Transformational Geometry Theory

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Geometric Transformations

Theorem 1. *Every line reflection is a transformation of the plane.*

Solution. Let M_s be a line reflection in the line s of the plane.

Is M_s a function?

If so, is M_s a transformation?

To decide if M_s is a function we need to answer 2 questions.

1) Does every point in the plane have some image?
To answer this we must prove every point P in the plane has an image.
In logic symbols: Prove: $\forall P \in \text{plane}, \exists M_s(P)$.
2) If points A and B in the plane are identical, is $M_s(A) = M_s(B)$?
To answer this we must prove if A and B are identical points, then $M_s(A) = M_s(B)$.
In logic symbols: Prove: $A = B \Rightarrow M_s(A) = M_s(B)$.

To decide if M_s is a transformation(bijection) we need to answer 2 questions.

1) Is M_s an onto function?
To answer this we must prove every point P in the plane(codomain of M_s) is the image of some point in the domain of M_s.
In logic symbols: Prove: $\forall P \in \text{plane}, \exists Q \in \text{plane} \ni M_s(Q) = P$.
2) Is M_s a one to one function?
To answer this we must prove M_s satisfies the conditions for a one to one function.

We have multiple approaches to consider:

a) We could use direct proof and prove: if $A \neq B$, then $M_s(A) \neq M_s(B)$.
b) We could use proof by contrapositive and prove: if $M_s(A) = M_s(B)$, then $A = B$.
c) We could use proof by contradiction and prove: $A \neq B \wedge M_s(A) = M_s(B)$ leads to some contradiction.

We must first prove M_s is a function from the plane to the plane. Then we prove M_s is both one to one and onto.

Thus our strategy is to prove the following:

1) prove M_s is a function of the plane
2) prove M_s is onto
3) prove M_s is 1 to 1 $\square$

Proof. Let M_s : plane $\mapsto$ plane be a line reflection in the line s.

We prove M_s is a function.

Let P be any point in the plane. Then P is either on s or not on s. The definition of M_s specifies a well defined rule for obtaining the image of P in either case, so $M_s(P)$ exists. Hence, the domain of M_s is the entire plane. Thus, every point in the domain of M_s has at least one image.

Let A and B be points in the domain of M_s such that $A = B = K$. Then A is either on s or not on s. We consider these cases separately.

Case 1: Suppose $A \in s$.
Then by definition of M_s, $M_s(A) = A$. Since $B = A$, then $B \in s$, so by definition of M_s, $M_s(B) = B = A$. Hence, $M_s(A) = M_s(B)$.

Case 2: Suppose $A \notin s$.
Then by definition of M_s, $M_s(A) = A'$ and $M_s(B) = B'$ and s is the perpendicular bisector of both $\overline{AA'}$ and $\overline{BB'}$. Thus, s is the perpendicular bisector of both $\overline{KA'}$ and $\overline{KB'}$.

Suppose for the sake of contradiction that $\overline{KA'} \neq \overline{KB'}$. Then $\overline{KA'}$ and $\overline{KB'}$ are distinct line segments that share a common endpoint K and s is the perpendicular bisector of both $\overline{KA'}$ and $\overline{KB'}$. But this is impossible since there exists no line that can be the perpendicular bisector to two distinct line segments that share a common endpoint. Thus, it cannot be true that $\overline{KA'} \neq \overline{KB'}$. Therefore, $\overline{KA'} = \overline{KB'}$, so it follows that $A' = B'$. Hence, $M_s(A) = M_s(B)$.

In either case we have shown $A = B$ implies $M_s(A) = M_s(B)$. This means two identical points map to the same point(i.e., they don't map to two different points). Since we have shown that each point in the domain of M_s has at least one image, then this implies M_s maps each point in the domain to at most one image. Hence, each point in the domain is mapped to exactly one image. Therefore, M_s is a function. $\square$

Proof. We prove M_s is a geometric transformation.

We first prove M_s is an onto function.

Let P be any point in the plane. Then P is either on s or not on s. We consider these cases separately.

