Linear Algebra Examples

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Vector Spaces

Example 1. trivial vector space

Let $V = \{0\}$.
Let F be a field.
Define addition on V by $0 + 0 = 0$.
Define the product $\lambda \cdot 0 = 0$ for all $\lambda \in F$.
Then $(V, +, \cdot)$ is a vector space, called the **trivial vector space**.

Proof. Addition defined on V is a function $V \times V \to V$ defined by $0 + 0 = 0$, so addition is a binary operation on V.

Since $\lambda \cdot 0 = 0$ for all $\lambda \in F$, then for every $\lambda \in F$ and $0 \in V$, scalar multiplication assigns the element $\lambda \cdot 0 = 0 \in V$, so scalar multiplication is a function $F \times V \to V$.

Observe that

$$\begin{aligned} (0+0)+0 &= 0+0 \\ &= 0+(0+0). \end{aligned}$$

Hence, $(0 + 0) + 0 = 0 + (0 + 0)$.
Since $0 \in V$ and $(0 + 0) + 0 = 0 + (0 + 0)$, then addition is associative.
Since $0 \in V$ and $0 + 0 = 0 + 0$, then addition is commutative.
Since $0 \in V$ and $0 + 0 = 0$, then $0 \in V$ is a right additive identity.
Since $0 \in V$ and $0 + 0 = 0$, then 0 is a right additive inverse of 0, so every element of V has a right additive inverse.

Let $\alpha, \beta \in F$.
Since F is closed under field multiplication, then $\alpha\beta \in F$.
Observe that

$$\begin{aligned} \alpha(\beta 0) &= \alpha 0 \\ &= 0 \\ &= (\alpha\beta)0. \end{aligned}$$

Therefore, $\alpha(\beta 0) = (\alpha\beta)0$, so associativity of scalar multiplication with field multiplication holds.

Let $\lambda \in F$.
Observe that

$$\begin{aligned} \lambda(0+0) &= \lambda 0 \\ &= 0 \\ &= 0+0 \\ &= \lambda 0 + \lambda 0. \end{aligned}$$

Therefore, $\lambda(0+0) = \lambda 0 + \lambda 0$, so the left distributive law of scalar multiplication over vector addition holds.

Let $\alpha, \beta \in F$.
Since F is closed under addition, then $\alpha + \beta \in F$.
Observe that

$$\begin{aligned} (\alpha+\beta)0 &= 0 \\ &= 0+0 \\ &= \alpha 0 + \beta 0. \end{aligned}$$

Therefore, $(\alpha+\beta)0 = \alpha 0 + \beta 0$, so the right distributive law of scalar multiplication over scalar addition holds.

Let $1 \in F$.
Then $1 \cdot 0 = 0$, so $1 \in F$ is a multiplicative identity for scalar multiplication.

Since addition is a binary operation on V, and scalar multiplication is a function $F \times V \to V$, and addition is associative, and addition is commutative, and $0 \in V$ is a right additive identity, and every element of V has a right additive inverse, and associativity of scalar multiplication with field multiplication holds, and the left distributive law of scalar multiplication over vector addition holds, and the right distributive law of scalar multiplication over scalar addition holds, and $1 \in F$ is a multiplicative identity for scalar multiplication, then $(V, +, \cdot)$ is a vector space over F. ☐

Example 2. The set $\{1\}$ is not a vector space.

Proof. Let $V = \{1\}$.
Since $1 + 1 = 2$ and $2 \notin V$, then V is not closed under addition.
Therefore, $(V, +, \cdot)$ is not a vector space. ☐

Vector Space F^n

Definition 3. list of length n
Let n be a nonnegative integer.
A **list of length** n is an ordered collection of n elements.

A list of length n is also called an **n-tuple**.
Order matters in a list.
A list may have duplicates.

Definition 4. empty list
The **empty list**, denoted (), is a list of length 0.

Let $n \in \mathbb{Z}^+$.
Let $(x_1, ..., x_n)$ be a list of length n.
Then $(x_1, ..., x_n)$ has n elements, so $(x_1, ..., x_n)$ has finite length.

Definition 5. equal lists
Let a and b be lists.
Then a **equals** b, denoted $a = b$, iff a and b have the same number of components and their corresponding components are equal.

Let n be a nonnegative integer.
Let $a = (a_1, a_2, ..., a_n)$ and $b = (b_1, b_2, ..., b_n)$ be two lists.
Then $a = b$ iff $a_1 = b_1$ and $a_2 = b_2$ and ... and $a_n = b_n$.
Therefore, two lists are equal iff they have the same length and the same elements in the same order.

Example 6. list of length 1
Let a be an object.
Then (a) is a list of length 1.

Example 7. list of length 2
Let a and b be distinct objects.
Then the ordered pair (a, b) is a list of length 2.
Observe that $a \neq b$, so $(a, b) \neq (b, a)$.
Therefore, the ordered pairs (a, b) and (b, a) are different.

