Linear Algebra Notes

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Linear algebra is the study of linear maps on finite dimensional vector spaces.

Matrix Theory

Definition 1. Real Matrix

A $m \times n$ **real matrix** is a rectangular array of m rows and n columns of real numbers.

Each $a_{ij} \in \mathbb{R}$ is an **entry** at row i and column j and $1 \leq i \leq m$ and $1 \leq j \leq n$.

Let $A = (a_{ij})_{m \times n}$ be a $m \times n$ matrix where $i \in \mathbb{N}_m$ and $j \in \mathbb{N}_n$.

Then

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

Definition 2. Equal Matrices

Two matrices are equal iff corresponding entries are equal.

Let $A = (a_{ij})_{m \times n}$ and $B = (b_{ij})_{m \times n}$.

Then $A = B$ iff $(\forall i \in \mathbb{N}_m)(\forall j \in \mathbb{N}_n)(a_{ij} = b_{ij})$.

Equal matrices have the same size.

Definition 3. Matrix Addition

The sum of two matrices is the sum of corresponding entries.

Let $A = (a_{ij})_{m \times n}$ and $B = (b_{ij})_{m \times n}$.

Then the matrix sum is defined by the rule $A + B = (c_{ij})_{m \times n}$ where $c_{ij} = a_{ij} + b_{ij}$.

Let $k \in \mathbb{R}$ be a scalar.

Then $kA = (ka_{ij})_{m \times n}$.

Definition 4. Matrix Multiplication

The entry at row i and column j of the matrix product is the dot product of the i^{th} row vector of matrix A with the j^{th} column vector of matrix B.

Let $A = (a_{ij})_{m\times n}$ and $B = (b_{ij})_{n\times p}$.

Then the matrix product is defined by the rule $AB = (c_{ij})_{m\times p}$ where $c_{ij} = \sum_{k=1}^{n} a_{ik}b_{kj}$.

Definition 5. Transpose of a Matrix

Let A be a matrix.

The transpose of A, denoted A^t, is the matrix obtained by transposing the rows and columns of A.

The i^{th} row of $A = i^{th}$ column of A^t.

The j^{th} column of $A = j^{th}$ row of A^t.

Let $A = (a_{ij})_{m\times n}$ with $i \in \mathbb{N}_m$ and $j \in \mathbb{N}_n$.

Then $A^t = (a_{ji})_{n\times m}$.

Let $A = (a_{ij})_{m\times n}$.

Then A^t is of size $n \times m$, so $(A^t)^t$ is of size $m \times n$.

Hence $(A^t)^t$ and A have the same size $m \times n$.

Let $i \in \mathbb{N}_m$ and $j \in \mathbb{N}_n$ be arbitrary.

Let a_{ij} be the ij^{th} entry of A.

Let b_{ij} be the ij^{th} entry of $(A^t)^t$.

Then

$$\begin{aligned} b_{ij} \text{ of } (A^t)^t &= b_{ji} \text{ of } A^t \\ &= b_{ij} \text{ of } A \\ &= a_{ij} \text{ of } A \end{aligned}$$

Therefore $(A^t)^t = A$.

Let $A = (a_{ij})_{m\times n}$ and $B = (b_{ij})_{m\times n}$.

Then $A + B$ is of size $m \times n$, so $(A + B)^t$ is of size $n \times m$.

Also, A^t and B^t are each of size $n \times m$, so $A^t + B^t$ is of size $n \times m$.

Hence $(A + B)^t$ and $A^t + B^t$ have the same size $n \times m$.

Let $i \in \mathbb{N}_n$ and $j \in \mathbb{N}_m$ be arbitrary.

Let c_{ij} be the ij^{th} entry of $(A + B)^t$.

Let d_{ij} be the ij^{th} entry of $A^t + B^t$.

Then

$$\begin{aligned} c_{ij} \text{ of } (A+B)^t &= c_{ji} \text{ of } A + B \\ &= a_{ji} + b_{ji} \text{ where } a_{ji} \text{ in } A \text{ and } b_{ji} \text{ in } B \\ &= a_{ij} + b_{ij} \text{ where } a_{ij} \text{ in } A^t \text{ and } b_{ij} \text{ in } B^t \\ &= d_{ij} \text{ of } A^t + B^t \end{aligned}$$

Therefore $(A + B)^t = A^t + B^t$.

