Logic Exercises

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Exercise 1. Let p and q be propositional variables.
Then $\neg q \vee (p \to q)$ is a tautology.

Solution. To prove this is a tautology, we devise the truth table for $\neg q \vee (p \to q)$.
The truth table demonstrates that under all truth values for p and q, the statement form $\neg q \vee (p \to q)$ is true.
Therefore, $\neg q \vee (p \to q)$ is a tautology. $\square$

Solution. An alternate solution is below.
The statement $p \to q$ is either true or false.
If $p \to q$ is true, then the disjunction $\neg q \vee (p \to q)$ is true.
If $p \to q$ is false, then p is true and q is false, so $\neg q$ is true.
Hence, the disjunction $\neg q \vee (p \to q)$ is true.
In all cases, $\neg q \vee (p \to q)$ is true, so $\neg q \vee (p \to q)$ is a tautology. $\square$

Exercise 2. Let p and q be propositional variables.
Then $p \vee (p \to q)$ is a tautology.

Solution. To prove this is a tautology, we devise the truth table for $p \vee (p \to q)$.
The truth table demonstrates that under all truth values for p and q, the statement form $p \vee (p \to q)$ is true.
Therefore, $p \vee (p \to q)$ is a tautology. $\square$

Solution. An alternate solution is below.
The statement p is either true or false.
If p is true, then the disjunction $p \vee (p \to q)$ is true.
If p is false, then $p \to q$ is vacuously true, so the disjunction $p \vee (p \to q)$ is true.
In all cases, $p \vee (p \to q)$ is true, so $p \vee (p \to q)$ is a tautology. $\square$

Exercise 3. Prove that $[p \to (q \vee r)] \leftrightarrow [(p \wedge \neg q) \to r]$ is a tautology.

Solution. To prove $p \to (q \vee r) \leftrightarrow [(p \wedge \neg q) \to r]$ is a tautology, we must show that $p \to (q \vee r)$ is logically equivalent to $(p \wedge \neg q) \to r$.
Observe that

$$\begin{aligned} p \to (q \vee r) &\Leftrightarrow \neg p \vee (q \vee r) \\ &\Leftrightarrow (\neg p \vee q) \vee r \\ &\Leftrightarrow \neg(p \wedge \neg q) \vee r \\ &\Leftrightarrow (p \wedge \neg q) \to r. \end{aligned}$$

An alternate proof would be to show that the truth values in the truth table of $p \to (q \vee r)$ match line for line with the truth values in the truth table for $(p \wedge \neg q) \to r$. $\square$

Exercise 4. Prove that $(p \wedge q) \to r \leftrightarrow (p \wedge \neg r) \to \neg q$ is a tautology.

Solution. To prove $(p \wedge q) \to r \leftrightarrow (p \wedge \neg r) \to \neg q$ is a tautology, we must show that $(p \wedge q) \to r$ is logically equivalent to $(p \wedge \neg r) \to \neg q$.

Observe that

$$\begin{aligned}
(p \wedge q) \to r &\Leftrightarrow (p \to r) \vee (q \to r) \\
&\Leftrightarrow (\neg p \vee r) \vee (\neg q \vee r) \\
&\Leftrightarrow \neg p \vee r \vee \neg q \\
&\Leftrightarrow \neg(p \wedge \neg r) \vee \neg q \\
&\Leftrightarrow (p \wedge \neg r) \to \neg q.
\end{aligned}$$

An more laborious proof would be to show that the truth values in the truth table of $(p \wedge q) \to r$ match line for line with the truth values in the truth table for $(p \wedge \neg r) \to \neg q$. $\square$

Exercise 5. modus ponens inference rule

Prove $P \wedge (P \to Q) \Rightarrow Q$.

Solution. We must prove $[P \wedge (P \to Q)] \to Q$ is a tautology.

We could use truth tables to prove this,but we choose an alternate method here.

Our hypothesis is $P \wedge (P \to Q)$.

Our conclusion is Q.

We note that this logical argument is the modus ponens logic inference rule, which we know is valid.

The statement means Q logically follows from $P \wedge (P \to Q)$.

Hence, the truth of Q follows from the truth of $P \wedge (P \to Q)$.

Therefore to prove Q we assume that $P \wedge (P \to Q)$ is true. $\square$

Proof. Suppose $P \wedge (P \to Q)$.

Then P is true and $P \to Q$ is true.

Observe that only one line in the truth table for $P \to Q$ satisfies these two conditions.

It is the line in which Q is also true. Therefore, whenever both P and $P \to Q$ are true, Q must also be true. $\square$

Exercise 6. Prove $P \to P$.

Proof. Assume P.

From P, we conclude P, by simplification.

From premise P and P, we conclude $P \to P$, by conditional introduction. $\square$

Exercise 7. Prove $P \to (Q \to P)$.