Case 1: Suppose $P \in s$.
Then by definition of M_s, $M_s(P) = P$.

Case 2: Suppose $P \notin s$.
Then there exists a unique point Q such that s is the perpendicular bisector of line segment $\overline{QP}$. Thus, $M_s(Q) = P$.

In either case P is the image of some point in the domain of M_s. Therefore, M_s is an onto function.

We prove the pre-image of any point $P \in s$ is on s. Let $P \in s$. Then by definition of M_s, $M_s(P) = P$. Thus, the pre-image of P is P. Since $P \in s$, then this means the pre-image of P is also on s.

We prove M_s is a one to one function.

We use proof by contradiction. Suppose M_s is not a one to one function. Then there exist distinct points A and B in the domain of M_s such that $A \neq B$

and $M_s(A) = M_s(B) = K$. K is either on s or not on s. We consider these cases separately.
Case 1: Suppose $K \notin s$.
Then by definition of M_s, s must be the perpendicular bisector of segments $\overline{AK}$ and $\overline{BK}$. But this cannot happen since no line can be the perpendicular bisector of two different segments sharing a common end point.
Case 2: Suppose $K \in s$.
Then the pre-images of K lie on s. Hence, $A \in s$ and $B \in s$. Therefore, by definition of M_s, $M_s(A) = A = K = M_s(B) = B$. Hence, $A = B$. But this contradicts the assumption that A and B are distinct points(i.e. $A \neq B$).

Both cases result in a contradiction when we assume there exist two distinct points that map to the same point. Thus, it cannot be true that there exist two distinct points which map to the same point. Hence, there do not exist distinct points which map to the same point. Therefore, M_s is one to one.

Since M_s is a function that is both one to one and onto, then M_s is a transformation of the plane onto the plane. □

Theorem 2. *Every line reflection is an isometry.*

Theorem 3. *The image of any line under an isometry is a line.*

Solution. The problem can be restated as follows. Let T be an isometry and let s, t be lines. Then $T(s) = t$.

We must prove $T(s) = t$.

To prove this let line $s = \overleftrightarrow{AB}$ where $A \neq B$. Since T is an isometry, then $T(A) = A'$ and $T(B) = B'$. Since $A \neq B$, then we know $A' \neq B'$ because T is one to one. So let $t = \overleftrightarrow{A'B'}$ be the line uniquely determined by A' and B'.

Let $T(s) = s'$ where s' is the set of all images of points on s. Is $s' = t$? To answer this question we must use the definition of equal sets since s' and t are sets of points. This means we must prove $s' \subseteq t$ and $t \subseteq s'$. If we can establish that $s' = t$, then we will have shown that $T(s) = s'$ is a line. □

Proof. Let T be an isometry of the plane onto the plane. Let points A and B exist in the domain of T such that $A \neq B$. Two distinct points determine a unique line, so let line $s = \overleftrightarrow{AB}$. T is a function so $T(A) = A'$ and $T(B) = B'$. Since T is one to one, then $A' \neq B'$. Thus, the distinct points A' and B' uniquely determine line $t = \overleftrightarrow{A'B'}$. Let $T(s) = s'$ where s' is the set of all images of points on s.

We prove $s' = t$.

We first prove $s' \subseteq t$. Let $P \in s$ such that $P \neq A$ and $P \neq B$. Then $T(P) = P'$ and $P' \in s'$. Either $P' \in t$ or $P' \notin t$. Suppose for the sake of contradiction that $P' \notin t$. Either P is between A and B or not. We consider these cases separately. There are three cases to consider.
Case 1: Suppose P is between A and B (A-P-B).
Since $P' \notin t$, then there exists $\triangle A'B'P'$. Since T is an isometry, then $A'B' = AB$, $A'P' = AP$ and $B'P' = BP$. Since P is between A and B, then $AB =$

$AP + PB$. Substituting, we obtain $A'B' = A'P' + P'B'$. By the triangle inequality, $A'P' + P'B' > A'B'$ since the sum of the lengths of two sides is greater than the length of the third side. So we have a contradiction $A'P' + P'B' = A'B'$ and $A'P' + P'B' > A'B'$. Thus, it cannot be true that $P' \notin t$. Therefore, $P' \in t$. We have shown that $P' \in s'$ implies $P' \in t$. Hence, $s' \subseteq t$ if P is between A and B.

