. Suppose $T(A) = T(B)$ where point $A = (a, b)$ and point $B = (c, d)$. Then $T(a, b) = T(c, d)$ so $(a, -b) = (c, -d)$. Thus, $a = c$ and $-b = -d$. Hence, $b = d$. Since

$a = c$ and $b = d$, then point $A = B$. Hence, $T(A) = T(B)$ implies $A = B$, so it follows that T is one to one.

Since T is a function that is one to one and onto,then it follows that T is a transformation of the plane onto the plane. $\square$

Proof. We prove T is an isometry.

Let $A = (a_1, a_2)$ and $B = (b_1, b_2)$ be any pair of points in the domain of T. Then $A' = T(A) = (a_1, -a_2)$ and $B' = T(B) = (b_1, -b_2)$. We must prove $A'B' = AB$. We compute the distance $A'B'$. Observe the following sequence of equalities:

$$\begin{aligned} A'B' &= \sqrt{(a_1 - b_1)^2 + (-a_2 - (-b_2))^2} \\ &= \sqrt{(a_1 - b_1)^2 + (b_2 - a_2)^2} \\ &= \sqrt{(a_1 - b_1)^2 + (a_2 - b_2)^2} \\ &= AB \end{aligned}$$

Thus, T is an isometry. $\square$

Exercise 10. Define mapping T : plane $\mapsto$ plane as follows: $T(P) = P$ if $P \in t$ and $T(P) = P'$ if $P \notin t$ such that P' is the midpoint of the perpendicular line segment from P to t.

What facts or relationships can we deduce about T? Certainly, we know that T is a relation(mapping) from the plane to the plane. Is T a function? If T is a function, is T a transformation? If T is a transformation, is T an isometry?

Solution. To decide if M_s is a function we need to answer 2 questions.
1) Does every point in the plane have some image?
To answer this we must prove every point P in the plane has an image.
In logic symbols: Prove: $\forall P \in \text{plane}, \exists M_s(P)$.
2) If points A and B in the plane are identical, is $M_s(A) = M_s(B)$?
To answer this we must prove if A and B are identical points, then $M_s(A) = M_s(B)$.
In logic symbols: Prove: $A = B \Rightarrow M_s(A) = M_s(B)$.

To decide if M_s is a transformation(bijection) we need to answer 2 questions.
1) Is M_s an onto function?
To answer this we must prove every point P in the plane(codomain of M_s) is the image of some point in the domain of M_s.
In logic symbols: Prove: $\forall P \in \text{plane}, \exists Q \in \text{plane} \ni M_s(Q) = P$.
2) Is M_s a one to one function?
To answer this we must prove M_s satisfies the conditions for a one to one function.

We have multiple approaches to consider:
a) We could use direct proof and prove: if $A \neq B$, then $M_s(A) \neq M_s(B)$.
b) We could use proof by contrapositive and prove: if $M_s(A) = M_s(B)$, then $A = B$.

c) We could use proof by contradiction and prove: $A \neq B \wedge M_s(A) = M_s(B)$ leads to some contradiction. To decide if T is an isometry, we need to answer 1 question.
1) Does $P'Q' = PQ$? That is, if P and Q are any points in the domain of T, does $P'Q' = PQ$ where $P' = T(P)$ and $Q' = T(Q)$?

Observations: T is a function and is a transformation,but is not an isometry. If $A'B' = AB$, then $\overline{AB} \parallel t$. □

Proof. We prove T is a function.

Let P be any point in the plane. Then P is either on t or not on t. In either case, the definition of T specifies a well defined rule for obtaining the image of P. Hence, the domain of T is the entire plane, so it follows that every point in the plane has some image.

Let A and B be points in the domain of T such that $A = B$. Point A is either on t or not on t.

If $A \in t$, then $T(A) = A$. Since $B = A$, then $B \in t$, so $T(B) = B$. Hence, $T(A) = A = B = T(B)$, so it follows that $T(A) = T(B)$.