Example 8. list of length 3
Let x, y, z be distinct objects.
Then the ordered triple (x, y, z) is a list of length 3.
Since x, y, and z are all distinct, then $(x, y, z) \neq (x, z, y) \neq (y, x, z) \neq (y, z, x) \neq (z, x, y) \neq (z, y, x)$.

Example 9. list of length n
Let $n \in \mathbb{Z}^+$.
Let $x_1, ..., x_n$ be n objects.
Then the n-tuple $(x_1, ..., x_n)$ is a list of length n.

Example 10. Order matters in a list.
The lists $(3, 5)$ and $(5, 3)$ are not equal, since 3 is the first element of the first list, whereas 5 is the first element of the second list.
Therefore, $(3, 5) \neq (5, 3)$.

Example 11. Duplicates are allowed in a list.

The lists $(4,4)$ and $(4,4,4)$ are not equal, since the first list has length 2, whereas the second list has length 3.

Therefore, $(4,4) \neq (4,4,4)$.

Definition 12. F^n

Let F be a field.

Let $n \in \mathbb{Z}^+$.

The set of all lists of n elements of F is $F^n = F \times F \times \ldots \times F = \{(v_1, ..., v_n) : v_k \in F \text{ for each } k \in \{1, ..., n\}\}$.

Let F be a field.

Let $n \in \mathbb{Z}^+$.

Let $(v_1, ..., v_n) \in F^n$.

Then $(v_1, ..., v_n)$ is a list of length n.

For each $k \in \{1, ..., n\}$, the k^{th} coordinate of $(v_1, ..., v_n)$ is v_k.

TODO: Determine the correct precise definition of coordinate of a vector . Is it defined relative to some basis? If so, add a new definition for it somewhere.

Definition 13. component-wise addition over F^n

Let $(F, +, \cdot)$ be a field.

Let $n \in \mathbb{Z}^+$.

Define **component-wise addition** on $F^n = F \times F \times \ldots F$ by $(v_1, ..., v_n) + (w_1, ..., w_n) = (v_1 + w_1, ..., v_n + w_n)$ for all $(v_1, ..., v_n), (w_1, ..., w_n) \in F^n$.

Let $(F, +, \cdot)$ be a field.

Let $n \in \mathbb{Z}^+$.

Let $(v_1, v_2, ..., v_n) \in F^n$.

Let $(w_1, w_2, ..., w_n) \in F^n$.

The **sum** is $(v_1, v_2, ..., v_n) + (w_1, w_2, ..., w_n) = (v_1 + w_1, v_2 + w_2, ..., v_n + w_n)$.

Proposition 14. $(F^n, +)$ ***is an abelian group under component-wise addition.***

Let $(F, +, \cdot)$ *be a field.*

Let $n \in \mathbb{Z}^+$.

Then $(F^n, +)$ *is an abelian group under component-wise addition.*

Proof. Since $(F, +, \cdot)$ is a field, then $(F, +)$ is an abelian group.

Addition on $F^n = F \times F \times \ldots F$ is defined component-wise by $(v_1, ..., v_n) + (w_1, ..., w_n) = (v_1 + w_1, ..., v_n + w_n)$ for all $(v_1, ..., v_n), (w_1, ..., w_n) \in F^n$.

Since $(F, +)$ is an abelian group, then the set F^n with component-wise addition is a direct sum of F with itself n times, so $(F^n, +)$ is a direct sum of abelian groups.

Since the direct sum of abelian groups is an abelian group, then $(F^n, +)$ is an abelian group. $\square$

Let $(F, +, \cdot)$ be a field.

Let $n \in \mathbb{Z}^+$.

Then $(F^n, +)$ is an abelian group under component-wise addition.

The additive identity is $(0, ..., 0)$, where $0 \in F$ is the additive identity of F.

The additive inverse of $(v_1, ..., v_n) \in F^n$ is $(-v_1, ..., -v_n)$, where $v_k \in F$ and $-v_k \in F$ for each $k \in \{1, ..., n\}$.

Definition 15. scalar multiplication in F^n

Let F be a field.

Let $n \in \mathbb{Z}^+$.

Let $(v_1, v_2, ..., v_n) \in F^n$.

Let $\lambda \in F$.

Define the **product** by $\lambda(v_1, v_2, ..., v_n) = (\lambda v_1, \lambda v_2, ..., \lambda v_n)$.

Theorem 16. $(F^n, +, \cdot)$ ***is a vector space over*** F**.**

Let $(F, +, \cdot)$ *be a field.*

Let $n \in \mathbb{Z}^+$*.*

The set F^n *with component-wise addition and scalar multiplication defined on* F^n *is a vector space over* F*.*

Proof. By proposition 14, $(F^n, +)$ is an abelian group under component-wise addition defined by $(v_1, ..., v_n) + (w_1, ..., w_n) = (v_1 + w_1, ..., v_n + w_n)$ for all $(v_1, ..., v_n), (w_1, ..., w_n) \in F^n$.

Let $\lambda \in F$ and $(v_1, v_2, ..., v_n) \in F^n$.

Then $\lambda(v_1, v_2, ..., v_n) = (\lambda v_1, \lambda v_2, ..., \lambda v_n)$.