Proof. Assume P and Q.

We conclude P, by simplification.

From premise Q and P, we conclude $Q \to P$, by conditional introduction.

From premise P and $Q \to P$, we conclude $P \to (Q \to P)$, by conditional introduction. □

Exercise 8. Prove $P \to ((P \to Q) \to Q)$.

Proof. Assume P and $P \to Q$.

We conclude Q, by modus ponens.

From premise $P \to Q$ and Q, we conclude $(P \to Q) \to Q$, by conditional introduction.

From premise P and $(P \to Q) \to Q$, we conclude $P \to ((P \to Q) \to Q)$, by conditional introduction. □

Exercise 9. Let P, Q, R be propositional variables.

Then

A. $\neg(P \wedge Q \wedge R) \Leftrightarrow \neg P \vee \neg Q \vee \neg R$.

This means that P, Q, R cannot all be true. At least one of them is not true.

B. $\neg(P \vee Q \vee R) \Leftrightarrow \neg P \wedge \neg Q \wedge \neg R$. This means P, Q, R are all not true.

Proof. We prove: $\neg(P \wedge Q \wedge R) \Leftrightarrow \neg P \vee \neg Q \vee \neg R$.

Observe that

$$\begin{aligned}
\neg(P \wedge Q \wedge R) &\Leftrightarrow \neg(P \wedge (Q \wedge R)) \\
&\Leftrightarrow \neg P \vee \neg(Q \wedge R) \\
&\Leftrightarrow \neg P \vee (\neg Q \vee \neg R) \\
&\Leftrightarrow \neg P \vee \neg Q \vee \neg R.
\end{aligned}$$

□

Proof. We prove: $\neg(P \vee Q \vee R) \Leftrightarrow \neg P \wedge \neg Q \wedge \neg R$.

Observe that

$$\begin{aligned}
\neg(P \vee Q \vee R) &\Leftrightarrow \neg(P \vee (Q \vee R)) \\
&\Leftrightarrow \neg P \wedge \neg(Q \vee R) \\
&\Leftrightarrow \neg P \wedge (\neg Q \wedge \neg R) \\
&\Leftrightarrow \neg P \wedge \neg Q \wedge \neg R.
\end{aligned}$$

□

Exercise 10. Prove $\neg(P \to Q) \to P$.

Proof. Assume $\neg(P \to Q)$.

We prove by contradiction: assume $\neg P$.

Assume P.

From $\neg P$ and P, we derive a contradiction, so we conclude Q.

From premise P and Q, we conclude $P \to Q$, by conditional introduction.

From $\neg(P \to Q)$ and $P \to Q$, we derive a contradiction.

Hence, P.

From premise $\neg(P \to Q)$ and P, we conclude $\neg(P \to Q) \to P$, by conditional introduction. □

Exercise 11. Prove $\neg(P \to Q) \to \neg Q$.

Proof. Assume $\neg(P \to Q)$.

Assume Q.

Assume P.

From premise P and Q, we conclude $P \to Q$, by conditional introduction.

From $\neg(P \to Q)$ and $P \to Q$, we derive a contradiction.

Hence, $\neg Q$.

From premise $\neg(P \to Q)$ and $\neg Q$, we conclude $\neg(P \to Q) \to \neg Q$, by conditional introduction. □

Exercise 12. Prove $\neg\neg P \to P$.

Proof. Assume $\neg\neg P$.

We prove by contradiction.

Assume $\neg P$.

From $\neg\neg P$ and $\neg P$, we derive a contradiction.

Hence, we conclude P.

From premise $\neg\neg P$ and P, we conclude $\neg\neg P \to P$, by conditional introduction. □

Exercise 13. Prove $P \to \neg\neg P$.

Proof. Assume P.

Assume $\neg P$.

From $\neg P$ and P, we derive a contradiction.

Hence, we conclude $\neg\neg P$.

From premise P and $\neg\neg P$, we conclude $P \to \neg\neg P$, by conditional introduction. □

Exercise 14. Prove $\neg(P \to Q) \to (P \land \neg Q)$.

Proof. Assume $\neg(P \to Q)$.

Assume P.

Assume $\neg P$.

From $\neg P$ and P, we derive a contradiction, so we conclude Q.

From premise P and Q, we conclude $P \to Q$, by conditional introduction.

From $\neg(P \to Q)$ and $P \to Q$, we derive a contradiction, so we conclude P.

Assume Q.
Assume P.
From Q and P, we conclude Q.
From premise P and Q, we conclude $P \to Q$, by conditional introduction.
From $\neg(P \to Q)$ and $P \to Q$, we derive a contradiction, so we conclude $\neg Q$.

