Vector Space Theory Notes

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Vector Spaces

Linear algebra is the study of linear maps on finite dimensional vector spaces. It originated as a theory for the solutions of systems of linear equations.

A vector space is an algebraic structure upon which addition and scalar multiplication are defined.

Definition 1. vector space

Let V be a set.

Define binary operation $+ : V \times V \to V$ by $\vec{v} + \vec{w} \in V$ for all $\vec{v}, \vec{w} \in V$. (**vector addition**)

Let F be a field.

Define function $\cdot : F \times V \to V$ by $\lambda\vec{v} \in V$ for all $\lambda \in F$ and for all $\vec{v} \in V$. (**scalar multiplication**)

A **vector space** $(V, +, \cdot)$ **over a field** F is a set V with two operations vector addition and scalar multiplication defined on V such that the following axioms hold:

V1. Vector addition is associative.
$(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$ for all $\vec{u}, \vec{v}, \vec{w} \in V$.

V2. Vector addition is commutative.
$\vec{v} + \vec{w} = \vec{w} + \vec{v}$ for all $\vec{v}, \vec{w} \in V$.

V3. There exists a right additive identity in V.
$(\exists\vec{0} \in V)(\forall\vec{v} \in V)(\vec{v} + \vec{0} = \vec{v})$.

V4. Every element of V has a right additive inverse.
$(\forall\vec{v} \in V)(\exists -\vec{v} \in V)[\vec{v} + (-\vec{v}) = \vec{0}]$.

V5. Associativity of scalar multiplication with field multiplication
$\alpha(\beta\vec{v}) = (\alpha\beta)\vec{v}$ for all $\vec{v} \in V$ and for all $\alpha, \beta \in F$.

V6. Left distributive law of scalar multiplication over vector addition
$\lambda(\vec{v} + \vec{w}) = \lambda\vec{v} + \lambda\vec{w}$ for all $\vec{v}, \vec{w} \in V$ and for all $\lambda \in F$.

V7. Right distributive law of scalar multiplication over scalar addition
$(\alpha + \beta)\vec{v} = \alpha\vec{v} + \beta\vec{v}$ for all $\vec{v} \in V$ and for all $\alpha, \beta \in F$.

V8. $1 \in F$ is a multiplicative identity for scalar multiplication.
$1 \cdot \vec{v} = \vec{v}$ for all $\vec{v} \in V$.

Theorem 2. ***alternate definition of a vector space***

Let V be a set.

Define binary operation $+ : V \times V \to V$ by $\vec{v} + \vec{w} \in V$ for all $\vec{v}, \vec{w} \in V$. ***(vector addition)***

Let F be a field.

Define function $\cdot : F \times V \to V$ by $\lambda\vec{v} \in V$ for all $\lambda \in F$ and for all $\vec{v} \in V$. ***(scalar multiplication)***

Then $(V, +, \cdot)$ is a ***vector space over a field*** *F iff*

1. $(V, +)$ is an abelian group.

2. Associativity of scalar multiplication with field multiplication
$\alpha(\beta\vec{v}) = (\alpha\beta)\vec{v}$ for all $\vec{v} \in V$ and for all $\alpha, \beta \in F$.

3. Left distributive law of scalar multiplication over vector addition
$\lambda(\vec{v} + \vec{w}) = \lambda\vec{v} + \lambda\vec{w}$ for all $\vec{v}, \vec{w} \in V$ and for all $\lambda \in F$.

4. Right distributive law of scalar multiplication over scalar addition
$(\alpha + \beta)\vec{v} = \alpha\vec{v} + \beta\vec{v}$ for all $\vec{v} \in V$ and for all $\alpha, \beta \in F$.

5. $1 \in F$ is a multiplicative identity for scalar multiplication.
$1 \cdot \vec{v} = \vec{v}$ for all $\vec{v} \in V$.

Definition 3. vector space terminology

Let $(V, +, \cdot)$ be a vector space over a field F.
A **vector**, denoted $\vec{v}$, is an element of V.
A **scalar** is an element of F.

Let $(V, +, \cdot)$ be a vector space over a field F.
Let $\vec{v} \in F$.
Then $\vec{v}$ is a vector.

Let $\lambda \in F$.
Then λ is a scalar.
Therefore, a scalar is just a number.

Since $(V, +, \cdot)$ is a vector space over a field F, then vector addition is a binary operation on V, and scalar multiplication is a function, and $(V, +)$ is an abelian group, and scalar multiplication satisfies the below axioms.