Let $A = (a_{ij})_{m\times n}$ and $B = (b_{ij})_{n\times p}$.
Then AB is of size $m \times p$, so $(AB)^t$ is of size $p \times m$.
Also, B^t is of size $p \times n$ and A^t is of size $n \times m$, so B^tA^t is of size $p \times m$.
Hence $(AB)^t$ and B^tA^t have the same size $p \times m$.
Let $i \in \mathbb{N}_p$ and $j \in \mathbb{N}_m$ be arbitrary.
Let c_{ij} be the ij^{th} entry of $(AB)^t$.
Let d_{ij} be the ij^{th} entry of B^tA^t.
Then

$$\begin{aligned}
c_{ij} \text{ of } (AB)^t &= c_{ji} \text{ of } AB \\
&= \sum_{k=1}^{n} a_{jk}b_{ki} \text{ where } a_{jk} \text{ in } A \text{ and } b_{ki} \text{ in } B \\
&= \sum_{k=1}^{n} a_{kj}b_{ik} \text{ where } a_{kj} \text{ in } A^t \text{ and } b_{ik} \text{ in } B^t \\
&= \sum_{k=1}^{n} b_{ik}a_{kj} \text{ where } b_{ik} \text{ in } B^t \text{ and } a_{kj} \text{ in } A^t \\
&= d_{ij} \text{ of } B^tA^t
\end{aligned}$$

Therefore $(AB)^t = B^tA^t$.

Definition 6. Square Matrix
Let $A = (a_{ij})_{m\times n}$ be a $m \times n$ matrix where $i \in \mathbb{N}_m$ and $j \in \mathbb{N}_n$.
A matrix is **square** iff $m = n$.
Therefore, a square matrix has the same number of rows as columns.

Suppose $A = (a_{ij})_{m\times m}$ is a square matrix.
A is **symmetric** iff $A^t = A$.
A is **antisymmetric** iff $A^t = -A$.
A is **diagonal** iff $(\forall i, j \in \mathbb{N}_m)(i \neq j \to a_{ij} = 0)$.
A is **upper triangular** iff $(\forall i, j \in \mathbb{N}_m)(i > j \to a_{ij} = 0)$.
A is **lower triangular** iff $(\forall i, j \in \mathbb{N}_m)(i < j \to a_{ij} = 0)$.

A square matrix is diagonal iff it is both upper and lower triangular.
The sum of two square symmetric matrices is symmetric.

Let $A = (a_{ij})_{m\times m}$.
Then $A + A^t$ is symmetric and $A - A^t$ is antisymmetric.
Observe that $(A + A^t)^t = A^t + (A^t)^t = A^t + A = A + A^t$.
Observe that $(A - A^t)^t = [A + (-A^t)]^t = A^t + (-A^t)^t = A^t + ((-A)^t)^t = A^t + (-A) = A^t - A = -(A - A^t)$.

Definition 7. Determinant of a Matrix
Let $A = (a_{ij})_{2\times 2}$.
The **determinant of matrix** A is defined by the rule $|A| = a_{11}a_{22} - a_{12}a_{21}$.

Let A and B be 2×2 matrices.
Then $|AB| = |A||B|$.

Definition 8. Identity Matrix
Let $n \in \mathbb{Z}^+$.
The **identity matrix**, denoted I_n, is an $n \times n$ matrix with ones along the principal diagonal and zeros everywhere else.
Therefore, $I_n = (\delta_{ij})_{n \times n}$ such that for every $i, j \in \mathbb{N}_n$,

$$\delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$$

The identity matrix is a square matrix.

Let A be a $n \times n$ matrix.
Then $A = AI = IA$.

Definition 9. Invertible Matrix
Let A be a square matrix.
Let I be the identity matrix.
Then A **is invertible** iff $\exists$ a matrix B such that $AB = BA = I$.

Suppose A is invertible.
Then $AB = BA = I$ for some matrix B.
Since A is invertible, then $A = (a_{ij})_{n \times n}$.
Let $B = (b_{ij})_{m \times p}$.
Since the product AB is defined, then $n \times n$ matrix multiplied by a $m \times p$ matrix implies $n = m$.
Since $I = (\delta_{ij})_{n \times n}$, then $n \times p = n \times n$, so $p = n$.
Hence, $B = (b_{ij})_{n \times n}$, so B is a square matrix.
The inverse of an invertible matrix is a square matrix.