From P and $\neg Q$, we conclude $P \wedge \neg Q$, by conjunction introduction.
From premise $\neg(P \to Q)$ and $P \wedge \neg Q$, we conclude $\neg(P \to Q) \to (P \wedge \neg Q)$, by conditional introduction. □

Exercise 15. Show that $P \to (Q \to R)$ is logically equivalent to $(P \wedge Q) \to R$.

Solution. We must prove that $(P \to (Q \to R)) \Leftrightarrow ((P \wedge Q) \to R)$.
We can either use truth tables(laborious way) or use logic tautologies/identities to prove this. □

Proof. Observe that

$$\begin{aligned}
P \to (Q \to R) &\Leftrightarrow \neg P \vee (Q \to R) \\
&\Leftrightarrow \neg P \vee (\neg Q \vee R) \\
&\Leftrightarrow (\neg P \vee \neg Q) \vee R \\
&\Leftrightarrow \neg(P \wedge Q) \vee R \\
&\Leftrightarrow (P \wedge Q) \to R.
\end{aligned}$$

□

Exercise 16. Show that the argument $P \wedge (P \to Q) \wedge (\neg Q \vee R) \Rightarrow R$ is valid.

Solution. We must show that the truth of the hypothesis $P \wedge (P \to Q) \wedge (\neg Q \vee R)$ implies the truth of the conclusion R. That is, we must show that $[P \wedge (P \to Q) \wedge (\neg Q \vee R)] \to R$ is a tautology.
In other words, show that $P \wedge (P \to Q) \wedge (\neg Q \vee R) \Rightarrow R$. □

Proof. We prove that we can validly deduce R from the assumed truth of the partial premises $P, P \to Q, \neg Q \vee R$.
We know that $\neg Q \vee R \Leftrightarrow Q \to R$.
Thus our premise becomes $P \wedge (P \to Q) \wedge (Q \to R)$.
From $P \wedge (P \to Q)$, we conclude Q, by modus ponens.
From Q and $Q \to R$, we conclude R, again by modus ponens, as desired.
Therefore, R does follow logically from the premise, so the argument is valid. □

Proof. Assume P and $P \to Q$ and $\neg Q \vee R$.
From P and $P \to Q$, we conclude Q, by modus ponens.
From Q and $\neg Q \vee R$, we conclude R, by disjunctive syllogism. □

Exercise 17. Let a and b be propositions.
Show that $(\neg(a \wedge b) \wedge (a \vee b)) \vee ((a \wedge b) \vee \neg(a \vee b))$ is a tautology.

Solution. Let $p = a \wedge b$.

Let $q = a \vee b$.

Let $r = p \vee \neg q$.

Then we have

$$\begin{aligned}
(\neg(a \wedge b) \wedge (a \vee b)) \vee ((a \wedge b) \vee \neg(a \vee b)) &\Leftrightarrow \\
(\neg p \wedge q) \vee (p \vee \neg q) &\Leftrightarrow \\
(\neg p \wedge \neg\neg q) \vee (p \vee \neg q) &\Leftrightarrow \\
\neg(p \vee \neg q) \vee (p \vee \neg q) &\Leftrightarrow \\
\neg r \vee r &\Leftrightarrow \\
T.
\end{aligned}$$

□

Exercise 18. Prove $(P \to Q) \wedge (Q \to R) \Rightarrow P \to R$.

Solution. We must prove $(P \to Q) \wedge (Q \to R)$ implies $P \to R$.

Our hypothesis is $(P \to Q) \wedge (Q \to R)$.

Our conclusion is $P \to R$.

We observe that this statement has the form $p \to (q \to r)$ which we know is logically equivalent to $(p \wedge q) \to r$. Hence, we can add p to the list of premises.

Thus our list of premises becomes $P \to Q, Q \to R, P$.

We must prove R.

We note that this is the hypothetical syllogism inference rule which we know is a valid argument. □

Proof. To prove $(P \to Q) \wedge (Q \to R)$ implies $P \to R$ we prove the equivalent statement $[(P \to Q) \wedge (Q \to R) \wedge P] \to R$.

To prove R is true we assume the truth of the premise $(P \to Q) \wedge (Q \to R) \wedge P$; that is, we assume the partial premises $P \to Q$, $Q \to R$, P are true.

From P and $P \to Q$, we conclude Q, by modus ponens.

From Q and $Q \to R$, we conclude R, again by modus ponens, as desired. □

Exercise 19. Show that the argument $(P \to Q) \wedge (Q \to R) \wedge (S \to R) \Rightarrow P \to S$ is not valid.

Solution. We analyze whether we can derive S from P, given the three hypotheses.

Since we know that $p \to (q \to r) \Leftrightarrow (p \wedge q) \to r$, we can add P to the list of hypotheses.