If $A \notin t$, then $T(A) = A'$ such that A' is the midpoint of the perpendicular segment from A to t. Since $B = A$, then $B \notin t$. Therefore, $T(B) = B'$ such that B' is the midpoint of the perpendicular segment from B to t. But $B = A$, so the segment from A to t is the same segment from B to t. Since the midpoint of a line segment is unique, then this implies the midpoint $A' = B'$. Hence, $T(A) = T(B)$.

In either case, we have shown that $A = B$ implies $T(A) = T(B)$. This means two identical points in the domain map to the same point(i.e., they don't map to two different points). Since we have shown that each point in the domain has at least one image, then this implies T maps each point in the domain to at most one image. Hence, each point in the domain is mapped to exactly one image. Therefore, we conclude T is a function. □

Proof. We prove T is a transformation.

We prove T is onto. Let P be any point in the plane. If $P \in t$, then by definition of T, $T(P) = P$. If $P \notin t$, then there exists a unique point Q such that P is the midpoint of the perpendicular line segment from Q to t . Thus, $T(Q) = P$. In either case, P is the image of at least one point in the domain of T. Hence, T is onto.

We prove T is one to one. We use proof by contradiction. Suppose T is not one to one. Then there exist points A and B in the domain of T such that $A \neq B$ and $T(A) = T(B) = K$. If $K \notin t$, then K is the midpoint of the perpendicular line segment from A to t and K is the midpoint of the perpendicular line segment from B to t. Since the midpoint of a line segment is unique, then the line segment from A to t must be the same as the line segment from B to t. Hence, $A = B$. But, this contradicts the assumption that $A \neq B$. But, if $K \in t$, then $A \in t$ and $B \in t$. Therefore, $T(A) = A = K = T(B) = B$. Hence, $A = B$. But this contradicts the assumption that A and B are distinct points (i.e. $A \neq B$). The contradiction implies that no distinct points are mapped to the same point. Hence, T is one to one.

Since T is a function that is both one to one and onto, we conclude that T is a transformation of the plane onto the plane. $\square$

Proof. We disprove that T is an isometry.

We can orient line t in the xy coordinate system so that t is the y axis. We demonstrate a counterexample to prove T is not an isometry.

Let $A = (12, 8)$ and $B = (14, 5)$. Then $T(A) = A' = (6, 8)$ and $T(B) = B' = (7, 5)$.

We compute the distance AB and $A'B'$. Observe that $AB = \sqrt{(12-14)^2 + (8-5)^2} = \sqrt{13}$. and $A'B' = \sqrt{(6-7)^2 + (8-5)^2} = \sqrt{10}$. Hence, $A'B' \neq AB$.

Thus, T is not an isometry. $\square$

Exercise 11. If $s = \{(x, y) : y = -x\}$ and $t = \{(x, y) : y = 2x - 3\}$, write an equation for $t' = M_s(t)$.

Solution. We know the line reflection defined by $M_s : R^2 \mapsto R^2$ in which s is the axis of reflection is an isometry. Every isometry maps lines onto lines. Thus, M_s maps lines onto lines. Since t is a line, then $M_s(t) = t'$ is also a line. Therefore, we must find the equation of the line t'.

In our analysis, we need to determine a formula for M_s. In other words, if $P = (x, y)$ is any point in the domain of M_s, then we need to find a rule to compute $M_s(P)$. Let $P = (x, y)$ be any point in the domain of M_s. Then the image of P is $P' = (p_1, p_2) = M_s(P) = M_s(x, y)$. We must find p_1 and p_2. Since s is $\perp$ bisector of $\overline{PP'}$, then let $M = (m_1, m_2) = s \cap \overleftrightarrow{PP'}$ be the midpoint of $\overline{PP'}$.