1. $\alpha(\beta\vec{v}) = (\alpha\beta)\vec{v}$ for all $\vec{v} \in V$ and for all $\alpha, \beta \in F$.
2. $\lambda(\vec{v} + \vec{w}) = \lambda\vec{v} + \lambda\vec{w}$ for all $\vec{v}, \vec{w} \in V$ and for all $\lambda \in F$.
3. $(\alpha + \beta)\vec{v} = \alpha\vec{v} + \beta\vec{v}$ for all $\vec{v} \in V$ and for all $\alpha, \beta \in F$.
4. $1 \cdot \vec{v} = \vec{v}$ for all $\vec{v} \in V$.

Since vector addition is a binary operation on V, then V is closed under vector addition.

Since $(V, +)$ is an abelian group, then vector addition is associative and commutative, and $\vec{0} \in V$ is the additive identity, and every vector in V has an additive inverse in V.

$\vec{0} \in V$ is called the **zero vector**, and the zero vector is the additive identity in V.

Since $\vec{0} \in V$, then $V \neq \emptyset$, so V contains at least one element.

Therefore, a vector space contains at least one element.

Since scalar multiplication is a function, then $\cdot$ assigns to each $\lambda \in F$ and each $\vec{v} \in V$ the product $\lambda v \in V$, so V is closed under scalar multiplication.

Therefore, in scalar multiplication, the product of a scalar and a vector is a vector.

Therefore, a vector space $(V+, \cdot)$ over a field F satisfies the following axioms:

V1. V is closed under vector addition.
$\vec{v} + \vec{w} \in V$ for all $\vec{v}, \vec{w} \in V$.
V2. Vector addition is associative.
$(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$ for all $\vec{u}, \vec{v}, \vec{w} \in V$.
V3. Vector addition is commutative.
$\vec{v} + \vec{w} = \vec{w} + \vec{v}$ for all $\vec{v}, \vec{w} \in V$.
V4. There exists an additive identity in V.
$(\exists \vec{0} \in V)(\forall \vec{v} \in V)(\vec{v} + \vec{0} = \vec{0} + \vec{v} = \vec{v})$.
V5. Every vector in V has an additive inverse.
$(\forall \vec{v} \in V)(\exists - \vec{v} \in V)[\vec{v} + (-\vec{v}) = -\vec{v} + \vec{v} = \vec{0}]$.
V6. V is closed under scalar multiplication.
$\lambda\vec{v} \in V$ for all $\lambda \in F$ and for all $\vec{v} \in V$.
V7. Associativity of scalar multiplication with field multiplication
$\alpha(\beta\vec{v}) = (\alpha\beta)\vec{v}$ for all $\vec{v} \in V$ and for all $\alpha, \beta \in F$.
V8. Left distributive law of scalar multiplication over vector addition
$\lambda(\vec{v} + \vec{w}) = \lambda\vec{v} + \lambda\vec{w}$ for all $\vec{v}, \vec{w} \in V$ and for all $\lambda \in F$.
V9. Right distributive law of scalar multiplication over scalar addition
$(\alpha + \beta)\vec{v} = \alpha\vec{v} + \beta\vec{v}$ for all $\vec{v} \in V$ and for all $\alpha, \beta \in F$.
V10. $1 \in F$ is a multiplicative identity for scalar multiplication.
$1 \cdot \vec{v} = \vec{v}$ for all $\vec{v} \in V$.

Observe that $\vec{0} + \vec{0} = \vec{0}$.

Definition 4. real vector space
A **real vector space** is a vector space over $\mathbb{R}$.

Definition 5. complex vector space
A **complex vector space** is a vector space over $\mathbb{C}$.

Theorem 6. ***basic properties of vector spaces***
Let $(V, +, \cdot)$ be a vector space over a field F.
1. Any scalar times the zero vector is the zero vector.
$\lambda\vec{0} = \vec{0}$ *for all* $\lambda \in F$.
2. Zero times any vector is the zero vector.
$0\vec{v} = \vec{0}$ *for all* $\vec{v} \in V$.
3. The scalar product is zero iff the scalar is zero or the vector is zero.
$\lambda\vec{v} = \vec{0}$ *iff* $\lambda = 0$ *or* $\vec{v} = \vec{0}$, *for all* $\vec{v} \in V$ *and for all* $\lambda \in F$.
4. Negative 1 *times any vector is the additive inverse of the vector.*
$(-1)\vec{v} = -\vec{v}$ *for all* $\vec{v} \in V$.

