Book Linear Algebra Done Right 4^{th} edition from 23 April 2025, Exercises

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Exercises 1A

Exercise 1. Additive identity of $\mathbb{C}$ is unique.

Show that for every $\alpha \in \mathbb{C}$, there exists a unique $\beta \in \mathbb{C}$ such that $\alpha + \beta = 0$.

Proof. Let α be an arbitrary complex number.

Then there exist real numbers a and b such that $\alpha = a + bi$.

Let $\beta = -a + (-b)i$.

Since $a \in \mathbb{R}$ and $b \in \mathbb{R}$, then $-a \in \mathbb{R}$ and $-b \in \mathbb{R}$, so $\beta \in \mathbb{C}$.

Observe that

$$\begin{aligned}
\alpha + \beta &= (a + bi) + [-a + (-b)i] \\
&= [a + (-a)] + [b + (-b)]i \\
&= 0 + 0i \\
&= 0 + 0 \\
&= 0.
\end{aligned}$$

Therefore, $\alpha + \beta = 0$, so β is an additive identity of α.

We prove an additive identity of α is unique.

Let $\beta_1 \in \mathbb{C}$ and $\beta_2 \in \mathbb{C}$ be additive identities of α.

Then $\beta_1 = c + di$ and$\beta_2 = e + fi$ for real numbers c, d, e, f, and $\alpha + \beta_1 = 0$ and $\alpha + \beta_2 = 0$.

Since $\alpha + \beta_1 = 0 = \alpha + \beta_2$, then $\alpha + \beta_1 = \alpha + \beta_2$.

Observe that

$$\begin{aligned}
\alpha + \beta_1 &= \alpha + \beta_2 \\
(a + bi) + (c + di) &= (a + bi) + (e + fi) \\
(a + c) + (b + d)i &= (a + e) + (b + f)i.
\end{aligned}$$

Thus, $a + c = a + e$ and $b + d = b + f$.

Observe that

$$\begin{aligned} e &= 0 + e \\ &= (-a + a) + e \\ &= -a + (a + e) \\ &= -a + (a + c) \\ &= (-a + a) + c \\ &= 0 + c \\ &= c. \end{aligned}$$

Therefore, $e = c$.
Observe that

$$\begin{aligned} f &= 0 + f \\ &= (-b + b) + f \\ &= -b + (b + f) \\ &= -b + (b + d) \\ &= (-b + b) + d \\ &= 0 + d \\ &= d. \end{aligned}$$

Therefore, $f = d$.
Hence, $\beta_2 = e + fi = c + di = \beta_1$, so $\beta_2 = \beta_1$.
Therefore, $\beta_1 = \beta_2$, so the additive inverse of α is unique. $\square$

Exercise 2. 3^{rd} **root of unity**

Show that $\dfrac{-1 + \sqrt{3}i}{2}$ is a cube root of 1.

Proof. Let $z = \dfrac{-1 + \sqrt{3}i}{2}$.

Observe that

$$\begin{aligned}
z^3 &= (\frac{-1+\sqrt{3}i}{2})^3 \\
&= (-\frac{1}{2}+\frac{i\sqrt{3}}{2})^3 \\
&= (-\frac{1}{2}+\frac{i\sqrt{3}}{2})^2 \cdot (-\frac{1}{2}+\frac{i\sqrt{3}}{2}) \\
&= [\frac{1}{4}+2\cdot(-\frac{1}{2})\cdot(\frac{i\sqrt{3}}{2})+\frac{3i^2}{4}]\cdot(-\frac{1}{2}+\frac{i\sqrt{3}}{2}) \\
&= [\frac{1}{4}+2\cdot(-\frac{1}{2})\cdot(\frac{i\sqrt{3}}{2})-\frac{3}{4}]\cdot(-\frac{1}{2}+\frac{i\sqrt{3}}{2}) \\
&= (-\frac{1}{2}-\frac{i\sqrt{3}}{2})\cdot(-\frac{1}{2}+\frac{i\sqrt{3}}{2}) \\
&= \frac{1}{4}-(\frac{i\sqrt{3}}{2})^2 \\
&= \frac{1}{4}-\frac{3i^2}{4} \\
&= \frac{1}{4}-\frac{(-1)(3)}{4} \\
&= \frac{1}{4}+\frac{3}{4} \\
&= 1.
\end{aligned}$$

Note that there are 3 roots to the equation $0 = z^3 - 1 = (z-1)(z^2+z+1)$. They are:

$z_1 = 1$

$z_2 = \frac{-1+i\sqrt{3}}{2} = cis(\frac{2\pi}{3})$.