Let $k \in \{1, 2, ..., n\}$.

Since F is closed under multiplication, and $\lambda \in F$ and $v_k \in F$, then $\lambda v_k \in F$.

Hence, $\lambda v_k \in F$ for all $k \in \{1, 2, ..., n\}$, so $(\lambda v_1, \lambda v_2, ..., \lambda v_n) \in F^n$.

Thus, for each $\lambda \in F$ and for each $(v_1, v_2, ..., v_n) \in F^n$ scalar multiplication in F^n assigns the product $(\lambda v_1, \lambda v_2, ..., \lambda v_n) \in F^n$.

Therefore, scalar multiplication in F^n is a function $F \times F^n \to F^n$ defined by $\lambda(v_1, v_2, ..., v_n) = (\lambda v_1, \lambda v_2, ..., \lambda v_n) \in F^n$ for all $\lambda \in F$ and for all $(v_1, v_2, ..., v_n) \in F^n$.

To prove associativity of scalar multiplication with field multiplication, we must prove $(\forall \vec{v} \in F^n)(\forall \alpha, \beta \in F)[\alpha(\beta\vec{v}) = (\alpha\beta)\vec{v}]$.

Let $\vec{v} \in F^n$.

Then there exist $v_1, v_2, ..., v_n \in F$ such that $\vec{v} = (v_1, v_2, ..., v_n)$.

Let $\alpha \in F$ and $\beta \in F$.

Observe that

$$\begin{aligned}
\alpha(\beta\vec{v}) &= \alpha[\beta(v_1, v_2, ..., v_n)] \\
&= \alpha(\beta v_1, \beta v_2, ..., \beta v_n) \\
&= (\alpha(\beta v_1), \alpha(\beta v_2), ..., \alpha(\beta v_n)) \\
&= ((\alpha\beta)v_1, (\alpha\beta)v_2, ..., (\alpha\beta)v_n) \\
&= (\alpha\beta)(v_1, v_2, ..., v_n) \\
&= (\alpha\beta)\vec{v}.
\end{aligned}$$

Therefore, $\alpha(\beta\vec{v}) = (\alpha\beta)\vec{v}$, as desired.

To prove the left distributive law of scalar multiplication over vector addition, we must prove $(\forall \vec{v}, \vec{w} \in F^n)(\forall \lambda \in F)[\lambda(\vec{v} + \vec{w}) = \lambda\vec{v} + \lambda\vec{w}]$.

Let $\vec{v} \in F^n$ and $\vec{w} \in F^n$.

Since $\vec{v} \in F^n$, then there exist $v_1, v_2, ..., v_n \in F$ such that $\vec{v} = (v_1, v_2, ..., v_n)$.

Since $\vec{w} \in F^n$, then there exist $w_1, w_2, ..., w_n \in F$ such that $\vec{w} = (w_1, w_2, ..., w_n)$.

Let $\lambda \in F$.

Observe that

$$\begin{aligned}
\lambda(\vec{v} + \vec{w}) &= \lambda[(v_1, v_2, ..., v_n) + (w_1, w_2, ..., w_n)] \\
&= \lambda(v_1 + w_1, v_2 + w_2, ..., v_n + w_n) \\
&= (\lambda(v_1 + w_1), \lambda(v_2 + w_2), ..., \lambda(v_n + w_n)) \\
&= ((\lambda v_1 + \lambda w_1), (\lambda v_2 + \lambda w_2), ..., (\lambda v_n + \lambda w_n)) \\
&= (\lambda v_1, \lambda v_2, ..., \lambda v_n) + (\lambda w_1, \lambda w_2, ..., \lambda w_n) \\
&= \lambda(v_1, v_2, ..., v_n) + \lambda(w_1, w_2, ..., w_n) \\
&= \lambda\vec{v} + \lambda\vec{w}.
\end{aligned}$$

Therefore, $\lambda(\vec{v} + \vec{w}) = \lambda\vec{v} + \lambda\vec{w}$, as desired.

To prove the right distributive law of scalar multiplication over scalar addition, we must prove $(\forall \vec{v} \in F^n)(\forall \alpha, \beta \in F)[(\alpha + \beta)\vec{v} = \alpha\vec{v} + \beta\vec{v}]$.

Let $\vec{v} \in F^n$.

Then there exist $v_1, v_2, ..., v_n \in F$ such that $\vec{v} = (v_1, v_2, ..., v_n)$.

Let $\alpha \in F$ and $\beta \in F$.

Observe that

$$\begin{aligned}
(\alpha + \beta)\vec{v} &= (\alpha + \beta)(v_1, v_2, ..., v_n) \\
&= ((\alpha + \beta)v_1, (\alpha + \beta)v_2, ..., (\alpha + \beta)v_n) \\
&= (\alpha v_1 + \beta v_1, \alpha v_2 + \beta v_2, ..., \alpha v_n + \beta v_n) \\
&= (\alpha v_1, \alpha v_2, ..., \alpha v_n) + (\beta v_1, \beta v_2, ..., \beta v_n) \\
&= \alpha(v_1, v_2, ..., v_n) + \beta(v_1, v_2, ..., v_n) \\
&= \alpha\vec{v} + \beta\vec{w}.
\end{aligned}$$

Therefore, $(\alpha + \beta)\vec{v} = \alpha\vec{v} + \beta\vec{v}$, as desired.

