Linear Algebra Exercises

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1 Linear Algebra Exercises

Exercise 1. Let A and B be $m \times n$ matrices with $A^t = B^t$. Prove $A = B$.

Solution. Our hypothesis is $A = (a_{ij})_{m\times n}$ and $B = (b_{ij})_{m\times n}$ and $A^t = B^t$. To prove our conclusion $A = B$, we must prove $(\forall i \in \mathbb{N}_m)(\forall j \in \mathbb{N}_n)(a_{ij} = b_{ij})$. Since this is a universally quantified statement, we let i, j be arbitrary integers such that $1 \leq i \leq m$ and $1 \leq j \leq n$. We must prove $a_{ij} = b_{ij}$. □

Proof. Let $A = (a_{ij})_{m\times n}$ and $B = (b_{ij})_{m\times n}$ and $A^t = B^t$. To prove $A = B$, we must prove $(\forall i \in \mathbb{N}_m)(\forall j \in \mathbb{N}_n)(a_{ij} = b_{ij})$. Matrices A and B have the same size $m \times n$. Let $i \in \mathbb{N}_m$ and $j \in \mathbb{N}_n$ be arbitrary integers. Let a_{ij} be the ij^{th} entry of A. Let b_{ij} be the ij^{th} entry of B. We must prove $a_{ij} = b_{ij}$.

Observe that

$$\begin{aligned} a_{ij} \text{ of } A &= a_{ji} \text{ of } A^t \\ &= b_{ji} \text{ of } B^t \\ &= b_{ij} \text{ of } B \end{aligned}$$

□

Exercise 2. If A and B are symmetric square matrices, then $A+B$ is symmetric.

Solution. Hypothesis is A and B are symmetric square matrices. To prove the conclusion $A + B$ is symmetric, we must prove $(A + B)^t = A + B$. In order that $A + B$ exist, it must be true that A and B must have the same size. Since A is symmetric then $A^t = A$. Since B is symmetric then $B^t = B$. □

Proof. Let A and B be arbitrary symmetric square matrices such that $A + B$ exists. Since A and B are symmetric, then $A^t = A$ and $B^t = B$. To prove $A+B$ is symmetric, we must prove $(A+B)^t = A+B$. Observe that $(A+B)^t = A^t + B^t = A + B$. □

Exercise 3. Let matrix ABC exist. Then $C^t B^t A^t = (ABC)^t$.

Solution. Our hypothesis is matrix ABC exists. To prove our conclusion $C^tB^tA^t = (ABC)^t$, we must prove $(\forall i)(\forall j)(d_{ij} = e_{ij})$. Since this is a universally quantified statement, we let i, j be arbitrary integers. We must prove $d_{ij} = e_{ij}$ where d_{ij} is the ij entry of $C^tB^tA^t$ and e_{ij} is the ij entry of $(ABC)^t$. □

Proof. Since matrix ABC exists and matrix multiplication is associative, then $ABC = (AB)C = A(BC)$. Since $ABC = A(BC)$ and ABC exists, then $A(BC)$ exists. Hence, A exists. Therefore, let $A = (a_{ij})_{m\times n}$. Then A has size $m \times n$.

Since $ABC = (AB)C$ and ABC exists, then $(AB)C$ exists. Hence, AB exists. Thus, B exists. Since AB exists and A has size $m \times n$, let $B = (b_{ij})_{n\times p}$. Then AB has size $m \times p$.

Since $(AB)C$ exists, then C exists. Since $(AB)C$ exists and AB has size $m \times p$, let $C = (c_{ij})_{p\times q}$. Then C has size $p \times q$.

Since AB has size $m \times p$ and C has size $p \times q$, then $(AB)C$ has size $m \times q$. Since $(AB)C = ABC$, then ABC has size $m \times q$. Thus, $(ABC)^t$ has size $q \times m$.