Thus we must show whether we can derive S from this expanded list.

From P and $P \to Q$, we conclude Q, by modus ponens.

From Q and $Q \to R$, we conclude R, again by modus ponens.

At this point we cannot deduce S from $R \wedge (S \to R)$.

Thus, we suspect that the argument is not valid.

Let's prove that the argument is not valid.

To prove that the argument is not valid we must devise a counterexample.

That is, we must devise a combination of truth values that cause the conclusion to be false.

In other words, we must find truth values for P, Q, R, S such that $(P \to Q) \land (Q \to R) \land (S \to R)$ is true while $P \to S$ is false.

In order for $P \to S$ to be false P must be true and S must be false.

In that case if $P \to Q$ is to be true then Q must be true.

But then R must be true in order for $Q \to R$ to be true.

Observe that if S is false and R is true then the partial premise $S \to R$ is true.

Thus, we have found a combination of truth values $P = T, Q = T, R = T, S = F$ for which the premise of the argument is true and the conclusion is false.

Therefore the argument is not valid. $\square$

Exercise 20. Show that $\forall x.p(x) \lor \forall x.q(x)$ is stronger than $\forall x.[p(x) \lor q(x)]$.

Solution. Let $p(x), q(x)$ be predicates defined over a domain of discourse U with truth sets P and Q, respectively.

We know that the statement $\forall x.p(x) \lor \forall x.q(x)$ is true if and only if $P = U$ or $Q = U$.

Thus, $P \cup Q = U$ if either P or Q is U.

Hence, $\forall x.[p(x) \lor q(x)]$ is true if either P or Q is U.

By transitivity of material implication $\to$, we conclude that $\forall x.p(x) \lor \forall x.q(x)$ logically implies $\forall x.[p(x) \lor q(x)]$. $\square$

Exercise 21. Show that $(\forall x)(\forall y)(\forall z)[p(x, y, z) \to q(x, y)]$ is stronger than $(\forall x)(\forall y)[(\forall z)p(x, y, z) \to q(x, y)]$.

Solution. Let U be a universal set (domain of discourse), $U \neq \emptyset$.

Let $p(x, y, z)$ and $q(x, y)$ be propositional functions where each variable $x, y, z \in U$.

The truth set of p is $P = \{(x, y, z) \in U^3 : p(x, y, z) \text{ is true } \}$.

The truth set of q is $Q = \{(x, y) \in U^2 : q(x, y) \text{ is true } \}$.

Define statements W, Z by

$W : (\forall x)(\forall y)(\forall z)[p(x, y, z) \to q(x, y)]$

$Z : (\forall x)(\forall y)[(\forall z)p(x, y, z) \to q(x, y)]$.

We know that in general, a statement W is stronger than a statement Z iff $W \Rightarrow Z$; that is, iff $W \to Z$ is true.

We know that $W \to Z \Leftrightarrow \neg Z \to \neg W$.

So, let's take the negation of each statement and show that $\neg Z$ is stronger than $\neg W$.

If we can show this, then we will have shown that W is stronger than Z.

Observe that

$$\begin{aligned}
\neg Z &\Leftrightarrow \neg(\forall x)(\forall y)[(\forall z)p(x, y, z) \to q(x, y)] \\
&\Leftrightarrow (\exists x)(\exists y)[(\forall z)p(x, y, z) \land \neg q(x, y)]
\end{aligned}$$

and

$$
\begin{aligned}
\neg W &\Leftrightarrow \neg(\forall x)(\forall y)(\forall z)[p(x,y,z) \to q(x,y)] \\
&\Leftrightarrow (\exists x)(\exists y)(\exists z)[p(x,y,z) \wedge \neg q(x,y)] \\
&\Rightarrow (\exists x)(\exists y)[(\exists z)p(x,y,z) \wedge (\exists z)\neg q(x,y)] \\
&\Leftrightarrow (\exists x)(\exists y)[(\exists z)p(x,y,z) \wedge \neg q(x,y)]
\end{aligned}
$$

The statements differ in that $\neg Z$ has the predicate $(\forall z)p(x,y,z)$ while $\neg W$ has the predicate $(\exists z)p(x,y,z)$.

A theorem from predicate logic states that $(\forall z)p(x,y,z) \Rightarrow (\exists z)p(x,y,z)$ for nonempty U.

Thus, $\neg Z$ is stronger than $\neg W$, so W is stronger than Z. □

Exercise 22. Let $\mathbb{R}$ be the domain of discourse.

Prove $(\exists! x)(x^2 + 8x + 16 = 0)$.

Solution. We define the predicate $p(x) : x^2 + 8x + 16 = 0$.

We must prove exactly one $x \in \mathbb{R}$ satisfies the equation $x^2 + 8x + 16 = 0$.