To prove $1 \in F$ is a multiplicative identity for scalar multiplication in F^n, we must prove $(\forall \vec{v} \in F^n)(1 \cdot \vec{v} = \vec{v})$.

Let $\vec{v} \in F^n$.

Then there exist $v_1, v_2, ..., v_n \in F$ such that $\vec{v} = (v_1, v_2, ..., v_n)$.

Observe that

$$\begin{aligned} 1 \cdot \vec{v} &= 1 \cdot (v_1, v_2, ..., v_n) \\ &= (1 \cdot v_1, 1 \cdot v_2, ..., 1 \cdot v_n) \\ &= (v_1, v_2, ..., v_n) \\ &= \vec{v}. \end{aligned}$$

Therefore, $1 \cdot \vec{v} = \vec{v}$, as desired.

Since $(F^n, +)$ is an abelian group under component-wise addition, and scalar multiplication in F^n is a function $F \times F^n \to F^n$ defined by $\lambda(v_1, v_2, ..., v_n) = (\lambda v_1, \lambda v_2, ..., \lambda v_n) \in F^n$ for all $\lambda \in F$ and for all $(v_1, v_2, ..., v_n) \in F^n$, and associativity of scalar multiplication with field multiplication holds, and the left distributive law of scalar multiplication over vector addition holds, and the right distributive law of scalar multiplication over scalar addition holds, and $1 \in F$ is a multiplicative identity for scalar multiplication in F^n, then $(F^n, +, \cdot)$ is a vector space over F. $\square$

Let $(F, +, \cdot)$ be a field.
Let $n \in \mathbb{Z}^+$.
Then $(F^n, +, \cdot)$ is a vector space over F.

Let $\vec{v} \in F^n$.
Then there exist $v_1, v_2, ..., v_n \in F$ such that $\vec{v} = (v_1, v_2, ..., v_n)$.
Therefore, $\vec{v} = (v_1, v_2, ..., v_n)$ is a vector.
Since $\vec{v}$ is a list of length n, then $\vec{v}$ is also called an **n-component vector**.
Therefore, $\vec{v} = (v_1, v_2, ..., v_n)$ is a vector of n-components.

Since $(F^n, +, \cdot)$ is a vector space, then $(F^n, +)$ is an abelian group.
The additive identity is the **zero vector** $\vec{0} = (0, ..., 0)$, a list of length n whose coordinates are all $0 \in F$.

The vector space $(F^n, +, \cdot)$ over field F satisfies the below axioms.
V1. F^n is closed under vector addition.
$\vec{v} + \vec{w} \in F^n$ for all $\vec{v}, \vec{w} \in F^n$.
V2. Vector addition is associative.
$(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$ for all $\vec{u}, \vec{v}, \vec{w} \in F^n$.
V3. Vector addition is commutative.
$\vec{v} + \vec{w} = \vec{w} + \vec{v}$ for all $\vec{v}, \vec{w} \in F^n$.
V4. There exists an additive identity in F^n.
$(\exists \vec{0} \in F^n)(\forall \vec{v} \in F^n)(\vec{v} + \vec{0} = \vec{0} + \vec{v} = \vec{v})$.
V5. Every vector in F^n has an additive inverse.
$(\forall \vec{v} \in F^n)(\exists -\vec{v} \in F^n)[\vec{v} + (-\vec{v}) = -\vec{v} + \vec{v} = \vec{0}]$.
V6. F^n is closed under scalar multiplication.
$\lambda \vec{v} \in F^n$ for all $\lambda \in F$ and for all $\vec{v} \in F^n$.

V7. Associativity of scalar multiplication with field multiplication
$\alpha(\beta\vec{v}) = (\alpha\beta)\vec{v}$ for all $\vec{v} \in F^n$ and for all $\alpha, \beta \in F$.
V8. Left distributive law of scalar multiplication over vector addition
$\lambda(\vec{v} + \vec{w}) = \lambda\vec{v} + \lambda\vec{w}$ for all $\vec{v}, \vec{w} \in F^n$ and for all $\lambda \in F$.
V9. Right distributive law of scalar multiplication over scalar addition
$(\alpha + \beta)\vec{v} = \alpha\vec{v} + \beta\vec{v}$ for all $\vec{v} \in F^n$ and for all $\alpha, \beta \in F$.
V10. $1 \in F$ is a multiplicative identity for scalar multiplication.
$1 \cdot \vec{v} = \vec{v}$ for all $\vec{v} \in F^n$.

Example 17. $(\mathbb{R}^n, +, \cdot)$ **is a real vector space.**

Proof. Let $n \in \mathbb{Z}^+$.

