Transformational Geometry Notes

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Geometric Transformations

Definition 1. Transformation
A **geometric transformation** is a one to one function from a set of points **onto** a set of points.

Let A be a set of points.
Let B be a set of points.
Let $f : A \to B$ be a geometric transformation.
Then f is a bijective map.

Definition 2. Transformation of the plane
A **transformation of the plane** is a one to one function from the plane **onto** the plane.

Therefore, a transformation of the plane is a bijective map.
Let $\mathbb{R}^2$ be a plane.
Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be a function.
Then f is a **transformation of the plane** iff
1. f is injective (one to one).
2. f is surjective (onto).

Definition 3. Line Reflection
A **line reflection** in a given line s is a function f defined for every point P of the plane such that:
1) if $P \in s$, then $f(P) = P$
2) if $P \notin s$, then $f(P) = P'$ such that s is the $\perp$ bisector of segment $\overline{PP'}$.

Let s be a line in $\mathbb{R}^2$.
Let $f_s : \mathbb{R}^2 \to \mathbb{R}^2$ be a function.
Let P be an arbitrary point of $\mathbb{R}^2$.
Then f_s is a line reflection iff
1. if $P \in s$, then $f(P) = P$
2. if $P \notin s$, then $f(P) = P'$ such that s is the $\perp$ bisector of $\overline{PP'}$.
The line s is the **axis of reflection**.
f_s is an isometry.

Example 4. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be a line reflection with axis of reflection s.

Let P be an arbitrary point of $\mathbb{R}^2$.

Then there exist $x, y \in \mathbb{R}$ such that $P = (x, y)$.

If s is the x axis, then $f(P) = f(x, y) = (x, -y)$.

If s is the y axis, then $f(P) = f(x, y) = (-x, y)$.

If s is the line $y = x$, then $f(P) = f(x, y) = (y, x)$.

If s is the line $y = -x$, then $f(P) = f(x, y) = (-y, -x)$.

Theorem 5. *Every line reflection is a transformation of the plane.*

Therefore every line reflection is a bijective map.

Definition 6. Isometry

An isometry is a distance preserving transformation.

Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a transformation of the plane.

Then T is an **isometry** iff for every pair of points P and Q, $P'Q' = PQ$ where $P' = T(P)$ and $Q' = T(Q)$.

PQ represents the distance between points P and Q.

The images of any two points are the same distance as the original two points.

Theorem 7. *Every line reflection is an isometry.*

Theorem 8. *The image of any line under an isometry is a line.*

Therefore, an isometry maps lines onto lines.

f maps lines onto lines.

Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be an isometry.

If s is a line, then $f(s)$ is a line.

Therefore, isometries preserve lines.

Theorem 9. *The image of any angle under an isometry has the same measure as the given angle.*

f preserves angle measures between lines. $\text{m } \angle A'B'C' = \text{m } \angle ABC$.

Therefore, isometries preserve angles.

Corollary 10. *The images of two lines under an isometry are perpendicular if and only if the given lines are perpendicular.*

f preserves perpendicularity between lines. $f(s) \perp f(t)$ iff $s \perp t$.

Therefore, isometries preserve perpendicularity between lines.

Theorem 11. *The images of two lines under an isometry are parallel if and only if the given lines are parallel.*

f preserves parallelism between lines. $f(s) \parallel f(t)$ iff $s \parallel t$.

Therefore, isometries preserve parallelism between lines.