In other words, the solution set P to $p(x)$ contains exactly one real number. □

Proof. Suppose $x^2 + 8x + 16 = 0$.

Then $(x + 4)^2 = 0$, so $x = -4$.

Hence, if $x^2 + 8x + 16 = 0$, then $x = -4$, so there is at most one solution.

We substitute -4 for x in $x^2 + 8x + 16$ to get $(-4)^2 + 8(-4) + 16 = 0$, so that, in fact, there is exactly one solution, $x = -4$, as desired. □

Exercise 23. Show that 'All men are mortal' and 'Socrates is a man' implies 'Socrates is mortal'.

Proof. Let U be the set of all men.

Define predicates $p(x) : x$ is a man and $q(x) : x$ is mortal over U.

Then we have the propositions : $\forall x.p(x) \to q(x)$ and $p(Socrates)$.

Hence, $p(Socrates) \to q(Socrates)$, by universal instantiation.

From $p(Socrates) \to q(Socrates)$ and $p(Socrates)$, we conclude $q(Socrates)$, by modus ponens.

Therefore, 'Socrates is mortal'. □

Exercise 24. Translate the following into English:

a. Let $p(x,y,z)$ be a propositional function.

$\forall x \exists y \forall z.p(x,y,z)$.

b. Let $p(v,w,x,y,z)$ be a propositional function.

$\exists v \exists w \forall x \forall y \exists z.p(v,w,x,y,z)$.

Solution. For a. the translation is 'to every x, there corresponds at least one y such that for every z, $p(x,y,z)$.

For b. the translation is 'there exist v and w having the property that, to every x and y, there corresponds at least one z such that $p(v,w,x,y,z)$. □

Exercise 25. Let $p(x, y)$ be a propositional function.

Discuss the conditions for the truth of $\forall x \exists y.p(x, y)$.

Solution. Let x have domain of discourse U_1 and y have domain of discourse U_2.

Then the domain of discourse for $p(x, y)$ is $U_1 \times U_2$.

Let P be the truth set of $p(x, y)$.

Then $P = \{(x, y) \in U_1 \times U_2 : p(x, y) \text{ is true } \}$, so $P \subseteq U_1 \times U_2$.

We know that $\forall x \exists y.p(x, y)$ is a statement because both x, y are bound variables.

The statement $\forall x \exists y.p(x, y)$ translates into English as 'to every x there corresponds at least one y such that $p(x, y)$.

When is this statement true?

We look at the innermost quantifier and work outwards.

Consider the predicate $\exists y.p(x, y)$.

We see that x is free and y is bound,so this predicate is a function of x.

Let $r(x) = \exists y.p(x, y)$ be a propositional function of x.

The truth set of $r(x)$ is $R = \{x \in U_1 : r(x) \text{ is true } \}$.

Thus, $R = \{x \in U_1 : p(x, y) \text{ is true for some } y \in U_2\} = \{x \in U_1 : (x, y) \in P$ for some $y \in U_2\}$.

Let $u_1 \in R$.

Then $u_1 \in U_1$ and $(u_1, y) \in P$ for some $y \in U_2$.

Hence, $u_1 \in R$ if and only if $(u_1, y) \in P$ for some $y \in U_2$.

This means that P must intersect the vertical line $x = u_1$.

Thus, $\forall x.r(x) = \forall x \exists y.p(x, y)$ means that P must intersect every vertical line. □

Exercise 26. Let $p(x, y)$ be a propositional function.

Show that $\forall x \forall y.p(x, y) \Leftrightarrow \forall y \forall x.p(x, y)$.

Solution. Let $U_1 \times U_2$ be the domain of discourse for $p(x, y)$.

We must prove:

1. $\forall x \forall y.p(x, y) \Rightarrow \forall y \forall x.p(x, y)$
2. $\forall y \forall x.p(x, y) \Rightarrow \forall x \forall y.p(x, y)$

Suppose $\forall x \forall y.p(x, y)$ is true.

Let $r(x) = \forall y.p(x, y)$.

Then $\forall x.r(x)$ is true.

Hence, $r(u_1)$ is true for an arbitrary element $u_1 \in U_1$.

Therefore, $\forall y.p(u_1, y)$ is true.

Hence, $p(u_1, u_2)$ is true for an arbitrary element $u_2 \in U_2$.

Since u_1 is arbitrary, then $p(x, u_2)$ is true for every $x \in U_1$.

Thus, $\forall x.p(x, u_2)$ is true.

Since u_2 is arbitrary, then $\forall x.p(x, y)$ is true for every $y \in U_2$.

Therefore, $\forall y \forall x.p(x, y)$ is true.

Note that the converse is also true and can be proved by reversing the argument above.

Hence, order does not matter when quantifiers are of the same kind.