Linear subspaces

Definition 7. linear subspace

Let $(V, +, \cdot)$ be a vector space.

A **subspace** of V is a subset of V that is a vector space under the operations of vector addition and scalar multiplication defined on V.

Let $(V, +, \cdot)$ be a vector space.

Let $W \subseteq V$.

Then W is a **subspace of** $(V, +, \cdot)$ iff $(W, +, \cdot)$ is a vector space under $+$ and $\cdot$ defined on V.

Let $(V, +, \cdot)$ be an arbitrary vector space over a field F with additive identity $0 \in V$.

Since $V \subseteq V$ and $(V, +, \cdot)$ is a vector space, then V is a subspace of V.

Therefore, every vector space is a subspace of itself.

Since $0 \in V$, then $\{0\} \subseteq V$.

Since $\{0\} \subseteq V$, and $0 + 0 = 0$, and $\lambda \cdot 0 = 0$ for all $\lambda \in F$, then the trivial vector space is a subspace of V.

Therefore, the **trivial vector space** $\{0\}$ is a subspace of every vector space.

Let W be a subspace of a vector space V with additive identity $0 \in V$.

Since $\{0\}$ is a subspace of every vector space, and W is a vector space, then $\{0\}$ is a subspace of W.

Hence, $\{0\} \subseteq W$, so $0 \in W$.

Therefore, every subspace of a vector space V contains the zero vector $0 \in V$.

Thus, if W is a subspace of V with additive identity $0 \in V$, then $0 \in W$.

Therefore, if $W \subseteq V$, but $0 \notin W$, then W cannot be a subspace of V.

TODO: Fix the below stuff and obtain the most common definition of proper subspace and update it and proper subgroup defn accordingly.

Definition 8. proper subspace

Let V be a vector space.

A **proper subspace** is a subspace of V other than V or the trivial subspace.

Let V be a vector space with additive identity $0 \in V$.

Let W be a subspace of V.

Then W is a proper subspace of V iff $W \neq V$ and $W \neq \{0\}$.

Theorem 9. ***Two-Step Subspace Test***

Let $(V, +, \cdot)$ be a vector space over a field F.

Let W be a nonempty subset of V.

Then W is a subspace of V iff

1. W is closed under vector addition.

$\vec{v} + \vec{w} \in W$ for all $\vec{v}, \vec{w} \in W$ (Closure under vector addition)

2. W is closed under scalar multiplication.
$\lambda\vec{v} \in W$ for all $\vec{v} \in W, \lambda \in F$ (Closure under scalar multiplication)

Linear Independence

Definition 10. linear combination
Let $(V, +, \cdot)$ be a vector space over a field F.
Let $n \in \mathbb{Z}^+$.
Let $\vec{v}_1, \vec{v}_2, ..., \vec{v}_n \in V$.
A vector $\vec{v} \in V$ is a **linear combination** of $\vec{v}_1, \vec{v}_2, ..., \vec{v}_n$ iff there exist scalars $a_1, a_2, ..., a_n \in F$ such that $\vec{v} = a_1\vec{v}_1 + a_2\vec{v}_2 + ... + a_n\vec{v}_n$.

Definition 11. linear span of a set of vectors
Let $(V, +, \cdot)$ be a vector space over a field F.
Let $n \in \mathbb{Z}^+$.
Let $\{\vec{v}_1, \vec{v}_2, ..., \vec{v}_n\} \subset V$.
The **span of** $\{\vec{v}_1, \vec{v}_2, ..., \vec{v}_n\}$ is defined as $span\{\vec{v}_1, \vec{v}_2, ..., \vec{v}_n\} = \{a_1\vec{v}_1 + a_2\vec{v}_2 + ... + a_n\vec{v}_n : a_1, a_2, ..., a_n \in F\}$.

Therefore, the span of a set of vectors of a vector space is the set of all linear combinations of the vectors.

Definition 12. finite-dimensional vector space
Let $(V, +, \cdot)$ be a vector space over a field F.
Let $n \in \mathbb{Z}^+$.
Then $(V, +, \cdot)$ is **finite-dimensional** iff $span\{\vec{v}_1, \vec{v}_2, ..., \vec{v}_n\} = V$, and we say that the set of vectors $\{\vec{v}_1, \vec{v}_2, ..., \vec{v}_n\}$ **spans** V.