We know $\overleftrightarrow{PP'} \perp s$, so $m_{\overleftrightarrow{PP'}} = \frac{-1}{m_s}$. Since $m_s = -1$, then $m_{\overleftrightarrow{PP'}} = 1$. Since $M = (m_1, m_2) \in s$, then $m_2 = -m_1$. Thus, $M = (m_1, -m_1)$. We need to find m_1. Since we know the slope of line $\overleftrightarrow{PP'}$ and a point $M \in \overleftrightarrow{PP'}$, we can use the formula for a slope to compute: $1 = \frac{y-(-m_1)}{x-m_1}$. Thus, $m_1 = \frac{x-y}{2}$, and so $M = (\frac{x-y}{2}, \frac{y-x}{2})$.

Since M is the midpoint of $\overleftrightarrow{PP'}$, we can use the midpoint formula to obtain: $\frac{x-y}{2} = \frac{p_1+x}{2}$ and $\frac{y-x}{2} = \frac{p_2+y}{2}$. Thus, $p_1 = -y$ and $p_2 = -x$. Hence, $P' = (-y, -x) = M_s(P) = M_s(x, y)$. Therefore, a formula for M_s is $M_s(P) = M_s(x, y) = (-y, -x)$.

To determine the equation for line t' we need two points on t'. Since M_s is an onto function, we know there exists points $A, B \in t$ such that $M_s(A) = A' \in t'$ and $M_s(B) = B' \in t'$. So, we may choose any arbitrary points of t. For convenience we choose the xy intercepts of t so let $A = (0, -3) \in t$ and $B = (3/2, 0) \in t$. We compute $M_s(A) = M_s(0, -3) = (3, 0) \in t'$ and $M_s(B) = M_s(3/2, 0) = (0, -3/2) \in t'$. We compute the slope of t' to be $m'_t = \frac{-3/2-0}{0-3} = \frac{1}{2}$. Substituting point $(0, -3/2)$ to determine the equation of the line t' we obtain $\frac{y-(-3/2)}{x-0} = \frac{1}{2}$. Thus, the equation of line t' is $y = \frac{x-3}{2}$. Therefore, $t' = \{(x, y) : y = \frac{x-3}{2}\}$. $\square$

Exercise 12. If $C = \{(x, y) : (x-2)^2 + (y-3)^2 = 4\}$ and T is an isometry which maps $(2, 3)$ onto $(1, -7)$, write an equation for $T(C)$.

Solution. Let $T : R^2 \mapsto R^2$ be an isometry from the plane onto the plane. Let $P = (2, 3)$ be the center of circle C. Then $T(P) = (1, -7)$. Let $T(C)$ be the set of points that is the image of circle C under transformation T. We ask how does T map each point on circle C to its corresponding image?

Let $Q \in C$ be any point on circle C. Since T is an isometry, we know T preserves distances, so $PQ = P'Q' = 2$ where $T(P) = P'$ and $T(Q) = Q'$. This means T maps any point Q on circle C to exactly one point Q' that is 2 units from P'. Thus, $T(C)$ is a circle whose center is $P' = (1, -7)$ and radius is 2. Hence, $T(C) = \{(x, y) : (x-1)^2 + (y+7)^2 = 4\}$. $\square$

Exercise 13. Given coplanar lines s, s', t, t', r such that $s' = M_r(s)$ and $t' = M_r(t)$. If $s' \parallel t'$, then $s \parallel t$.

Solution. Let M_r be a line reflection such that $s' = M_r(s)$ and $t' = M_r(t)$. We must prove $s' \parallel t'$ implies $s \parallel t$. We try direct proof. $\square$

Proof. Let M_r be a line reflection from the plane onto the plane such that $s' = M_r(s)$ and $t' = M_r(t)$.

We prove if $s' \parallel t'$, then $s \parallel t$. We use direct proof. Suppose $s' \parallel t'$. Then, by substitution, $M_r(s) \parallel M_r(t)$. Since M_r is an isometry, then M_r preserves parallelism between lines. Thus, $M_r(s) \parallel M_r(t)$ if and only if $s \parallel t$. Since $M_r(s) \parallel M_r(t)$, then it follows that $s \parallel t$. Hence, $s' \parallel t'$ implies $s \parallel t$. $\square$

Exercise 14. Given coplanar lines s, s', t such that $s' = M_t(s)$. If $s' \parallel s$, then $s \parallel t$.