Since $(\mathbb{R}, +, \cdot)$ is a field, then the set $\mathbb{R}^n$ with component-wise addition defined by $(v_1, ..., v_n)+(w_1, ..., w_n) = (v_1+w_1, ..., v_n+w_n)$ for all $(v_1, ..., v_n), (w_1, ..., w_n) \in \mathbb{R}^n$, and scalar multiplication defined by $\lambda(v_1, v_2, ..., v_n) = (\lambda v_1, \lambda v_2, ..., \lambda v_n)$ for all $\lambda \in \mathbb{R}$ and for all $(v_1, ..., v_n) \in \mathbb{R}^n$, is a vector space over $\mathbb{R}$.

Therefore, $(\mathbb{R}^n, +, \cdot)$ is a vector space over $\mathbb{R}$, so $(\mathbb{R}^n, +, \cdot)$ is a real vector space. $\square$

The vector space $(\mathbb{R}^n, +, \cdot)$ over $\mathbb{R}$ satisfies the below axioms.
V1. $\mathbb{R}^n$ is closed under vector addition.
$\vec{v} + \vec{w} \in \mathbb{R}^n$ for all $\vec{v}, \vec{w} \in \mathbb{R}^n$.
V2. Vector addition is associative.
$(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$ for all $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^n$.
V3. Vector addition is commutative.
$\vec{v} + \vec{w} = \vec{w} + \vec{v}$ for all $\vec{v}, \vec{w} \in \mathbb{R}^n$.
V4. There exists an additive identity in $\mathbb{R}^n$.
$(\exists\vec{0} \in \mathbb{R}^n)(\forall\vec{v} \in \mathbb{R}^n)(\vec{v} + \vec{0} = \vec{0} + \vec{v} = \vec{v})$.
V5. Every vector in $\mathbb{R}^n$ has an additive inverse.
$(\forall\vec{v} \in \mathbb{R}^n)(\exists - \vec{v} \in \mathbb{R}^n)[\vec{v} + (-\vec{v}) = -\vec{v} + \vec{v} = \vec{0}]$.
V6. $\mathbb{R}^n$ is closed under scalar multiplication.
$\lambda\vec{v} \in \mathbb{R}^n$ for all $\lambda \in \mathbb{R}$ and for all $\vec{v} \in \mathbb{R}^n$.
V7. Associativity of scalar multiplication with field multiplication
$\alpha(\beta\vec{v}) = (\alpha\beta)\vec{v}$ for all $\vec{v} \in \mathbb{R}^n$ and for all $\alpha, \beta \in \mathbb{R}$.
V8. Left distributive law of scalar multiplication over vector addition
$\lambda(\vec{v} + \vec{w}) = \lambda\vec{v} + \lambda\vec{w}$ for all $\vec{v}, \vec{w} \in \mathbb{R}^n$ and for all $\lambda \in \mathbb{R}$.
V9. Right distributive law of scalar multiplication over scalar addition
$(\alpha + \beta)\vec{v} = \alpha\vec{v} + \beta\vec{v}$ for all $\vec{v} \in \mathbb{R}^n$ and for all $\alpha, \beta \in \mathbb{R}$.
V10. $1 \in \mathbb{R}$ is a multiplicative identity for scalar multiplication.
$1 \cdot \vec{v} = \vec{v}$ for all $\vec{v} \in \mathbb{R}^n$.

Example 18. $\mathbb{R}$ **is a** 1**-dimensional real vector space.**

$\mathbb{R} = \{x : x \text{ is a real number}\}$.

$\mathbb{R}$ is the real-coordinate space of dimension 1 and corresponds to Euclidean 1-dimensional space (Euclidean line).

Example 19. $\mathbb{R}^2$ **is a 2-dimensional real vector space.**

The set of all ordered pairs of real numbers is $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R} = \{(x_1, x_2) : x_k \in \mathbb{R}$ for each $k \in \{1, 2\}\} = \{(x_1, x_2) : x_1 \in \mathbb{R}$ and $x_2 \in \mathbb{R}\}$.

Therefore, $\mathbb{R}^2 = \{(x, y) : x, y \in \mathbb{R}\}$.

$\mathbb{R}^2$ is the set of all lists of 2 real numbers.

$\mathbb{R}^2$ is the real-coordinate space of dimension 2 and corresponds to Euclidean 2-dimensional space (Euclidean plane).

Example 20. $\mathbb{R}^3$ **is a 3-dimensional real vector space.**

The set of all ordered triples of real numbers is $\mathbb{R}^3 = \mathbb{R}\times\mathbb{R}\times\mathbb{R} = \{(x_1, x_2, x_3) : x_k \in \mathbb{R}$ for each $k \in \{1, 2, 3\}\} = \{(x_1, x_2, x_3) : x_1 \in \mathbb{R}$ and $x_2 \in \mathbb{R}$ and $x_3 \in \mathbb{R}\}$.

Therefore, $\mathbb{R}^3 = \{(x, y, z) : x, y, z \in \mathbb{R}\}$.

$\mathbb{R}^3$ is the set of all lists of 3 real numbers.

$\mathbb{R}^3$ is the real-coordinate space of dimension 3 and corresponds to Euclidean 3-dimensional space (Euclidean space).