$z_3 = \frac{-1-i\sqrt{3}}{2} = cis(\frac{4\pi}{3})$.

Each of the roots are 120 degrees apart on the unit circle in the complex plane. □

Exercise 3. Find $x \in \mathbb{R}^4$ such that $(4,-3,1,7)+2x = (5,9,-6,8)$.

Solution. Let $x \in \mathbb{R}^4$ such that $(4,-3,1,7)+2x = (5,9,-6,8)$.

Observe that

$$\begin{aligned}
(4,-3,1,7)+2x &= (5,9,-6,8) \\
-(4,-3,1,7)+(4,-3,1,7)+2x &= -(4,-3,1,7)+(5,9,-6,8) \\
\vec{0}+2x &= (-4,3,-1,-7)+(5,9,-6,8) \\
2x &= (1,12,-7,1) \\
x &= \frac{1}{2}(1,12,-7,1) \\
x &= (\frac{1}{2},6,-\frac{7}{2},\frac{1}{2}).
\end{aligned}$$

Therefore, $x = (\frac{1}{2}, 6, -\frac{7}{2}, \frac{1}{2})$.

We verify the solution.
Observe that

$$\begin{aligned} (4,-3,1,7) + 2x &= (4,-3,1,7) + 2(\frac{1}{2}, 6, -\frac{7}{2}, \frac{1}{2}) \\ &= (4,-3,1,7) + (1,12,-7,1) \\ &= (5,9,-6,8). \end{aligned}$$

The solution is $x = (\frac{1}{2}, 6, -\frac{7}{2}, \frac{1}{2})$. $\square$

Exercise 4. Explain why there does not exist $\lambda \in \mathbb{C}$ such that
$\lambda(2-3i, 5+4i, -6+7i) = (12-5i, 7+22i, -32-9i)$.

Solution. Suppose there exists $\lambda \in \mathbb{C}$ such that $\lambda(2-3i, 5+4i, -6+7i) = (12-5i, 7+22i, -32-9i)$.

Since $\lambda \in \mathbb{C}$, then there exist real numbers a and b such that $\lambda = a + bi$.
Observe that

$$\begin{aligned} \lambda(2-3i, 5+4i, -6+7i) &= (12-5i, 7+22i, -32-9i) \\ (\lambda(2-3i), \lambda(5+4i), \lambda(-6+7i)) &= (12-5i, 7+22i, -32-9i). \end{aligned}$$

Thus, $\lambda(2-3i) = 12-5i$ and $\lambda(5+4i) = 7+22i$ and $\lambda(-6+7i) = -32-9i$.

Since $\lambda(2-3i) = 12-5i$ and $\lambda = a+bi$, then $(a+bi)(2-3i) = 12-5i$, so $2a - 3ai + 2bi - 3bi^2 = 2a - (3a-2b)i - 3b(-1) = (2a+3b) - (3a-2b)i = 12-5i$.

Hence, $2a + 3b = 12$ and $3a - 2b = 5$, so $a = 3$ and $b = 2$.

Since $\lambda(5+4i) = 7+22i$ and $\lambda = a+bi$, then $(a+bi)(5+4i) = 7+22i$, so $5a + 4ai + 5bi + 4bi^2 = 5a + (4a+5b)i + 4b(-1) = (5a-4b) + (4a+5b)i = 7+22i$.

Hence, $5a - 4b = 7$ and $4a + 5b = 22$, so $a = 3$ and $b = 2$.

Since $\lambda(-6+7i) = -32-9i$ and $\lambda = a+bi$, then $(a+bi)(-6+7i) = -32-9i$, so $-6a + 7ai - 6bi + 7bi^2 = -6a + (7a-6b)i + 7b(-1) = (-6a-7b) + (7a-6b)i = -32-9i$.

Hence, $-6a - 7b = -32$ and $7a - 6b = -9$, so $a = \dfrac{129}{85}$ and $b = \dfrac{278}{85}$.

Thus, we have $a = 3$ and $a = \dfrac{129}{85}$, and $b = 2$ and $b = \dfrac{278}{85}$.

There is no such complex number $\lambda = a + bi$ that has two different values of a and two different values of b for real numbers a and b.

Therefore, λ does not exist, so there is no complex number λ that satisfies the equation $\lambda(2-3i, 5+4i, -6+7i) = (12-5i, 7+22i, -32-9i)$. $\square$