Book Elementary Linear Algebra 2^{nd} edition by Stanley Grossman, Exercises

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May 28, 2025

Chapter 5 Vector Spaces

5.2 Definition and Basic Properties

Example 1. Let $L = \{(x, y) \in \mathbb{R}^2 : y = 2x + 1\}$.

The set L under addition and scalar multiplication defined on $\mathbb{R}^2$ is not a vector space.

Proof. Let $p, q \in L$.

Since $p \in L$, then there exist $x_1, y_1 \in \mathbb{R}$ such that $p = (x_1, y_1)$ and $y_1 = 2x_1 + 1$.

Since $q \in L$, then there exist $x_2, y_2 \in \mathbb{R}$ such that $q = (x_2, y_2)$ and $y_2 = 2x_2 + 1$.

Since $p = (x_1, y_1)$ and $q = (x_2, y_2)$, then $p + q = (x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2) \in \mathbb{R}^2$.

Observe that

$$\begin{aligned} y_1 + y_2 &= (2x_1 + 1) + (2x_2 + 1) \\ &= 2x_1 + 2x_2 + 2 \\ &= 2(x_1 + x_2) + 2 \\ &\neq 2(x_1 + x_2) + 1. \end{aligned}$$

Hence, $y_1 + y_2 \neq 2(x_1 + x_2) + 1$.

Since $p + q = (x_1 + x_2, y_1 + y_2) \in \mathbb{R}^2$, but $y_1 + y_2 \neq 2(x_1 + x_2) + 1$, then $p + q \notin L$, so L is not closed under addition defined on $\mathbb{R}^2$.

Therefore, $(L, +, \cdot)$ is not a vector space. $\square$

Proof. Observe that $(\mathbb{R}^2, +, \cdot)$ is a vector space over $\mathbb{R}$, and the additive identity is the point $(0, 0) \in \mathbb{R}^2$, and $L \subset \mathbb{R}^2$.

Since $(0, 0) \in \mathbb{R}^2$, but $0 \neq 2 \cdot 0 + 1 = 1$, then $(0, 0) \notin L$.

Since $L \subset \mathbb{R}^2$, but $(0, 0) \notin L$, then L is not a subspace of $\mathbb{R}^2$, so $(L, +, \cdot)$ is not a vector space. $\square$