Example 21. $\mathbb{R}^n$ **is an** n**-dimensional real space.**

Let $n \in \mathbb{Z}^+$.

The set of all ordered n-tuples of real numbers is $\mathbb{R}^n = \mathbb{R} \times \mathbb{R} \times \ldots \times \mathbb{R} = \{(x_1, ..., x_n) : x_k \in \mathbb{R}$ for each $k \in \{1, ..., n\}\}$.

$\mathbb{R}^n$ is the set of all lists of n real numbers.

$\mathbb{R}^n$ is the real-coordinate space of dimension n and corresponds to Euclidean n-dimensional space.

Example 22. $(\mathbb{C}^n, +, \cdot)$ **is a complex vector space.**

Proof. Let $n \in \mathbb{Z}^+$.

Since $(\mathbb{C}, +, \cdot)$ is a field, then the set $\mathbb{C}^n$ with component-wise addition defined by $(v_1, ..., v_n)+(w_1, ..., w_n) = (v_1+w_1, ..., v_n+w_n)$ for all $(v_1, ..., v_n), (w_1, ..., w_n) \in \mathbb{C}^n$, and scalar multiplication defined by $\lambda(v_1, v_2, ..., v_n) = (\lambda v_1, \lambda v_2, ..., \lambda v_n)$ for all $\lambda \in \mathbb{C}$ and for all $(v_1, ..., v_n) \in \mathbb{C}^n$, is a vector space over $\mathbb{C}$.

Therefore, $(\mathbb{C}^n, +, \cdot)$ is a vector space over $\mathbb{C}$, so $(\mathbb{C}^n, +, \cdot)$ is a complex vector space. □

The vector space $(\mathbb{C}^n, +, \cdot)$ over $\mathbb{C}$ satisfies the below axioms.

V1. $\mathbb{C}^n$ is closed under vector addition.

$\vec{v} + \vec{w} \in \mathbb{C}^n$ for all $\vec{v}, \vec{w} \in \mathbb{C}^n$.

V2. Vector addition is associative.

$(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$ for all $\vec{u}, \vec{v}, \vec{w} \in \mathbb{C}^n$.

V3. Vector addition is commutative.

$\vec{v} + \vec{w} = \vec{w} + \vec{v}$ for all $\vec{v}, \vec{w} \in \mathbb{C}^n$.

V4. There exists an additive identity in $\mathbb{C}^n$.

$(\exists\vec{0} \in \mathbb{C}^n)(\forall\vec{v} \in \mathbb{C}^n)(\vec{v} + \vec{0} = \vec{0} + \vec{v} = \vec{v})$.

V5. Every vector in $\mathbb{C}^n$ has an additive inverse.

$(\forall\vec{v} \in \mathbb{C}^n)(\exists - \vec{v} \in \mathbb{C}^n)[\vec{v} + (-\vec{v}) = -\vec{v} + \vec{v} = \vec{0}]$.

V6. $\mathbb{C}^n$ is closed under scalar multiplication.

$\lambda\vec{v} \in \mathbb{C}^n$ for all $\lambda \in \mathbb{C}$ and for all $\vec{v} \in \mathbb{C}^n$.

V7. Associativity of scalar multiplication with field multiplication

$\alpha(\beta\vec{v}) = (\alpha\beta)\vec{v}$ for all $\vec{v} \in \mathbb{C}^n$ and for all $\alpha, \beta \in \mathbb{C}$.
V8. Left distributive law of scalar multiplication over vector addition
$\lambda(\vec{v} + \vec{w}) = \lambda\vec{v} + \lambda\vec{w}$ for all $\vec{v}, \vec{w} \in \mathbb{C}^n$ and for all $\lambda \in \mathbb{C}$.
V9. Right distributive law of scalar multiplication over scalar addition
$(\alpha + \beta)\vec{v} = \alpha\vec{v} + \beta\vec{v}$ for all $\vec{v} \in \mathbb{C}^n$ and for all $\alpha, \beta \in \mathbb{C}$.
V10. $1 \in \mathbb{C}$ is a multiplicative identity for scalar multiplication.
$1 \cdot \vec{v} = \vec{v}$ for all $\vec{v} \in \mathbb{C}^n$.

Example 23. $\mathbb{C}^n$ is an n-dimensional complex vector space.
Let $n \in \mathbb{Z}^+$.
The set of all ordered n-tuples of complex numbers is $\mathbb{C}^n = \mathbb{C} \times \mathbb{C} \times \ldots \times \mathbb{C} = \{(z_1, ..., z_n) : z_k \in \mathbb{C} \text{ for each } k \in \{1, ..., n\}\}$.
$\mathbb{C}^n$ is the set of all lists of n complex numbers.
$\mathbb{C}^n$ is the complex-coordinate space of dimension n.

Example 24. $\mathbb{C}^1$ is a 1-dimensional complex space.
The set of all ordered 1-tuples of complex numbers is $\mathbb{C}^1 = \{(z) : z \in \mathbb{C}\}$.
$\mathbb{C}^1$ is the set of all lists of 1 complex number.
$\mathbb{C}^1$ is the complex-coordinate space of dimension 1.
$\mathbb{C}^1$ corresponds to the Euclidean plane $\mathbb{R}^2$.