Let $n \in \mathbb{Z}^+$.
Let GL_n be the set of all $n \times n$ invertible matrices.
Then $GL_n = \{X : X \text{ is an } n \times n \text{ invertible matrix } \}$ = **general linear group**.
$(GL_n, \cdot)$ is a non-abelian group where $\cdot$ = matrix multiplication.
Identity of GL_n is I_n = identity matrix.
Inverse of matrix A is matrix A^{-1}, where $A^{-1} \in GL_n$ and $AA^{-1} = A^{-1}A = I$.
Since $n \times n$ invertible matrices $\subset$ $n \times n$ matrices $\subset$ $m \times n$ matrices, then
$GL_n \subset$ square matrices $\subset M_{m \times n}$.

Vector Spaces

Definition 10. Vector Space
Let $V = \{\vec{v} : \vec{v}$ is a **vector**$\}$.
Define binary operation $+ : V \times V \to V$ by $\vec{v} + \vec{w} \in V$ for all $\vec{v}, \vec{w} \in V$. (**vector addition**)
Let F be a field such that $F = \{\alpha : \alpha$ is a **scalar**$\}$.
Define binary operation $\cdot : F \times V \to V$ by $\alpha\vec{v} \in V$ for all $\alpha \in F, \vec{v} \in V$ (**scalar multiplication**).
A **vector space** V is an abelian group $(V, +)$ with a binary operation $\cdot$ **scalar multiplication** defined on V such that for all $\vec{v}, \vec{w} \in V$ and for all $\alpha, \beta \in F$ the following axioms hold:
V1. **Associative** $\alpha(\beta\vec{v}) = (\alpha\beta)\vec{v}$
V2. **Left Distributive** $\alpha(\vec{v} + \vec{w}) = \alpha\vec{v} + \alpha\vec{w}$
V3. **Right Distributive** $(\alpha + \beta)\vec{v} = \alpha\vec{v} + \beta\vec{v}$
V4. **Identity** $1 \cdot \vec{v} = \vec{v}$

Let V be a vector space over a field F.
Then V is closed under vector addition and scalar multiplication.
The **zero vector** $\vec{0}$ is additive identity.
$\vec{0} + \vec{0} = \vec{0}$

Proposition 11. *Let V be a vector space over field K.*
Let $\vec{v} \in V$ and $\alpha \in K$ be arbitrary.
Then the following are true:
1. $0\vec{v} = \vec{0}$
2. $\alpha\vec{0} = \vec{0}$
3. $\alpha\vec{v} = \vec{0}$ *iff* $\alpha = 0$ *or* $\vec{v} = \vec{0}$ *(or both)*
4. $(-1)\vec{v} = -\vec{v}$

Example 12. trivial vector space
The **trivial vector space** is $\{\vec{0}\}$.

Example 13. $\mathbb{R}^n$ is a vector space over $\mathbb{R}$
Let $n \in \mathbb{Z}^+$.
Let $\alpha \in \mathbb{R}$.
Let $\mathbb{R}^n = \{(r_1, r_2, r_3, ..., r_n) : r_i \in \mathbb{R}\}$.
Let $\vec{v}, \vec{w} \in \mathbb{R}^n$ such that $\overrightarrow{v} = (v_1, v_2, ..., v_n)$ and $\overrightarrow{w} = (w_1, w_2, ..., w_n)$.
Define $+$ by:

$$\overrightarrow{v} + \overrightarrow{w} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} + \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{pmatrix} = \begin{pmatrix} v_1 + w_1 \\ v_2 + w_2 \\ \vdots \\ v_n + w_n \end{pmatrix}$$

Define $\cdot$ by:

$$\alpha\vec{v} = \alpha \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} = \begin{pmatrix} \alpha v_1 \\ \alpha v_2 \\ \vdots \\ \alpha v_n \end{pmatrix}$$

Then $\mathbb{R}^n$ under vector addition and scalar multiplication is an n dimensional **real vector space**.

$\mathbb{R}^1 = \mathbb{R} = \{x : x \in \mathbb{R}\}$ is the real number line.
Subspaces of $\mathbb{R}^1$ are trivial vector space and $\mathbb{R}^1$.
Therefore, $\mathbb{R}^1$ does not have any proper subspaces.