Definition 13. infinite-dimensional vector space
A vector space that is not finite-dimensional is called **infinite-dimensional**.

Definition 14. Linear Independence of vectors
Let V be a vector space over a field K.
Let $\{\vec{v}_1, \vec{v}_2, ..., \vec{v}_n\}$ be a set of vectors in V.
The set of vectors is **linearly independent** iff $(\forall_{k=1}^{n}\alpha_k \in K)[(\sum_{k=1}^{n} \alpha_k\vec{v}_k = \vec{0}) \rightarrow (\forall_{k=1}^{n}k)(\alpha_k = 0)]$.
A set of vectors is **linearly dependent** iff it is not linearly independent.

Observe that $\neg(\forall_{k=1}^{n}\alpha_k \in K)[(\sum_{k=1}^{n} \alpha_k\vec{v}_k = \vec{0}) \rightarrow (\forall_{k=1}^{n}k)(\alpha_k = 0)] \Leftrightarrow (\exists_{k=1}^{n}\alpha_k \in K)[(\sum_{k=1}^{n} \alpha_k\vec{v}_k = \vec{0}) \wedge \neg(\forall_{k=1}^{n}k)(\alpha_k = 0)] \Leftrightarrow (\exists_{k=1}^{n}\alpha_k \in K)[(\sum_{k=1}^{n} \alpha_k\vec{v}_k = \vec{0}) \wedge (\exists_{k=1}^{n}k)(\alpha_k \neq 0)]$.

Therefore a set of vectors is **linearly dependent** iff $(\exists_{k=1}^{n}\alpha_k \in K)[(\sum_{k=1}^{n} \alpha_k\vec{v}_k = \vec{0}) \wedge (\exists_{k=1}^{n}k)(\alpha_k \neq 0)]$.

$\emptyset$ is linearly independent.
A subset of a linearly independent set of vectors is linearly independent.
A superset of a linearly dependent set of vectors is linearly dependent.

Linear Transformations

Definition 15. Linear Map

Let V, W be arbitrary vector spaces over a field K.

A **linear map(linear operator)** is a function $T : V \mapsto W$ that assigns to each vector $\vec{v} \in V$ a unique vector $T\vec{v} \in W$ such that, for all $\vec{u}, \vec{v} \in V$ and for all $\alpha \in K$:

1. $T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v})$ (preserves vector addition)
2. $T(\alpha\vec{v}) = \alpha T(\vec{v})$ (preserves scalar multiplication)

A linear map is a homomorphism of vector spaces.

Let V and W be vector spaces over a field K.

Let T be a linear transformation from V to W.

Let $\vec{v}, \vec{w} \in V$ be arbitrary.

Let $\alpha, \beta \in K$ be arbitrary.

Then the following are true:

1. $T(\alpha\vec{v} + \beta\vec{w}) = \alpha T(\vec{v}) + \beta T(\vec{w})$. (preserves linear combinations)
2. $T(\vec{0}) = \vec{0}$. (preserves zero vector)
3. $T(\vec{u} - \vec{v}) = T\vec{u} - T\vec{v}$. (preserves vector subtraction)

Let $T : V \mapsto W$ be defined by $T(\vec{v}) = \vec{0}$ for all $\vec{v} \in V$.

Then $T(\vec{v}_1 + \vec{v}_2) = \vec{0} = \vec{0} + \vec{0} = T(\vec{v}_1) + T(\vec{v}_2)$ and $T(\alpha\vec{v}) = \vec{0} = \alpha\vec{0} = \alpha T(\vec{v})$.

Therefore T is a linear map. $T(\vec{v}) = \vec{0}$ is the **zero transformation**.

Let $T : V \mapsto V$ be defined by $T(\vec{v}) = \vec{v}$ for all $\vec{v} \in V$.

Then $T(\vec{u} + \vec{v}) = \vec{u} + \vec{v} = T(\vec{u}) + T(\vec{v})$ and $T(\alpha\vec{v}) = \alpha\vec{v} = \alpha T(\vec{v})$.

Therefore T is a linear map.

$T(\vec{v}) = \vec{v}$ is the **identity linear transformation**.

Let $A = m \times n$ matrix.

Let $T : \mathbb{R}^n \mapsto \mathbb{R}^m$ be defined by $T(\vec{x}) = Ax$ for all $\vec{x} \in \mathbb{R}^n$.