Example 25. $\mathbb{C}^4$ is a 4-dimensional complex space.
The set of all ordered 4-tuples of complex numbers is $\mathbb{C}^4 = \{(z_1, z_2, z_3, z_4) : z_1, z_2, z_3, z_4 \in \mathbb{C}\}$.
$\mathbb{C}^4$ is the set of all lists of 4 complex numbers.
$\mathbb{C}^4$ is the complex-coordinate space of dimension 4.

Example 26. Vectors in $\mathbb{R}^2$
Since $(\mathbb{R}, +, \cdot)$ is a field, then $(\mathbb{R}^2, +)$ is an abelian group under vector addition.
Let $\vec{v} = (x, y)$ be a vector in $\mathbb{R}^2$.
The **length of** v is $\sqrt{x^2 + y^2}$.
Two vectors in $\mathbb{R}^2$ are the 'same' if they have the same length and same direction.

Example 27. geometric meaning of scalar multiplication in $\mathbb{R}^2$
Let F be a field.
Let $\lambda \in F$ and $\vec{v}$ be a vector in $\mathbb{R}^2$.
Then $\lambda\vec{v}$ is the vector that points in the same direction as $\vec{v}$.
If $\lambda > 1$, $\vec{v}$ is stretched by a factor of λ.
If $\lambda < 1$, $\vec{v}$ is shrunk by a factor of λ.
If $\lambda < 0$, then $\lambda\vec{v}$ is the vector that points in the opposite direction to that of $\vec{v}$ and whose length is $|\lambda|$ times the length of $\vec{v}$.

TODO: Rework the sections below.

Example 28. $\mathbb{R}^n$ **is a vector space over** $\mathbb{R}$

Let $n \in \mathbb{Z}^+$.
Let $\alpha \in \mathbb{R}$.
Let $\mathbb{R}^n = \{(r_1, r_2, r_3, ..., r_n) : r_i \in \mathbb{R}\}$.
Let $\vec{v}, \vec{w} \in \mathbb{R}^n$ such that $\overrightarrow{v} = (v_1, v_2, ..., v_n)$ and $\overrightarrow{w} = (w_1, w_2, ..., w_n)$.
Define $+$ by:

$$\overrightarrow{v} + \overrightarrow{w} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} + \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{pmatrix} = \begin{pmatrix} v_1 + w_1 \\ v_2 + w_2 \\ \vdots \\ v_n + w_n \end{pmatrix}$$

Define $\cdot$ by:

$$\alpha\vec{v} = \alpha \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} = \begin{pmatrix} \alpha v_1 \\ \alpha v_2 \\ \vdots \\ \alpha v_n \end{pmatrix}$$

Then $\mathbb{R}^n$ under vector addition and scalar multiplication is an n dimensional **real vector space**.

$\mathbb{R}^1 = \mathbb{R} = \{x : x \in \mathbb{R}\}$ is the real number line.
Subspaces of $\mathbb{R}^1$ are trivial vector space and $\mathbb{R}^1$.
Therefore, $\mathbb{R}^1$ does not have any proper subspaces.

Example 29. Every line in $\mathbb{R}^2$ **passing through the origin with slope** m **is a vector space.**

Let $m \in \mathbb{R}$ be fixed.
Let L be a line in $\mathbb{R}^2$ passing through the origin with slope m.
Then $L = \{(x, y) \in \mathbb{R}^2 : y = mx\}$.
Therefore, $(L, +, \cdot)$ is a vector space.

Proof. We prove $(L, +, \cdot)$ is a vector space using the two-step subspace test.
Observe that $(\mathbb{R}^2, +, \cdot)$ is a vector space over $\mathbb{R}$ and $L \subset \mathbb{R}^2$.
Since $0 \in \mathbb{R}$, then $(0, 0) \in \mathbb{R}^2$.
Since $(0, 0) \in \mathbb{R}^2$ and $0 = m \cdot 0$, then $(0, 0) \in L$, so $L \neq \emptyset$.
Therefore, $L \subset \mathbb{R}^2$ and $L \neq \emptyset$, so L is a nonempty subset of $\mathbb{R}^2$.

We prove L is closed under vector addition defined on $\mathbb{R}^2$.
Let $p, q \in L$.
Since $p \in L$, then there exist real numbers x_1 and y_1 such that $p = (x_1, y_1)$ and $y_1 = mx_1$.
Since $q \in L$, then there exist real numbers x_2 and y_2 such that $q = (x_2, y_2)$ and $y_2 = mx_2$.
Thus, $p+q = (x_1, y_1)+(x_2, y_2) = (x_1+x_2, y_1+y_2)$ and $y_1+y_2 = mx_1+mx_2 = m(x_1 + x_2)$.

Since $(\mathbb{R}^2, +, \cdot)$ is a vector space, then $\mathbb{R}^2$ is closed under addition.