$\mathbb{R}^2 = \mathbb{R} \times \mathbb{R} = \{(x, y) : x, y \in \mathbb{R}\}$ is the Euclidean 2 dimensional plane.
Let $V = \{(x, y) \in \mathbb{R}^2 : y = mx\}$ for some $m \in \mathbb{R}$.
Then V is the line with slope m passing through the origin.
Every line in $\mathbb{R}^2$ with slope m passing through the origin is a vector space.

$\mathbb{R}^3 = \mathbb{R} \times \mathbb{R} \times \mathbb{R} = \{(x, y, z) : x, y, z \in \mathbb{R}\}$ is the Euclidean 3 dimensional space.
Let $V = \{(x, y, z) \in \mathbb{R}^3 : ax + by + cz = 0\}$ for fixed $a, b, c \in \mathbb{R}$.
Then V is the plane passing through the origin with normal vector (a, b, c).
Every plane in $\mathbb{R}^3$ passing through the origin is a vector space.

Example 14. $\mathbb{C}^n$ is a vector space over $\mathbb{C}$
Let $n \in \mathbb{Z}^+$.
Let $\mathbb{C}^n = \{(z_1, z_2, z_3, ..., z_n) : z_i \in \mathbb{C}\}$.
Then $\mathbb{C}^n$ under vector addition and scalar multiplication is an n dimensional **complex vector space**.

Example 15. vector space P_n of polynomials
Let $n \in \mathbb{Z}^+$.
Let $P_n =$ the set of polynomials with real coefficients of degree $\leq n$.
Let $p, q \in P_n$.
Then
$p(x) = a_n x^n + a_{n-1}x^{n-1} + ... + a_1 x + a_0$ and
$q(x) = b_n x^n + b_{n-1}x^{n-1} + ... + b_1 x + b_0$ for $a_i, b_i \in \mathbb{R}$.
Define $p + q$ by $p(x) + q(x) = (a_n + b_n)x^n + (a_{n-1} + b_{n-1})x^{n-1} + ... + (a_1 + b_1)x + (a_0 + b_0)$.
Then P_n is a real vector space.

Example 16. vector space of all $m \times n$ matrices with real entries
Let $M_{m\times n}(\mathbb{R})$ be the set of all $m \times n$ matrices with real entries.
Then $M_{m\times n}(\mathbb{R}) = \{(a_{ij})_{m\times n} : a_{ij} \in \mathbb{R}\}$.
$M_{m\times n}(\mathbb{R})$ under matrix addition and scalar multiplication by $k \in \mathbb{R}$ is a vector space over the field $\mathbb{R}$.
$M_{m\times n}(\mathbb{R})$ is a **real vector space**.

Example 17. vector space of all $m \times n$ matrices with complex entries

Let $M_{m\times n}(\mathbb{C})$ be the set of all $m \times n$ matrices with complex entries.

Then $M_{m\times n}(\mathbb{C}) = \{(a_{ij})_{m\times n} : a_{ij} \in \mathbb{C}\}$.

$M_{m\times n}(\mathbb{C})$ under matrix addition and scalar multiplication by $k \in \mathbb{C}$ is a vector space over the field $\mathbb{C}$.

$M_{m\times n}(\mathbb{C})$ is a **complex vector space**.

Example 18. F^n is a vector space over the field F.

Let $n \in \mathbb{Z}^+$.

Let F be a field.

Let $F^n = \{(\alpha_1, \alpha_2, ..., \alpha_n) : \alpha_i \in F\}$.

Then F^n under vector addition and scalar multiplication is an **n-dimensional vector space over F**.

Definition 19. Linear Subspace

Let V be a vector space.

Then W is a **subspace of V** iff

1. $W \subseteq V$
2. W is a vector space under $+$ and $\cdot$ defined on V

Let V be a vector space with additive identity $\vec{0} \in V$.

V is a subspace of V (since $V \subseteq V$ and V is a vector space under $+$ and $\cdot$ defined on V)

$\{\vec{0}\}$ is a subspace of V (since $\vec{0} \in V$ and $\vec{0} + \vec{0} = \vec{0} \in \{\vec{0}\}$ and $\alpha\vec{0} = \vec{0} \in \{\vec{0}\}$)

A **proper subspace** is any subspace of V other than V or the trivial subspace.

Let V be a vector space over a field K.

Let $W \subseteq V$ and $W \neq \emptyset$.