Thus, $\forall x \forall y.p(x,y) \Leftrightarrow \forall y \forall x.p(x,y)$. □

Proof. Let $U_1 \times U_2$ be the domain of discourse for $p(x,y)$.

We prove $\forall x \forall y.p(x,y) \Rightarrow \forall y \forall x.p(x,y)$.

Suppose $\forall x \forall y.p(x,y)$ is true.

Consider the inner predicate $\forall y.p(x,y)$.

Observe that y is bound and x is free, so $\forall y.p(x,y)$ is a function of x.

Let $r(x) = \forall y.p(x,y)$.

Then $\forall x.r(x)$ is true.

By universal instantiation, $r(u_1)$ is true for an arbitrary element $u_1 \in U_1$.

Thus, $\forall y.p(u_1,y)$ is true.

By universal instantiation, $p(u_1,u_2)$ is true for an arbitrary element $u_2 \in U_2$.

Since u_1 is arbitrary, then by universal generalization, $p(x,u_2)$ is true for every $x \in U_1$.

Thus, $\forall x.p(x,u_2)$ is true.

Since u_2 is arbitrary, then by universal generalization, $\forall x.p(x,y)$ is true for every $y \in U_2$.

Thus, $\forall y \forall x.p(x,y)$ is true.

Therefore, $\forall x \forall y.p(x,y)$ logically implies $\forall y \forall x.p(x,y)$.

Conversely, we prove $\forall y \forall x.p(x,y) \Rightarrow \forall x \forall y.p(x,y)$.

Suppose $\forall y \forall x.p(x,y)$ is true.

Consider the inner predicate $\forall x.p(x,y)$.

Observe that x is bound and y is free, so $\forall x.p(x,y)$ is a function of y.

Let $s(y) = \forall x.p(x,y)$.

Then $\forall y.s(y)$ is true.

By universal instantiation, $s(u_2)$ is true for an arbitrary $u_2 \in U_2$.

Thus, $\forall x.p(x,u_2)$ is true.

By universal instantiation, $p(u_1,u_2)$ is true for an arbitrary $u_1 \in U_1$.

Since u_2 is arbitrary, then by universal generalization, $p(u_1,y)$ is true for every $y \in U_2$.

Thus, $\forall y.p(u_1,y)$ is true.

Since u_1 is arbitrary, then by universal generalization, $\forall y.p(x,y)$ is true for every $x \in U_1$.

Thus, $\forall x \forall y.p(x,y)$ is true.

Therefore, $\forall y \forall x.p(x,y)$ logically implies $\forall x \forall y.p(x,y)$.

Since $\forall x \forall y.p(x,y)$ implies $\forall y \forall x.p(x,y)$ and $\forall y \forall x.p(x,y)$ implies $\forall x \forall y.p(x,y)$, then it follows that $\forall x \forall y.p(x,y)$ is logically equivalent to $\forall y \forall x.p(x,y)$. □

Exercise 27. Let $p(x,y)$ be a propositional function.

Show that $\exists x \exists y.p(x,y) \Leftrightarrow \exists y \exists x.p(x,y)$.

Solution. We must prove:

1. $\exists x \exists y.p(x,y) \Rightarrow \exists y \exists x.p(x,y)$
2. $\exists y \exists x.p(x,y) \Rightarrow \exists x \exists y.p(x,y)$ □

Proof. Let $U_1 \times U_2$ be the domain of discourse for $p(x, y)$.

We first prove $\exists x \exists y.p(x, y) \Rightarrow \exists y \exists x.p(x, y)$.

Suppose $\exists x \exists y.p(x, y)$ is true.

Consider the inner predicate $\exists y.p(x, y)$.

Observe that the variable y is bound and x is free, so $\exists y.p(x, y)$ is a function of x.

Let $r(x) = \exists y.p(x, y)$.

Then $\exists x.r(x)$ is true.

By existential instantiation, $r(u_1)$ is true for a specific element $u_1 \in U_1$.

Hence, $\exists y.p(u_1, y)$ is true.

By existential instantiation, $p(u_1, u_2)$ is true for a specific object $u_2 \in U_2$.

Since u_1 is a particular element in U_1, then by existential generalization, $p(x, u_2)$ is true for some $x \in U_1$.

Thus, $\exists x.p(x, u_2)$ is true.

Since u_2 is a particular element in U_2, then by existential generalization, $\exists x.p(x, y)$ is true for some $y \in U_2$.

Thus, $\exists y \exists x.p(x, y)$ is true.

Therefore, $\exists x \exists y.p(x, y)$ logically implies $\exists y \exists x.p(x, y)$.

Conversely, we prove $\exists y \exists x.p(x, y) \Rightarrow \exists x \exists y.p(x, y)$.