Since C has size $p \times q$, then C^t has size $q \times p$. Since B has size $n \times p$, then B^t has size $p \times n$. Thus, C^tB^t has size $q \times n$. Since A has size $m \times n$, then A^t has size $n \times m$. Thus, $(C^tB^t)A^t$ has size $q \times m$. Since matrix multiplication is associative, then $(C^tB^t)A^t = C^t(B^tA^t) = C^tB^tA^t$. Hence, $C^tB^tA^t$ has size $q \times m$.

Therefore, matrices $C^tB^tA^t$ and $(ABC)^t$ have the same size $q \times m$.

To prove $C^tB^tA^t = (ABC)^t$, we must prove $(\forall i \in \mathbb{N}_q)(\forall j \in \mathbb{N}_m)(d_{ij} = e_{ij})$. Thus, we let $i \in \mathbb{N}_q$ and $j \in \mathbb{N}_m$ be arbitrary integers where d_{ij} is the ij entry of $C^tB^tA^t$ and e_{ij} is the ij entry of $(ABC)^t$. We must prove $d_{ij} = e_{ij}$.

Observe that

$$\begin{aligned}
e_{ij} \text{ of } (ABC)^t &= e_{ji} \text{ of } ABC \\
&= e_{ji} \text{ of } (AB)C \\
&= \sum_{k=1}^{p} f_{jk}c_{ki} \text{ where } f_{jk} \text{ in } AB \text{ and } c_{ki} \text{ in } C \\
&= \sum_{k=1}^{p} c_{ki}f_{jk} \text{ where } c_{ki} \text{ in } C \text{ and } f_{jk} \text{ in } AB \\
&= \sum_{k=1}^{p} c_{ki}f_{kj} \text{ where } c_{ki} \text{ in } C \text{ and } f_{kj} \text{ in } (AB)^t \\
&= \sum_{k=1}^{p} c_{ik}f_{kj} \text{ where } c_{ik} \text{ in } C^t \text{ and } f_{kj} \text{ in } (AB)^t \\
&= \sum_{k=1}^{p} c_{ik}f_{kj} \text{ where } c_{ik} \text{ in } C^t \text{ and } f_{kj} \text{ in } B^tA^t \\
&= d_{ij} \text{ of } C^t(B^tA^t) \\
&= d_{ij} \text{ of } C^tB^tA^t
\end{aligned}$$

□

Exercise 4. Let $A = (a_{ij})_{n\times n}$ and $B = (B_{ij})_{n\times n}$ be square diagonal matrices. Then AB is diagonal.

Solution. Our hypothesis is: $A = (a_{ij})_{n\times n}$ is a square diagonal matrix and $B = (B_{ij})_{n\times n}$ is a square diagonal matrix. To prove our conclusion $AB = (c_{ij})_{n\times n}$ is diagonal, where $c_{ij} = \sum_{k=1}^{n} a_{ik}b_{kj}$, we let $i, j \in \mathbb{N}_n$ be arbitrary such that $i \neq j$. We must prove $(c_{ij})_{n\times n} = 0$. □

Proof. Let $A = (a_{ij})_{n\times n}$ be a square diagonal matrix.

Let $B = (b_{ij})_{n\times n}$ be a square diagonal matrix.

Let $AB = (c_{ij})_{n\times n}$ where $c_{ij} = \sum_{k=1}^{n} a_{ik}b_{kj}$.

To prove AB is diagonal, we let $i, j \in \mathbb{N}_n$ be arbitrary such that $i \neq j$.

We must prove $c_{ij} = 0$.

Since A is diagonal, then for every $x, y \in \mathbb{N}_n$ such that $x \neq y$, then $a_{xy} = 0$. Since B is diagonal, then for every $x, y \in \mathbb{N}_n$ such that $x \neq y$, then $b_{xy} = 0$.