Then $T(\vec{x} + \vec{y}) = A(\vec{x} + \vec{y}) = A\vec{x} + A\vec{y} = T\vec{x} + T\vec{y}$ and $T(\alpha\vec{x}) = A(\alpha\vec{x}) = \alpha A\vec{x} = \alpha T(\vec{x})$.

Therefore T is a linear map.

Hence every $m \times n$ matrix gives rise to a linear map from $\mathbb{R}^n$ to $\mathbb{R}^m$.

Let $\vec{v} = (x, y) \in \mathbb{R}^2$ be a vector with angle α with the x axis.

Then $x = r\cos\alpha$ and $y = r\sin\alpha$ where $r = \sqrt{x^2 + y^2}$.

Let $\vec{v}$ be rotated counter clockwise by θ.

Let $\vec{v}' = (x', y') \in \mathbb{R}^2$ be the final position of $\vec{v}$.

Then $x' = r\cos(\alpha + \theta)$ and $y' = r\sin(\alpha + \theta)$.

Since $\sin(\alpha + \theta) = \sin\alpha\cos\theta + \cos\alpha\sin\theta$ and $\cos(\alpha + \theta) = \cos\alpha\cos\theta - \sin\alpha\sin\theta$ then $x' = r\cos\alpha\cos\theta - r\sin\alpha\sin\theta = x\cos\theta - y\sin\theta$ and $y' = r\sin\alpha\cos\theta + r\cos\alpha\sin\theta = y\cos\theta + x\sin\theta = x\sin\theta + y\cos\theta$.

Let

$$A_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Then $\vec{v}' = (x', y') = A_\theta \vec{v}$. A_θ is the **rotation matrix**.

Example:

Let $T_A : \mathbb{R}^2 \mapsto \mathbb{R}^2$.

The associated matrix is

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Let $\overrightarrow{x} \in R^2$.

Then $\overrightarrow{x} = (x_1, x_2)$.

Thus, $T_A(\overrightarrow{x}) = A\overrightarrow{x} = A(x_1, x_2) = (ax_1 + bx_2, cx_1 + dx_2)$.

Definition 16. Line Reflection

A **line reflection** in a given line s is a function f defined for every point P of the plane so that:

1) if $P \in s$, then $f(P) = P$.
2) if $P \notin s$, then $f(P) = P'$ such that s is the $\perp$ bisector of segment $\overline{PP'}$.

Notes: $f_s : R^2 \to R^2$.

f_s is a transformation of the plane, so f_s is bijective map.

f_s is an isometry.

$s =$ axis of reflection

Examples: if axis of reflection is x axis, $f(P) = f(x, y) = (x, -y)$

if axis of reflection is y axis, $f(P) = f(x, y) = (-x, y)$

if axis of reflection is line $y = x$, $f(P) = f(x, y) = (y, x)$

if axis of reflection is line $y = -x$, $f(P) = f(x, y) = (-y, -x)$

Definition 17. Isometry

Transformation f is an **isometry** iff for every pair of points P and Q, $P'Q' = PQ$ where $P' = f(P)$ and $Q' = f(Q)$.

Notes:

$f : R^2 \mapsto R^2$

f is a transformation of the plane.

An **isometry** is a geometric transformation of the plane that preserves distance.

The images of any two points are the same distance as the original two points.

Facts:

f maps lines onto lines. If s is a line, then $f(s)$ is a line.

f preserves angle measures between lines. m $\angle A'B'C'$ = m $\angle ABC$.
f preserves perpendicularity between lines. $f(s) \perp f(t)$ iff $s \perp t$.
f preserves parallelism between lines. $f(s) \parallel f(t)$ iff $s \parallel t$.

Definition 18. Dot product (scalar product)
Let $a, b \in \mathbb{R}^n$.
The **dot product** of $a = [a_1, a_2, ..., a_n]$ and $b = [b_1, b_2, ..., b_n]$ is defined as $a \cdot b = \sum_{k=1}^{n} a_k b_k$.

Matrix Theory

Definition 19. Real Matrix
A $m \times n$ **real matrix** is a rectangular array of m rows and n columns of real numbers.
Each $a_{ij} \in \mathbb{R}$ is an **entry** at row i and column j and $1 \leq i \leq m$ and $1 \leq j \leq n$.
Let $A = (a_{ij})_{m \times n}$ be a $m \times n$ matrix where $i \in \mathbb{N}_m$ and $j \in \mathbb{N}_n$.
Then

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

Definition 20. Equal Matrices
Two matrices are equal iff corresponding entries are equal.
Let $A = (a_{ij})_{m \times n}$ and $B = (b_{ij})_{m \times n}$.
Then $A = B$ iff $(\forall i \in \mathbb{N}_m)(\forall j \in \mathbb{N}_n)(a_{ij} = b_{ij})$.
Equal matrices have the same size.