Suppose $\exists y \exists x.p(x, y)$ is true.

Consider the inner predicate $\exists x.p(x, y)$.

Observe that the variable x is bound and y is free, so $\exists x.p(x, y)$ is a function of y.

Let $s(y) = \exists x.p(x, y)$.

Then $\exists y.s(y)$ is true.

By existential instantiation, $s(u_2)$ is true for a specific object $u_2 \in U_2$.

Thus, $\exists x.p(x, u_2)$ is true.

By existential instantiation, $p(u_1, u_2)$ is true for a specific object $u_1 \in U_1$.

Since u_2 is a particular element in U_2,then by existential generalization, $p(u_1, y)$ is true for some $y \in U_2$.

Thus, $\exists y.p(u_1, y)$ is true.

Since u_1 is a particular element in U_1, then by existential generalization, $\exists y.p(x, y)$ is true for some $x \in U_1$.

Thus, $\exists x \exists y.p(x, y)$ is true.

Therefore, $\exists y \exists x.p(x, y)$ logically implies $\exists x \exists y.p(x, y)$.

Since $\exists x \exists y.p(x, y) \Rightarrow \exists y \exists x.p(x, y)$ and $\exists y \exists x.p(x, y) \Rightarrow \exists x \exists y.p(x, y)$, then it follows that $\exists x \exists y.p(x, y) \Leftrightarrow \exists y \exists x.p(x, y)$. □

Exercise 28. Let $p(x, y)$ be a propositional function.

Discuss the conditions for the truth of $\exists y \forall x.p(x, y)$.

Solution. Let x have domain of discourse U_1 and y have domain of discourse U_2.

Then the domain of discourse for $p(x, y)$ is $U_1 \times U_2$.

Let P be the truth set of $p(x, y)$.

Then $P = \{(x, y) \in U_1 \times U_2 : p(x, y) \text{ is true } \}$, so $P \subseteq U_1 \times U_2$.

We know that $\exists y \forall x . p(x, y)$ is a statement because x and y are bound variables.

The statement $\exists y \forall x . p(x, y)$ translates into English as 'there exists y such that for every x, $p(x, y)$'.

When is this statement true?

We look at the innermost quantifier and work outwards.

Consider the predicate $\forall x . p(x, y)$.

We see that x is bound and y is free, so this predicate is a function of y.

Let $s(y) = \forall x . p(x, y)$.

Thus, $s(y)$ is a propositional function of y.

How can we describe the truth set of $s(y)$?

The truth set of $s(y)$ is $S = \{y \in U_2 : s(y) \text{ is true }\} = \{y \in U_2 : p(x, y) \text{ is true for every } x \in U_1\} = \{y \in U_2 : (x, y) \in P \text{ for each } x \in U_1\}$.

Let $u_2 \in S$.

Then $u_2 \in U_2$ and $(x, u_2) \in P$ for every $x \in U_1$.

This means that the horizontal line $y = u_2$ is contained in P.

Thus, $\exists y \forall x . p(x, y)$ means that some horizontal line is contained in P.

Therefore, $\exists y \forall x . p(x, y)$ means that some horizontal line is in the truth set of $p(x, y)$. □

Exercise 29. Let $p(x, y)$ be a predicate with domain of discourse $U_1 \times U_2$.

Let $A \subset U_1$ and $B \subset U_2$.

Show that $\neg(\forall x \in A)(\exists y \in B).p(x, y) \Leftrightarrow (\exists x \in A)(\forall y \in B).\neg p(x, y)$.

Solution. We use logic inference rules. □

Proof. Observe that

$$\begin{aligned}
\neg(\forall x \in A)(\exists y \in B).p(x, y) &\Leftrightarrow \neg(\forall x \in A)\exists y.[y \in B \wedge p(x, y)] \\
&\Leftrightarrow \neg\forall x.[(x \in A) \rightarrow \exists y.(y \in B \wedge p(x, y))] \\
&\Leftrightarrow \exists x.\neg[(x \in A) \rightarrow \exists y.(y \in B \wedge p(x, y))] \\
&\Leftrightarrow \exists x.[(x \in A) \wedge \neg\exists y.(y \in B \wedge p(x, y))] \\
&\Leftrightarrow (\exists x \in A)\neg\exists y[(y \in B) \wedge p(x, y)] \\
&\Leftrightarrow (\exists x \in A)\forall y.\neg[(y \in B) \wedge p(x, y)] \\
&\Leftrightarrow (\exists x \in A)\forall y.[(y \notin B) \vee \neg p(x, y)] \\
&\Leftrightarrow (\exists x \in A)\forall y.[(y \in B) \rightarrow \neg p(x, y)] \\
&\Leftrightarrow (\exists x \in A)(\forall y \in B).\neg p(x, y).
\end{aligned}$$

□

Exercise 30. Show that $(\forall x \neq 0)(\exists y)(xy = 1)$ is true for domain of discourse $\mathbb{R}$.