Thus, in particular, if we let $x = i$ and $y = j$, then if $i \neq j$, then $a_{ij} = b_{ij} = 0$. Since $c_{ij} = \sum_{k=1}^{n} a_{ik}b_{kj}$, then k goes from 1 to n. Thus, either $k = i$ or $k \neq i$ or $k = j$ or $k \neq j$. If $k = i$, then since $i \neq j$, then $k \neq j$. If $k = j$, then since $j \neq i$, then $k \neq i$. Hence, either $k \neq i$ or $k \neq j$.

There are 2 cases to consider.

Case 1: Suppose $k \neq i$.

Then $a_{ik} = 0$. Then $a_{ik}b_{kj} = 0 * b_{kj} = 0$.

Case 2: Suppose $k \neq j$.

Then $b_{kj} = 0$. Then $a_{ik}b_{kj} = a_{ik} * 0 = 0$.

Hence, in all cases, the term $a_{ik}b_{kj}$ is zero. Since c_{ij} is the sum of n such terms, then c_{ij} is the sum of n zeros. Hence, $c_{ij} = 0$, as desired. □

Exercise 5. Let S be the set of all $n \times n$ matrices with real entries. For all $A, B \in S$, define a relation $\sim$ by $A \sim B$ iff there exists an invertible matrix P such that $B = PAP^{-1}$. Then $\sim$ is an equivalence relation on S.

Proof. Let $A \in S$. Then A is an $n \times n$ matrix of real entries. Let I be the $n \times n$ identity matrix. Then I is invertible and $AI = IA$, so $A = IAI^{-1}$. Thus, $A \sim A$, so $\sim$ is reflexive.

Let $A, B \in S$ such that $A \sim B$. Then A and B are $n \times n$ matrices of real entries and there exists an invertible matrix P such that $B = PAP^{-1}$. Thus, $P^{-1}B = AP^{-1}$, so $P^{-1}B(P^{-1})^{-1} = A$. Since $A = P^{-1}B(P^{-1})^{-1}$ and P^{-1} is invertible, then $B \sim A$. Hence, $\sim$ is symmetric.

Let $A, B, C \in S$ such that $A \sim B$ and $B \sim C$. Then A, B, C are $n \times n$ matrices of real entries and there exists an invertible matrix P such that $B = PAP^{-1}$ and there exists an invertible matrix Q such that $C = QBQ^{-1}$. Thus,

$$\begin{aligned} C &= Q(PAP^{-1})Q^{-1} \\ &= (QP)A(P^{-1}Q^{-1}) \\ &= (QP)A(QP)^{-1}. \end{aligned}$$

The product of invertible matrices is an invertible matrix. Hence, QP is an invertible matrix. Since QP is an invertible matrix and $C = (QP)A(QP)^{-1}$, then $A \sim C$. Hence, $\sim$ is transitive.

Since $\sim$ is reflexive, symmetric, and transitive, then $\sim$ is an equivalence relation on S. □

Exercise 6. What are the subspaces of $\mathbb{R}^1$?

Solution. We know $\mathbb{R}^1$ is a vector space and $\mathbb{R}^1$ and the trivial vector space $\{\vec{0}\}$ are subspaces of $\mathbb{R}^1$. Are these the only subspaces of $\mathbb{R}^1$? We know $\mathbb{R}^1 = \{(r) : r \in \mathbb{R}\}$ and if $\vec{r} \in \mathbb{R}^1$, then $\vec{r} = (r) = r$ and $r \in \mathbb{R}$. Let's assume there exists a nontrivial subspace of $\mathbb{R}^1$ and see what we can deduce. □