Solution. We must prove if $s' \parallel s$, then $s \parallel t$. We could try proof by contradiction by assuming $s' \parallel s$ and $s \nparallel t$. $\square$

Proof. Let M_t be a line reflection from the plane onto the plane such that $s' = M_t(s)$ for lines s, s', t.

We prove if $s' \parallel s$, then $s \parallel t$. We use proof by contradiction. Suppose $s' \parallel s$ and $s \nparallel t$. Then s and t intersect in some point X. Thus, $\{X\} = s \cap t$, so $X \in s$ and $X \in t$. Since $X \in s$, then $M_t(X) \in M_t(s)$. Thus, $M_t(X) \in s'$. Since $X \in t$, then $M_t(X) = X$, by definition of M_t. Therefore, $X \in s'$. Since $X \in s$ and $X \in s'$, then it follows that $X \in s \cap s'$. Consequently, $s \nparallel s'$. But, this contradicts the assumption that $s' \parallel s$. Hence, it cannot be true that $s \nparallel t$. Thus, $s \parallel t$. $\square$

Exercise 15. Given coplanar lines s, s', t such that $s' = M_t(s)$. Prove or disprove: If $s' = s$, then $s = t$.

Solution. We consider how lines s and s' could be related under a line reflection M_t in t. Since M_t is an isometry, we know M_t preserves perpendicularity and parallelism of lines. Thus, $M_t(t) \parallel M_t(s)$ iff $t \parallel s$ and $M_t(t) \perp M_t(s)$ iff $t \perp s$. We know $M_t(t) = t$ since a line reflection reflects the axis of reflection onto itself.

If $t \parallel s$, then the only way $s' = s$ is if $t = s$. But, if $t \perp s$, then $M_t(t) \perp M_t(s)$, so $t \perp M_t(s)$ which implies $t \perp s'$. Thus, it is possible for $s = s'$ but $s \neq t$. This provides us the counter example. $\square$

Proof. The conjecture is false. Let M_t be a line reflection in line t such that $s' = M_t(s)$. Suppose $s = s'$. Let $t \perp s$. Then $t \neq s$. Thus, $s' = s$, but $t \neq s$. $\square$

Exercise 16. Given coplanar lines s, s', t such that $s' = M_t(s)$. Prove or disprove: If $s' \cap s = \{A\}$, then $A \in t$.

Solution. We must prove if $s' \cap s = \{A\}$, then $A \in t$. We could try proof by contradiction by assuming $s' \cap s = \{A\}$ and $A \notin t$. $\square$

Proof. We use proof by contradiction. Suppose $s' \cap s = \{A\}$ and $A \notin t$. Then $A \in s'$ and $A \in s$. Since $A \in s$, then $A' \in s'$. Since $A \notin t$, then by definition of M_t, t is the perpendicular bisector of $\overline{AA'}$. Thus, A' is distinct from A, so $A' \neq A$. Hence, the distinct points A and A' determine the unique line $s' = \overline{AA'}$. Since t is the perpendicular bisector of $\overline{AA'}$, then $t \perp s'$. Since M_t is an isometry, then M_t preserves perpendicularity of lines. Therefore, $M_t(s) \perp M_t(t)$ if and only if $s \perp t$. Since $s' = M_t(s)$ and $M_t(t) = t$, then this means $s' \perp t$ if and only if $s \perp t$. Thus, since $t \perp s'$, then it follows that $s \perp t$. Since $s \perp t$ and $s' \perp t$, then $s \parallel s'$. Consequently, $s \cap s' = \emptyset$. But, this contradicts the assumption that $s \cap s' = \{A\}$. Hence, it cannot be true that $A \notin t$. Thus, $A \in t$. $\square$