Case 2: Suppose B is between A and P (A-B-P).
Since $P' \notin t$, then there exists $\triangle A'B'P'$. Since T is an isometry, then $A'B' = AB$, $A'P' = AP$ and $B'P' = BP$. Since B is between A and P, then $AP = AB + BP$. Substituting, we obtain $A'P' = A'B' + B'P'$. By the triangle inequality, $A'B' + B'P' > A'P'$ since the sum of the lengths of two sides is greater than the length of the third side. So we have a contradiction $A'B' + B'P' = A'P'$ and $A'B' + B'P' > A'P'$. Thus, it cannot be true that $P' \notin t$. Therefore, $P' \in t$. We have shown that $P' \in s'$ implies $P' \in t$. Hence, $s' \subseteq t$ if B is between A and P.

Case 3: Suppose A is between P and B (P-A-B).
Since $P' \notin t$, then there exists $\triangle A'B'P'$. Since T is an isometry, then $A'B' = AB$, $A'P' = AP$ and $B'P' = BP$. Since A is between P and B, then $PB = PA + AB$. Substituting, we obtain $P'B' = P'A' + A'B'$. By the triangle inequality, $P'A' + A'B' > P'B'$ since the sum of the lengths of two sides is greater than the length of the third side. So we have a contradiction $P'A' + A'B' = P'B'$ and $P'A' + A'B' > P'B'$. Thus, it cannot be true that $P' \notin t$. Therefore, $P' \in t$. We have shown that $P' \in s'$ implies $P' \in t$. Hence, $s' \subseteq t$ if A is between P and B.

In all three cases we conclude that $s' \subseteq t$.

Now we prove $t \subseteq s'$. Let $C' \in t$ such that $C' \neq A'$ and $C' \neq B'$. Since T is an onto function, then there exists some C in the plane such that $T(C) = C'$. Either $C \in s$ or $C \notin s$. Suppose for the sake of contradiction that $C \notin s$. Either C' is between A' and B' or not. We consider these cases separately. There are three cases to consider.

Case 1: Suppose C' is between A' and B' (A'-C'-B').
Since $C \notin s$, then there exists $\triangle ABC$. Since T is an isometry, then $AB = A'B'$, $BC = B'C'$ and $AC = A'C'$. Since C' is between A' and B', then $A'B' = A'C' + C'B'$. Substituting, we obtain $AB = AC + CB$. By the triangle inequality, $AC + CB > AB$ since the sum of the lengths of two sides is greater than the length of the third side. So we have a contradiction $AC + CB = AB$ and $AC + CB > AB$. Thus, it cannot be true that $C \notin s$. Therefore, $C \in s$, so $T(C) = C'$. Hence, $C' \in s'$. We have shown that $C' \in t$ implies $C' \in s'$. Hence, $t \subseteq s'$ if C' is between A' and B'.

Case 2: Suppose B' is between A' and C' (A'-B'-C').
Since $C \notin s$, then there exists $\triangle ABC$. Since T is an isometry, then $AB = A'B'$, $BC = B'C'$ and $AC = A'C'$. Since B' is between A' and C', then $A'C' = A'B' + B'C'$. Substituting, we obtain $AC = AB + BC$. By the triangle inequality, $AB + BC > AC$ since the sum of the lengths of two sides is greater than the length of the third side. So we have a contradiction $AB + BC = AC$ and $AB + BC > AC$. Thus, it cannot be true that $C \notin s$. Therefore, $C \in s$, so