Proof. Suppose W is a nontrivial subspace of $\mathbb{R}^1$. Then by definition of subspace, $W \subseteq \mathbb{R}^1$ and W is a vector space. Since W is not the trivial vector space, then $\exists \vec{w} \in W$ that is non-zero. Let $\vec{w} \in W$. Then $\vec{w} \neq \vec{0}$. Since $W \subseteq \mathbb{R}^1$ then $\vec{w} \in \mathbb{R}^1$. Hence $\vec{w} = (w) = w$ and $w \in \mathbb{R}$, by definition of $\mathbb{R}^1$. Since $\vec{w} \neq \vec{0}$ and $\vec{0} = (0) = 0$ then $w \neq 0$. Since $\mathbb{R}$ is a field then by definition of field, $\mathbb{R}$ is a ring with unity $1 \in \mathbb{R}$. Since $\mathbb{R}$ is a field and $w \in \mathbb{R}$ is nonzero, then w is a unit. Hence $\exists \frac{1}{w} \in \mathbb{R}$ such that $w \cdot \frac{1}{w} = \frac{1}{w} \cdot w = 1$ by definition of unit. Thus, $\frac{1}{w} \cdot w = 1$, so $\frac{1}{w} \cdot \vec{w} = 1$. Since W is a vector space then W is closed under scalar multiplication. Hence, $\frac{1}{w} \cdot \vec{w} = \vec{1}$ so $\vec{1} \in W$.

Let $r \in \mathbb{R}$. Then $r = (r) = \vec{r}$, so $\vec{r} \in \mathbb{R}^1$. Since $(\mathbb{R}, +, \cdot)$ is a field then $1 \in \mathbb{R}$ is multiplicative identity. Hence, $1 \cdot r = r \cdot 1 = r$ by definition of multiplicative identity. Thus, $r \cdot 1 = r$. Consequently, $r \cdot \vec{1} = r$. Since W is closed under scalar multiplication then $r \cdot \vec{1} = \vec{r}$. Hence, $\vec{r} \in W$. Therefore, $\vec{r} \in \mathbb{R}^1$ implies $\vec{r} \in W$. Thus, $\mathbb{R}^1 \subseteq W$ by definition of subset.

Since $W \subseteq \mathbb{R}^1$ and $\mathbb{R}^1 \subseteq W$ then $W = \mathbb{R}^1$. Hence any nontrivial subspace of $\mathbb{R}^1$ must be $\mathbb{R}^1$ itself. Therefore, the only subspaces of $\mathbb{R}^1$ are the trivial subspace and $\mathbb{R}^1$. Thus, $\mathbb{R}^1$ does not have any proper subspaces. □

Exercise 7. Let $\vec{v}_1, \vec{v}_2$ and $\vec{v}_3$ be linearly independent vectors in a vector space V and let c be a nonzero scalar. Then $\{\vec{v}_1, c\vec{v}_2, \vec{v}_3\}$ is linearly independent.

Solution. Our hypothesis is V is a vector space over a field K and $\vec{v}_1, \vec{v}_2$ and $\vec{v}_3$ are linearly independent vectors of V and $c \in K$ and $c \neq 0$.

To prove our conclusion $\{\vec{v}_1, c\vec{v}_2, \vec{v}_3\}$ is linearly independent, we must prove for every $\alpha_1, \alpha_2, \alpha_3 \in K$, if $\alpha_1\vec{v}_1 + \alpha_2(c\vec{v}_2) + \alpha_3\vec{v}_3 = \vec{0}$, then $\alpha_1 = \alpha_2 = \alpha_3 = 0$.

Thus, we assume $\alpha_1, \alpha_2, \alpha_3 \in K$ are arbitrary scalars such that $\alpha_1\vec{v}_1 + \alpha_2(c\vec{v}_2) + \alpha_3\vec{v}_3 = \vec{0}$. We must prove $\alpha_1 = \alpha_2 = \alpha_3 = 0$.

Since $\vec{v}_1, \vec{v}_2$ and $\vec{v}_3$ is linearly independent, then for every $\lambda_1, \lambda_2, \lambda_3 \in K$, if $\lambda_1\vec{v}_1 + \lambda_2\vec{v}_2 + \lambda_3\vec{v}_3 = \vec{0}$, then $\lambda_1 = \lambda_2 = \lambda_3 = 0$.