Since $(x_1, y_1) \in \mathbb{R}^2$ and $(x_2, y_2) \in \mathbb{R}^2$, then $p \in \mathbb{R}^2$ and $q \in \mathbb{R}^2$, so $p+q \in \mathbb{R}^2$.

Since $p + q \in \mathbb{R}^2$ and $p + q = (x_1 + x_2, y_1 + y_2)$ and $y_1 + y_2 = m(x_1 + x_2)$, then $p + q \in L$.

Therefore, $p \in L$ and $q \in L$ imply $p + q \in L$, so L is closed under vector addition defined on $\mathbb{R}^2$.

We prove L is closed under scalar multiplication defined on $\mathbb{R}^2$.

Let $\lambda \in \mathbb{R}$ and $p \in L$.

Since $p \in L$, then there exist real numbers x_1 and y_1 such that $p = (x_1, y_1)$ and $y_1 = mx_1$.

Since $p \in L$ and $L \subset \mathbb{R}^2$, then $p \in \mathbb{R}^2$.

Since $(\mathbb{R}^2, +, \cdot)$ is a vector space, then $\mathbb{R}^2$ is closed under scalar multiplication.

Since $\lambda \in \mathbb{R}$ and $p \in \mathbb{R}^2$, then we conclude $\lambda p \in \mathbb{R}^2$.

Observe that

$$\begin{aligned} \lambda p &= \lambda(x_1, y_1) \\ &= (\lambda x_1, \lambda y_1). \end{aligned}$$

Thus, $\lambda p = (\lambda x_1, \lambda y_1)$.

Observe that

$$\begin{aligned} \lambda y_1 &= \lambda(m x_1) \\ &= (\lambda m) x_1 \\ &= (m \lambda) x_1 \\ &= m(\lambda x_1). \end{aligned}$$

Hence, $\lambda y_1 = m(\lambda x_1)$.

Since $\lambda p \in \mathbb{R}^2$ and $\lambda p = (\lambda x_1, \lambda y_1)$ and $\lambda y_1 = m(\lambda x_1)$, then $\lambda p \in L$.

Therefore, $\lambda \in \mathbb{R}$ and $p \in L$ imply $\lambda p \in L$, so L is closed under scalar multiplication defined on $\mathbb{R}^2$.

Since $(\mathbb{R}^2, +, \cdot)$ is a vector space, and L is a nonempty subset of $\mathbb{R}^2$, and L is closed under vector addition and scalar multiplication defined on $(\mathbb{R}^2, +, \cdot)$, then by the two-step subspace test, L is a subspace of V, so $(L, +, \cdot)$ is a vector space. $\square$

Example 30. Every plane in $\mathbb{R}^3$ passing through the origin with normal vector (a, b, c) is a vector space.

Let P be a plane in $\mathbb{R}^3$ passing through the origin with normal vector (a, b, c).

Then $P = \{(x, y, z) \in \mathbb{R}^3 : ax + by + cz = 0\}$ for fixed $a, b, c \in \mathbb{R}$.

Therefore, $(P, +, \cdot)$ is a vector space.

Proof. TODO: Start here. Page 174. $\square$

Example 31. vector space P_n of polynomials

Let $n \in \mathbb{Z}^+$.

Let P_n = the set of polynomials with real coefficients of degree $\leq n$.

Let $p, q \in P_n$.

Then

$p(x) = a_n x^n + a_{n-1}x^{n-1} + ... + a_1 x + a_0$ and

$q(x) = b_n x^n + b_{n-1}x^{n-1} + ... + b_1 x + b_0$ for $a_i, b_i \in \mathbb{R}$.

Define $p + q$ by $p(x) + q(x) = (a_n + b_n)x^n + (a_{n-1} + b_{n-1})x^{n-1} + ... + (a_1 + b_1)x + (a_0 + b_0)$.

Then P_n is a real vector space.

Example 32. vector space of all $m \times n$ matrices with real entries

Let $M_{m\times n}(\mathbb{R})$ be the set of all $m \times n$ matrices with real entries.

Then $M_{m\times n}(\mathbb{R}) = \{(a_{ij})_{m\times n} : a_{ij} \in \mathbb{R}\}$.

$M_{m\times n}(\mathbb{R})$ under matrix addition and scalar multiplication by $k \in \mathbb{R}$ is a vector space over the field $\mathbb{R}$.

$M_{m\times n}(\mathbb{R})$ is a **real vector space**.

Example 33. vector space of all $m \times n$ matrices with complex entries

Let $M_{m\times n}(\mathbb{C})$ be the set of all $m \times n$ matrices with complex entries.

Then $M_{m\times n}(\mathbb{C}) = \{(a_{ij})_{m\times n} : a_{ij} \in \mathbb{C}\}$.

$M_{m\times n}(\mathbb{C})$ under matrix addition and scalar multiplication by $k \in \mathbb{C}$ is a vector space over the field $\mathbb{C}$.

$M_{m\times n}(\mathbb{C})$ is a **complex vector space**.

Linear Subspaces

Linear Independence

Linear Transformations