Definition 21. Matrix Addition
The sum of two matrices is the sum of corresponding entries.
Let $A = (a_{ij})_{m \times n}$ and $B = (b_{ij})_{m \times n}$.
Then the matrix sum is defined by the rule $A + B = (c_{ij})_{m \times n}$ where $c_{ij} = a_{ij} + b_{ij}$.

Let $k \in \mathbb{R}$ be a scalar.
Then $kA = (ka_{ij})_{m \times n}$.

Definition 22. Matrix Multiplication
The entry at row i and column j of the matrix product is the dot product of the i^{th} row vector of matrix A with the j^{th} column vector of matrix B.
Let $A = (a_{ij})_{m \times n}$ and $B = (b_{ij})_{n \times p}$.
Then the matrix product is defined by the rule $AB = (c_{ij})_{m \times p}$ where $c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$.

Definition 23. Transpose of a Matrix

Let A be a matrix.

The transpose of A, denoted A^t, is the matrix obtained by transposing the rows and columns of A.

The i^{th} row of $A = i^{th}$ column of A^t.

The j^{th} column of $A = j^{th}$ row of A^t.

Let $A = (a_{ij})_{m\times n}$ with $i \in \mathbb{N}_m$ and $j \in \mathbb{N}_n$.

Then $A^t = (a_{ji})_{n\times m}$.

Let $A = (a_{ij})_{m\times n}$.

Then A^t is of size $n \times m$, so $(A^t)^t$ is of size $m \times n$.

Hence $(A^t)^t$ and A have the same size $m \times n$.

Let $i \in \mathbb{N}_m$ and $j \in \mathbb{N}_n$ be arbitrary.

Let a_{ij} be the ij^{th} entry of A.

Let b_{ij} be the ij^{th} entry of $(A^t)^t$.

Then

$$\begin{aligned} b_{ij} \text{ of } (A^t)^t &= b_{ji} \text{ of } A^t \\ &= b_{ij} \text{ of } A \\ &= a_{ij} \text{ of } A \end{aligned}$$

Therefore $(A^t)^t = A$.

Let $A = (a_{ij})_{m\times n}$ and $B = (b_{ij})_{m\times n}$.

Then $A + B$ is of size $m \times n$, so $(A + B)^t$ is of size $n \times m$.

Also, A^t and B^t are each of size $n \times m$, so $A^t + B^t$ is of size $n \times m$.

Hence $(A + B)^t$ and $A^t + B^t$ have the same size $n \times m$.

Let $i \in \mathbb{N}_n$ and $j \in \mathbb{N}_m$ be arbitrary.

Let c_{ij} be the ij^{th} entry of $(A + B)^t$.

Let d_{ij} be the ij^{th} entry of $A^t + B^t$.

Then

$$\begin{aligned} c_{ij} \text{ of } (A+B)^t &= c_{ji} \text{ of } A+B \\ &= a_{ji} + b_{ji} \text{ where } a_{ji} \text{ in } A \text{ and } b_{ji} \text{ in } B \\ &= a_{ij} + b_{ij} \text{ where } a_{ij} \text{ in } A^t \text{ and } b_{ij} \text{ in } B^t \\ &= d_{ij} \text{ of } A^t + B^t \end{aligned}$$

Therefore $(A + B)^t = A^t + B^t$.

Let $A = (a_{ij})_{m\times n}$ and $B = (b_{ij})_{n\times p}$.

Then AB is of size $m \times p$, so $(AB)^t$ is of size $p \times m$.

Also, B^t is of size $p \times n$ and A^t is of size $n \times m$, so B^tA^t is of size $p \times m$.

Hence $(AB)^t$ and B^tA^t have the same size $p \times m$.

Let $i \in \mathbb{N}_p$ and $j \in \mathbb{N}_m$ be arbitrary.