$T(C) = C'$. Hence, $C' \in s'$. We have shown that $C' \in t$ implies $C' \in s'$. Hence, $t \subseteq s'$ if B' is between A' and C'.
Case 3: Suppose A' is between C' and B' (C'-A'-B').
Since $C \notin s$, then there exists $\triangle ABC$. Since T is an isometry, then $AB = A'B'$, $BC = B'C'$ and $AC = A'C'$. Since A' is between C' and B', then $C'B' = C'A' + A'B'$. Substituting, we obtain $CB = CA + AB$. By the triangle inequality, $CA + AB > CB$ since the sum of the lengths of two sides is greater than the length of the third side. So we have a contradiction $CA + AB = CB$ and $CA + AB > CB$. Thus, it cannot be true that $C \notin s$. Therefore, $C \in s$, so $T(C) = C'$. Hence, $C' \in s'$. We have shown that $C' \in t$ implies $C' \in s'$. Hence, $t \subseteq s'$ if A' is between C' and B'.

In all three cases we conclude that $t \subseteq s'$.

Since $s' \subseteq t$ and $t \subseteq s'$, then it follows that $s' = t$. Hence, $T(s) = t$. This means the image of line s under the isometry T is a line. □

Theorem 4. *The image of any angle under an isometry has the same measure as the given angle.*

Solution. The problem can be restated as follows. Let T be an isometry and let $\angle ABC$ be any angle. Then m $\angle A'B'C' =$ m $\angle ABC$.

We must prove m $\angle A'B'C' =$ m $\angle ABC$. Either A, B, C are all collinear or not, so we consider these cases separately. □

Proof. Let T be an isometry of the plane onto the plane. Let A, B, C be points in the plane and let $\angle ABC$ be any angle. Either A, B, C are all collinear or not. We consider these cases separately. There are two cases to consider.
Case 1: Suppose A, B, C are collinear.
Let line s be the line that contains A, B, and C. Then m $\angle ABC$ is either 0 degrees or 180 degrees. Since T maps lines onto lines, let line $t = T(s)$. Then the points $T(A) = A', T(B) = B', T(C) = C'$ are on t. So, m $\angle A'B'C'$ is either 0 degrees or 180 degrees. Thus, m $\angle A'B'C' =$ m $\angle ABC$.
Case 2: Suppose A, B, C are not collinear.
Then there exists $\triangle ABC$. Let line $s = AB$ and line $t = BC$. Since T maps lines onto lines, let line $s' = T(s)$ and line $t' = T(t)$. Since $A \in s$, then $T(A) = A'$ is on s'. Since $C \in t$, then $T(C) = C'$ is on t'. Since $B \in s \cap t$, then $T(B) = B'$ is on both s' and t'. Thus, $B' \in s' \cap t'$, so $s' \nparallel t'$. Therefore, $\triangle A'B'C'$ exists. Since T preserves distance, $AB = A'B', AC = A'C', BC = B'C'$. Therefore, $\overline{AB} \cong \overline{A'B'}, \overline{AC} \cong \overline{A'C'}, \overline{BC} \cong \overline{B'C'}$. By SSS(side-side-side postulate), $\triangle ABC \cong \triangle A'B'C'$. By CPCTC(corresponding parts of congruent triangles are congruent), $\angle ABC \cong A'B'C'$. Therefore, m $\angle A'B'C' =$ m $\angle ABC$. □

Corollary 5. *The images of two lines under an isometry are perpendicular if and only if the given lines are perpendicular.*

Solution. The problem can be restated as follows. Let T be an isometry and let s, t be lines with $T(s) = s'$ and $T(t) = t'$. Then $s' \perp t'$ iff $s \perp t$.

Since this is a biconditional we must prove the implication and its converse:
1) prove if $s' \perp t'$, then $s \perp t$.
2) prove if $s \perp t$, then $s' \perp t'$. $\square$

Proof. Let T be an isometry of the plane onto the plane. Let s and t be lines in the plane with $T(s) = s'$ and $T(t) = t'$.