Observe that $\vec{0} = \alpha_1\vec{v}_1 + \alpha_2(c\vec{v}_2) + \alpha_3\vec{v}_3 = \alpha_1\vec{v}_1 + (\alpha_2 c)\vec{v}_2 + \alpha_3\vec{v}_3$. Since $\alpha_2 c \in K$, then let $\lambda_1 = \alpha_1$ and $\lambda_2 = \alpha_2 c$ and $\lambda_3 = \alpha_3$.

□

Proof. Let V be a vector space over a field K. Let $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ be a linearly independent set of vectors of V. Let $c \in K$ and $c \neq 0$.

Suppose $\alpha_1, \alpha_2, \alpha_3 \in K$ are arbitrary scalars such that $\alpha_1\vec{v}_1 + \alpha_2(c\vec{v}_2) + \alpha_3\vec{v}_3 = \vec{0}$.

Observe that $\vec{0} = \alpha_1\vec{v}_1 + \alpha_2(c\vec{v}_2) + \alpha_3\vec{v}_3 = \alpha_1\vec{v}_1 + (\alpha_2 c)\vec{v}_2 + \alpha_3\vec{v}_3$.

Since $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is linearly independent, then for every $\lambda_1, \lambda_2, \lambda_3 \in K$, if $\lambda_1\vec{v}_1 + \lambda_2\vec{v}_2 + \lambda_3\vec{v}_3 = \vec{0}$, then $\lambda_1 = \lambda_2 = \lambda_3 = 0$.

Observe that $\alpha_1 \in K$ and $\alpha_2 c \in K$, and $\alpha_3 \in K$.

Hence, in particular, if we let $\lambda_1 = \alpha_1$ and $\lambda_2 = \alpha_2 c$ and $\lambda_3 = \alpha_3$, then we have if $\alpha_1\vec{v}_1 + (\alpha_2 c)\vec{v}_2 + \alpha_3\vec{v}_3 = \vec{0}$, then $\alpha_1 = \alpha_2 c = \alpha_3 = 0$.

Since $\alpha_1\vec{v}_1 + (\alpha_2 c)\vec{v}_2 + \alpha_3\vec{v}_3 = \vec{0}$, then by modus ponens, $\alpha_1 = \alpha_2 c = \alpha_3 = 0$. Hence, $\alpha_1 = 0$ and $\alpha_3 = 0$ and $\alpha_2 c = 0$.

Since $\alpha_2 c = 0$, then either $\alpha_2 = 0$ or $c = 0$. Since $c \neq 0$, by hypothesis, then it follows that $\alpha_2 = 0$, by disjunctive syllogism.

Since $\alpha_1 = \alpha_2 = \alpha_3 = 0$, then $\{\vec{v}_1, c\vec{v}_2, \vec{v}_3\}$ is linearly independent.

□

Exercise 8. Let A be any square matrix. Then A is the sum of a symmetric and antisymmetric matrix.

Solution.

We must prove:

$(\forall A)(\exists B)(\exists C)(A = B + C)$, where A is a square matrix and B is a symmetric matrix and C is an anti-symmetric matrix.

Let A be an arbitrary square matrix.

We must find a specific matrix B that is symmetric and a specific matrix C that is anti-symmetric such that $A = B + C$.

We can work backwards.

Suppose $A = B + C$ for some symmetric matrix B and some anti-symmetric matrix C. Since B is symmetric, then $B^t = B$. Since C is anti-symmetric, then $C^t = -C$.

Since $C^t = -C$, then $C + C^t = 0$. Let $A = (a_{ij})_{n\times n}$. Let $B = (b_{ij})_{n\times n}$. Let $C = (c_{ij})_{n\times n}$. Assume arbitrary $i, j \in \mathbb{N}_n$. Then $c_{ij} + c_{ji} = 0$. Either $i = j$ or $i \neq j$. If $i = j$, then $0 = c_{ij} + c_{ji} = c_{ij} + c_{ij} = 2c_{ij}$, so $c_{ij} = 0$. This implies the principal diagonal of matrix C must be all zeros. If $i \neq j$, then $c_{ij} = -c_{ji}$.