Then W is a subspace of V iff

1. Closure under Vector Addition: $\vec{v} + \vec{w} \in W$ for all $\vec{v}, \vec{w} \in W$
2. Closure under Scalar Multiplication: $\alpha\vec{v} \in W$ for all $\vec{v} \in W, \alpha \in K$

Every subspace of V contains $\vec{0}$.

Thus, if W is a subspace of V, then $\vec{0} \in W$.

Therefore, if $W \subset V$ but $\vec{0} \notin W$, then W cannot be a subspace of V.

Linear Independence

Definition 20. Linear Independence of vectors

Let V be a vector space over a field K.

Let $\{\vec{v}_1, \vec{v}_2, ..., \vec{v}_n\}$ be a set of vectors in V.

The set of vectors is **linearly independent** iff $(\forall_{k=1}^{n} \alpha_k \in K)[(\sum_{k=1}^{n} \alpha_k \vec{v}_k = \vec{0}) \to (\forall_{k=1}^{n} k)(\alpha_k = 0)]$.

A set of vectors is **linearly dependent** iff it is not linearly independent.

Observe that $\neg(\forall_{k=1}^{n}\alpha_k \in K)[(\sum_{k=1}^{n}\alpha_k\vec{v}_k = \vec{0}) \to (\forall_{k=1}^{n}k)(\alpha_k = 0)] \Leftrightarrow (\exists_{k=1}^{n}\alpha_k \in K)[(\sum_{k=1}^{n}\alpha_k\vec{v}_k = \vec{0}) \wedge \neg(\forall_{k=1}^{n}k)(\alpha_k = 0)] \Leftrightarrow (\exists_{k=1}^{n}\alpha_k \in K)[(\sum_{k=1}^{n}\alpha_k\vec{v}_k = \vec{0}) \wedge (\exists_{k=1}^{n}k)(\alpha_k \neq 0)]$.

Therefore a set of vectors is **linearly dependent** iff $(\exists_{k=1}^{n}\alpha_k \in K)[(\sum_{k=1}^{n}\alpha_k\vec{v}_k = \vec{0}) \wedge (\exists_{k=1}^{n}k)(\alpha_k \neq 0)]$.

$\emptyset$ is linearly independent.
A subset of a linearly independent set of vectors is linearly independent.
A superset of a linearly dependent set of vectors is linearly dependent.

Linear Transformations

Definition 21. Linear Map
Let V, W be arbitrary vector spaces over a field K.
A **linear map(linear operator)** is a function $T : V \mapsto W$ that assigns to each vector $\vec{v} \in V$ a unique vector $T\vec{v} \in W$ such that, for all $\vec{u}, \vec{v} \in V$ and for all $\alpha \in K$:
1. $T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v})$ (preserves vector addition)
2. $T(\alpha\vec{v}) = \alpha T(\vec{v})$ (preserves scalar multiplication)

A linear map is a homomorphism of vector spaces.

Let V and W be vector spaces over a field K.
Let T be a linear transformation from V to W.
Let $\vec{v}, \vec{w} \in V$ be arbitrary.
Let $\alpha, \beta \in K$ be arbitrary.
Then the following are true:
1. $T(\alpha\vec{v} + \beta\vec{w}) = \alpha T(\vec{v}) + \beta T(\vec{w})$. (preserves linear combinations)
2. $T(\vec{0}) = \vec{0}$. (preserves zero vector)
3. $T(\vec{u} - \vec{v}) = T\vec{u} - T\vec{v}$. (preserves vector subtraction)

Let $T : V \mapsto W$ be defined by $T(\vec{v}) = \vec{0}$ for all $\vec{v} \in V$.
Then $T(\vec{v}_1 + \vec{v}_2) = \vec{0} = \vec{0} + \vec{0} = T(\vec{v}_1) + T(\vec{v}_2)$ and $T(\alpha\vec{v}) = \vec{0} = \alpha\vec{0} = \alpha T(\vec{v})$.
Therefore T is a linear map. $T(\vec{v}) = \vec{0}$ is the **zero transformation**.

Let $T : V \mapsto V$ be defined by $T(\vec{v}) = \vec{v}$ for all $\vec{v} \in V$.
Then $T(\vec{u} + \vec{v}) = \vec{u} + \vec{v} = T(\vec{u}) + T(\vec{v})$ and $T(\alpha\vec{v}) = \alpha\vec{v} = \alpha T(\vec{v})$.
Therefore T is a linear map.
$T(\vec{v}) = \vec{v}$ is the **identity linear transformation**.