We prove if $s \perp t$, then $s' \perp t'$. We use direct proof. Suppose $s \perp t$. Let $A \in s, B \in s \cap t, C \in t$ be points such that m $\angle ABC = 90°$. Then the following are true: $T(A) = A'$ and $T(s) = s'$. Since $A \in s$, then $T(A) \in T(s)$, so $A' \in s'$. $T(C) = C'$ and $T(t) = t'$. Since $C \in t$, then $T(C) \in T(t)$, so $C' \in t'$. $B = s \cap t$. $T(B) = T(s \cap t) = B' = T(s) \cap T(t) = s' \cap t'$. Thus, $\angle A'B'C' = T(\angle ABC)$. Therefore, m $\angle A'B'C' =$ m $\angle ABC$, by theorem 4. Since m $\angle ABC = 90°$, then it follows that m $\angle A'B'C' = 90°$. Hence, $s' \perp t'$.

Conversely, we prove if $s' \perp t'$, then $s \perp t$. We use direct proof. Suppose $s' \perp t'$. Let $A' \in s', B' \in s' \cap t', C' \in t'$ be points such that m $\angle A'B'C' = 90°$. Since T is onto, there exists $A \in s$ such that $T(A) = A'$. Since T is onto, there exists $C \in t$ such that $T(C) = C'$. Since T is onto, there exists B such that $T(B) = B'$. Since $B' \in s' \cap t'$, then $B' \in T(s \cap t)$, so $B \in s \cap t$. Thus, $\angle A'B'C' = T(\angle ABC)$. Therefore, m $\angle A'B'C' =$ m $\angle ABC$, by theorem 4. Since m $\angle A'B'C' = 90°$, then it follows that m $\angle ABC = 90°$. Hence, $s \perp t$. $\square$

Theorem 6. *The images of two lines under an isometry are parallel if and only if the given lines are parallel.*

Solution. The problem can be restated as follows. Let T be an isometry and let s, t be lines with $T(s) = s'$ and $T(t) = t'$. Then $s' \parallel t'$ iff $s \parallel t$.

Since this is a biconditional we must prove the implication and its converse:
1) prove if $s' \parallel t'$, then $s \parallel t$.
2) prove if $s \parallel t$, then $s' \parallel t'$. $\square$

Proof. Let T be an isometry of the plane onto the plane. Let s and t be lines in the plane with $T(s) = s'$ and $T(t) = t'$.

We prove if $s \parallel t$, then $s' \parallel t'$. We use proof by contradiction. Suppose $s \parallel t$ and $s' \nparallel t'$. Then s' and t' intersect at some point X. Thus, $\{X\} = s' \cap t'$. Therefore, $X \in s'$ and $X \in t'$. Since T is an onto function, then there exists $P \in s$ such that $T(P) = X$ and there exists $Q \in t$ such that $T(Q) = X$. Since $s \parallel t$, then P and Q are distinct points. Thus there exist distinct points $P \neq Q$ such that $T(P) = T(Q)$. Hence, T is not one to one. But, T is a transformation, so T is one to one. Therefore, we have a contradiction T is one to one and T is not one to one. Consequently, it cannot be true that $s' \nparallel t'$. Thus, $s' \parallel t'$.

Conversely, we prove if $s' \parallel t'$, then $s \parallel t$. We use proof by contradiction. Suppose $s' \parallel t'$ and $s \nparallel t$. Then s and t intersect in some point K. Thus, $\{K\} = s \cap t$. Therefore, $K \in s$ and $K \in t$. Since T is a function, then there exists $P' \in s'$ such that $T(K) = P'$ and there exists $Q' \in t'$ such that $T(K) = Q'$. Since T is a function, then $P' = Q'$. Thus, $P' \in s'$ and $P' \in t'$, so $P' \in s' \cap t'$. Hence, $s' \nparallel t'$. But, this contradicts the assumption that $s' \parallel t'$. Consequently, it cannot be true that $s \nparallel t$. Thus, $s \parallel t$. $\square$

Proposition 7. *If $t \perp s$, then $M_s(t) = t$.*

Solution. Let t' be the set of all images of line t under the line reflection M_s. That is, $t' = M_s(t)$. Since M_s is an isometry, we know M_s maps lines onto lines. Thus, M_s maps line t onto line t'.