Since $A = B + C$, then $a_{ij} = b_{ij} + c_{ij}$. Either $i = j$ or $i \neq j$.

Suppose $i = j$. Then $c_{ij} = 0$, so $a_{ij} = b_{ij}$. This means the principal diagonal of matrix B must be the same as the principal diagonal of matrix A.

Suppose $i \neq j$. Then $c_{ij} = -c_{ji}$. Since $B = B^t$, then $b_{ij} = b_{ji}$. Hence, $a_{ij} = b_{ij} + c_{ij}$ and $a_{ji} = b_{ji} + c_{ji}$. Thus, $a_{ij} = b_{ij} + c_{ij}$ and $a_{ji} = b_{ij} + c_{ji}$. Subtracting both equations, we obtain $a_{ij} - a_{ji} = c_{ij} - c_{ji}$. Thus, $a_{ij} - a_{ji} = c_{ij} + c_{ij} = 2c_{ij}$, so $c_{ij} = \frac{a_{ij} - a_{ji}}{2}$. Therefore, $C = \frac{1}{2}(A - A^t)$.

Since $a_{ij} = b_{ij} + c_{ij}$, then $b_{ij} = a_{ij} - c_{ij} = a_{ij} - \frac{1}{2}(a_{ij} - a_{ji}) = \frac{1}{2}(a_{ij} + a_{ji})$. Therefore, $B = \frac{1}{2}(A + A^t)$.

We should devise some concrete examples for matrices A, B, C that satisfy the criteria and verify this is the case.

Now, how can we actually prove this? We must show that $B = \frac{1}{2}(A + A^t)$ is a symmetric matrix and $C = \frac{1}{2}(A - A^t)$ is an anti-symmetric matrix.

We know that in general, for any square matrix X, $X + X^t$ is symmetric and $X - X^t$ is anti-symmetric. Hence, in particular, $A + A^t$ must be symmetric and $A - A^t$ must be anti-symmetric. Thus,we must now prove $\frac{1}{2}(A + A^t)$ is a symmetric and $\frac{1}{2}(A - A^t)$ is anti-symmetric.

This suggest a conjecture: If X is symmetric, is kX symmetric for some scalar k? Similarly, if X is antisymmetric, is kX anti-symmetric?

We need to prove these lemmas.

Let $k \in \mathbb{R}$. Suppose X is a symmetric matrix. Then $X^t = X$ and $(kX)^t = k(X^t) = kX$. Therefore, kX is symmetric.

Suppose X is an anti-symmetric matrix. Then $X^t = -X$ and $(kX)^t = k(X^t) = k(-X) = -kX$. Therefore, kX is anti-symmetric.

$\square$

Proof. Let A be an arbitrary square matrix.

We must find a specific matrix B that is symmetric and a specific matrix C that is anti-symmetric such that $A = B + C$.

Let $B = \frac{1}{2}(A + A^t)$ and $C = \frac{1}{2}(A - A^t)$.

If X is any square matrix, then $X + X^t$ is symmetric and $X - X^t$ is anti-symmetric. Hence, in particular, $A + A^t$ is symmetric and $A - A^t$ is anti-symmetric.

If X is any symmetric matrix, then any scalar multiple of X is symmetric. Hence, in particular, $\frac{1}{2}(A + A^t)$ is symmetric.

If X is any antisymmetric matrix, then any scalar multiple of X is antisymmetric. Hence, in particular, $\frac{1}{2}(A - A^t)$ is antisymmetric.

Therefore, B is symmetric and C is anti-symmetric.

Observe that

$$\begin{aligned} A &= \frac{2A}{2} \\ &= \frac{A + A}{2} \\ &= \frac{(A + A^t + A - A^t)}{2} \\ &= \frac{(A + A^t) + (A - A^t)}{2} \\ &= \frac{A + A^t}{2} + \frac{A - A^t}{2} \\ &= B + C, \text{ as desired.} \end{aligned}$$

$\square$