Let c_{ij} be the ij^{th} entry of $(AB)^t$.
Let d_{ij} be the ij^{th} entry of B^tA^t.
Then

$$\begin{aligned}
c_{ij} \text{ of } (AB)^t &= c_{ji} \text{ of } AB \\
&= \sum_{k=1}^{n} a_{jk}b_{ki} \text{ where } a_{jk} \text{ in } A \text{ and } b_{ki} \text{ in } B \\
&= \sum_{k=1}^{n} a_{kj}b_{ik} \text{ where } a_{kj} \text{ in } A^t \text{ and } b_{ik} \text{ in } B^t \\
&= \sum_{k=1}^{n} b_{ik}a_{kj} \text{ where } b_{ik} \text{ in } B^t \text{ and } a_{kj} \text{ in } A^t \\
&= d_{ij} \text{ of } B^tA^t
\end{aligned}$$

Therefore $(AB)^t = B^tA^t$.

Definition 24. Square Matrix
Let $A = (a_{ij})_{m\times n}$ be a $m \times n$ matrix where $i \in \mathbb{N}_m$ and $j \in \mathbb{N}_n$.
A matrix is **square** iff $m = n$.
Therefore, a square matrix has the same number of rows as columns.

Suppose $A = (a_{ij})_{m\times m}$ is a square matrix.
A is **symmetric** iff $A^t = A$.
A is **antisymmetric** iff $A^t = -A$.
A is **diagonal** iff $(\forall i, j \in \mathbb{N}_m)(i \neq j \to a_{ij} = 0)$.
A is **upper triangular** iff $(\forall i, j \in \mathbb{N}_m)(i > j \to a_{ij} = 0)$.
A is **lower triangular** iff $(\forall i, j \in \mathbb{N}_m)(i < j \to a_{ij} = 0)$.

A square matrix is diagonal iff it is both upper and lower triangular.
The sum of two square symmetric matrices is symmetric.

Let $A = (a_{ij})_{m\times m}$.
Then $A + A^t$ is symmetric and $A - A^t$ is antisymmetric.
Observe that $(A + A^t)^t = A^t + (A^t)^t = A^t + A = A + A^t$.
Observe that $(A - A^t)^t = [A + (-A^t)]^t = A^t + (-A^t)^t = A^t + ((-A)^t)^t = A^t + (-A) = A^t - A = -(A - A^t)$.

Definition 25. Determinant of a Matrix
Let $A = (a_{ij})_{2\times 2}$.
The **determinant of matrix** A is defined by the rule $|A| = a_{11}a_{22} - a_{12}a_{21}$.
Let A and B be 2×2 matrices.
Then $|AB| = |A||B|$.

Definition 26. Identity Matrix

Let $n \in \mathbb{Z}^+$.

The **identity matrix**, denoted I_n, is an $n \times n$ matrix with ones along the principal diagonal and zeros everywhere else.

Therefore, $I_n = (\delta_{ij})_{n \times n}$ such that for every $i, j \in \mathbb{N}_n$,

$$\delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$$

The identity matrix is a square matrix.

Let A be a $n \times n$ matrix.

Then $A = AI = IA$.

Definition 27. Invertible Matrix

Let A be a square matrix.

Let I be the identity matrix.

Then A **is invertible** iff $\exists$ a matrix B such that $AB = BA = I$.

Suppose A is invertible.

Then $AB = BA = I$ for some matrix B.

Since A is invertible, then $A = (a_{ij})_{n \times n}$.

Let $B = (b_{ij})_{m \times p}$.

Since the product AB is defined, then $n \times n$ matrix multiplied by a $m \times p$ matrix implies $n = m$.

Since $I = (\delta_{ij})_{n \times n}$, then $n \times p = n \times n$, so $p = n$.

Hence, $B = (b_{ij})_{n \times n}$, so B is a square matrix.

The inverse of an invertible matrix is a square matrix.

Let $n \in \mathbb{Z}^+$.

Let GL_n be the set of all $n \times n$ invertible matrices.

Then $GL_n = \{X : X \text{ is an } n \times n \text{ invertible matrix }\} =$ **general linear group**.

$(GL_n, \cdot)$ is a non-abelian group where $\cdot =$ matrix multiplication.

Identity of GL_n is $I_n =$ identity matrix.

Inverse of matrix A is matrix A^{-1}, where $A^{-1} \in GL_n$ and $AA^{-1} = A^{-1}A = I$.

Since $n \times n$ invertible matrices $\subset n \times n$ matrices $\subset m \times n$ matrices, then $GL_n \subset$ square matrices $\subset M_{m \times n}$.