Let $A = m \times n$ matrix.

Let $T : \mathbb{R}^n \mapsto \mathbb{R}^m$ be defined by $T(\vec{x}) = Ax$ for all $\vec{x} \in \mathbb{R}^n$.

Then $T(\vec{x} + \vec{y}) = A(\vec{x} + \vec{y}) = A\vec{x} + A\vec{y} = T\vec{x} + T\vec{y}$ and $T(\alpha\vec{x}) = A(\alpha\vec{x}) = \alpha A\vec{x} = \alpha T(\vec{x})$.

Therefore T is a linear map.

Hence every $m \times n$ matrix gives rise to a linear map from $\mathbb{R}^n$ to $\mathbb{R}^m$.

Let $\vec{v} = (x, y) \in \mathbb{R}^2$ be a vector with angle α with the x axis.

Then $x = r\cos\alpha$ and $y = r\sin\alpha$ where $r = \sqrt{x^2 + y^2}$.

Let $\vec{v}$ be rotated counter clockwise by θ.

Let $\vec{v}' = (x', y') \in \mathbb{R}^2$ be the final position of $\vec{v}$.

Then $x' = r\cos(\alpha + \theta)$ and $y' = r\sin(\alpha + \theta)$.

Since $\sin(\alpha + \theta) = \sin\alpha\cos\theta + \cos\alpha\sin\theta$ and $\cos(\alpha + \theta) = \cos\alpha\cos\theta - \sin\alpha\sin\theta$ then $x' = r\cos\alpha\cos\theta - r\sin\alpha\sin\theta = x\cos\theta - y\sin\theta$ and $y' = r\sin\alpha\cos\theta + r\cos\alpha\sin\theta = y\cos\theta + x\sin\theta = x\sin\theta + y\cos\theta$.

Let

$$A_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Then $\vec{v}' = (x', y') = A_\theta\vec{v}$. A_θ is the **rotation matrix**.

Example:

Let $T_A : \mathbb{R}^2 \mapsto \mathbb{R}^2$.

The associated matrix is

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Let $\overrightarrow{x} \in R^2$.

Then $\overrightarrow{x} = (x_1, x_2)$.

Thus, $T_A(\overrightarrow{x}) = A\overrightarrow{x} = A(x_1, x_2) = (ax_1 + bx_2, cx_1 + dx_2)$.

Definition 22. Line Reflection

A **line reflection** in a given line s is a function f defined for every point P of the plane so that:

1) if $P \in s$, then $f(P) = P$.

2) if $P \notin s$, then $f(P) = P'$ such that s is the $\perp$ bisector of segment $\overline{PP'}$.

Notes: $f_s : R^2 \to R^2$.

f_s is a transformation of the plane, so f_s is bijective map.

f_s is an isometry.

s = axis of reflection

Examples: if axis of reflection is x axis, $f(P) = f(x, y) = (x, -y)$

if axis of reflection is y axis, $f(P) = f(x, y) = (-x, y)$

if axis of reflection is line $y = x$, $f(P) = f(x, y) = (y, x)$

if axis of reflection is line $y = -x$, $f(P) = f(x, y) = (-y, -x)$

Definition 23. Isometry

Transformation f is an **isometry** iff for every pair of points P and Q, $P'Q' = PQ$ where $P' = f(P)$ and $Q' = f(Q)$.

Notes:

$f : R^2 \mapsto R^2$

f is a transformation of the plane.

An **isometry** is a geometric transformation of the plane that preserves distance.

The images of any two points are the same distance as the original two points.

Facts:

f maps lines onto lines. If s is a line, then $f(s)$ is a line.

f preserves angle measures between lines. m $\angle A'B'C'$ = m $\angle ABC$.

f preserves perpendicularity between lines. $f(s) \perp f(t)$ iff $s \perp t$.

f preserves parallelism between lines. $f(s) \parallel f(t)$ iff $s \parallel t$.

Definition 24. Dot product (scalar product)

Let $a, b \in \mathbb{R}^n$.

The **dot product** of $a = [a_1, a_2, ..., a_n]$ and $b = [b_1, b_2, ..., b_n]$ is defined as $a \cdot b = \sum_{k=1}^{n} a_k b_k$.