We must prove $t' = t$. Since we know t' and t are sets of points in the plane, we can use the definition of set equality. Thus, our strategy is to prove both $t' \subseteq t$ and $t \subseteq t'$. $\square$

Proof. Let $t' = M_s(t)$ be the set of all images of line t under the line reflection M_s and $t \perp s$. We prove $t' = t$.

We prove $t \subseteq t'$. Let P be any point on line t. Either P is on s or not on s. We consider these cases separately.

Case 1: Suppose $P \in s$.

Then $M_s(P) = P' = P$, by definition of M_s. Since $P' \in t'$, then this means $P \in t'$. Thus, $P \in t$ implies $P \in t'$.

Case 2: Suppose $P \notin s$.

Then s is the perpendicular bisector of $\overline{PP'}$, by definition of M_s. Thus P' lies on line t, so $P' \in t$. Since $P' \in t'$ and $P' \in t$, then $P' \in t \cap t'$. Since M_s is an isometry, then M_s preserves perpendicularity of lines. Thus, $M_s(t) \perp M_s(s)$ if and only if $t \perp s$. By assumption $t \perp s$, so this implies $M_s(t) \perp M_s(s)$. Since $M_s(t) = t'$ and $M_s(s) = s$, then this means $t' \perp s$. Since $s \perp t$ and $s \perp t'$, then $t \parallel t'$. This implies either $t = t'$ or t and t' are distinct lines that are parallel. Suppose for the sake of contradiction that $t \parallel t'$ and $t \neq t'$. Then $t \cap t' = \emptyset$. But we concluded $t \cap t' = \{P'\}$. Thus, it cannot be true that $t \neq t'$. Therefore, $t = t'$. Since $P \in t$, this means $P \in t'$.

In both cases we have shown $P \in t$ implies $P \in t'$. Hence, we conclude $t \subseteq t'$.

We prove $t' \subseteq t$. Let $P' \in t'$. Since M_s is an onto function, then there exists $P \in t$ such that $M_s(P) = P'$. Either P' is on s or not on s. We consider these cases separately.

Case 1: Suppose $P' \in s$.

Then there exists $P \in t$ such that $M_s(P) = P'$. Since $P' \in s$, then by definition of M_s, $P = P'$. Since $P \in t$, then this means $P' \in t$. Hence, $P' \in t'$ implies $P' \in t$.

Case 2: Suppose $P' \notin s$.

Then s is the perpendicular bisector of line segment $\overline{PP'}$ such that $P \in t$ and $P' = M_s(P)$, by definition of M_s. Thus, $P' \in t$ and $P' \in \overleftrightarrow{PP'}$, so $P' \in t \cap \overleftrightarrow{PP'}$. Since s is perpendicular to $\overline{PP'}$, then $s \perp \overleftrightarrow{PP'}$. Thus, $s \perp t$ and $s \perp \overleftrightarrow{PP'}$, so it follows that $t \parallel \overleftrightarrow{PP'}$. Therefore, either $t = \overleftrightarrow{PP'}$ or t and $\overleftrightarrow{PP'}$ are distinct lines that are parallel. Suppose for the sake of contradiction that $t \parallel \overleftrightarrow{PP'}$ and $t \neq \overleftrightarrow{PP'}$. Then $t \cap \overleftrightarrow{PP'} = \emptyset$. But, we concluded $t \cap \overleftrightarrow{PP'} = \{P\}$. Thus, it cannot be true that $t \neq \overleftrightarrow{PP'}$. Hence, $t = \overleftrightarrow{PP'}$. Therefore, $P' \in t$.

In both cases we have shown $P' \in t'$ implies $P' \in t$. Hence, we conclude $t' \subseteq t$.

Since $t \subseteq t'$ and $t' \subseteq t$, then it follows that $t = t'$. Hence, $M_s(t) = t$. $\square$