Solution. Let $p(x,y) : xy = 1$ be a predicate defined over $\mathbb{R} \times \mathbb{R}$.

We must prove that $(\forall x \neq 0)(\exists y).p(x,y)$ is true.

Let a be an arbitrary nonzero real number.

Then $a \in \mathbb{R}^*$.

Let $b = \frac{1}{a}$.

Since a is nonzero, then $\frac{1}{a}$ is a real number.

Hence, $b \in \mathbb{R}$.

Observe that $ab = a(1/a) = 1$.

In English, this true statement says to each nonzero real number, there corresponds at least one real number that is its multiplicative inverse. In other words, the assertion is that all nonzero real numbers have a multiplicative inverse. □

Exercise 31. Express the English statements below in formal logic:

1. Zero things are human.
2. At least one thing is human.
3. Exactly one thing is human.
4. At least two things are human.
5. Exactly two things are human.

Solution. We must define the predicate.

In this case, we define

$H(x) : x$ is human or more compactly, $Hx : x$ is human

For example, $H(Jason)$ means 'Jason' is human, which is true.

$H(cat)$ means 'a cat is human', which is false.

We must translate each statement by considering its meaning.

'Zero things are human' means 'nothing is human' which means 'there is no thing that is human' which is logically equivalent to 'all things are not human'.

Thus, $\neg\exists x.Hx$ which is logically equivalent to $\forall x.\neg Hx$.

'At least one thing is human' means 'there is some thing x such that x is human'.

Thus, $\exists x.Hx$.

'Exactly one thing is human' means

'there is at least one thing, x, such that x is human and, if there is anything else y, that is human, then y is the same as x'.

Thus, we have $\exists x.Hx$ and $\forall y.Hy \to y = x$.

Hence, $\exists x \forall y.Hx \wedge (Hy \to y = x)$.

'At least two things are human' means 'there are some things x and y such that x is human and y is human and y is not identical to x' which means 'there is a human x and there is a human y and y is not identical to x'

Thus, $\exists x.Hx$ and $\exists y.Hy$ and $y \neq x$. Hence, $\exists x \exists y.(Hx \wedge Hy) \wedge (y \neq x)$.

'Exactly two things are human' means 'there are some things x and y such that x is human and y is human and y is not identical to x, and if there is anything else z, that is human, then z is identical to x or identical to y'.

Thus we have $\exists x.Hx$ and $\exists y.Hy$ and $y \neq x$ and $\forall z.Hz \to (z = x \vee z = y)$.

Hence, $\exists x \exists y. \forall z. Hx \wedge Hy \wedge (y \neq x) \wedge (Hz \to (z = x \vee z = y))$. ☐

Exercise 32. Write the definition of even number using logical symbols.

Solution. Our domain of discourse is the set of integers $\mathbb{Z}$.

By definition of even number, any integer n is even if and only if $n = 2k$ for some integer k.

Let P : Any integer n is even if and only if $n = 2k$ for some integer k.

Define propositions below.

p : any integer n is even

q : $n = 2k$ for some integer k

Then P : $p \leftrightarrow q$.

We decompose proposition p further because it expresses universal quantification.

p : $\forall n, n \in \mathbb{Z} \to n$ is even

We decompose proposition q further because it expresses existential quantification.

q : $\exists k \in \mathbb{Z}, n = 2k$

Therefore, P : $\forall n \in \mathbb{Z}, n$ is even $\leftrightarrow (\exists k \in \mathbb{Z}, n = 2k)$. ☐

Exercise 33. Define predicate $odd(n)$: n is odd iff there exists integer k such that $n = 2k + 1$.

Prove the statement $odd(n) \to odd(n + 2)$.

Proof. Assume $odd(n)$.

Then n is odd, so there exists an integer k such that $n = 2k + 1$.

Thus, $n + 2 = (2k + 1) + 2 = 2k + 1 + 2 = 2k + 2 + 1 = 2(k + 1) + 1$.

Since $k + 1 \in \mathbb{Z}$ and $n + 2 = 2(k + 1) + 1$, then $n + 2$ is odd, so $odd(n + 2)$.

From premise $odd(n)$ and $odd(n + 2)$, we conclude $odd(n) \to odd(n + 2)$, by conditional introduction. ☐

Exercise 34. Define propositional function $d(a, b)$: a divides b over the domain of discourse $\mathbb{Z} \times \mathbb{Z}$ by $\exists (k \in \mathbb{Z})(b = ak)$.

Let $a|b$ denote the statement a divides b.

Then $a|b$ means $\exists (k \in \mathbb{Z})(b = ak)$.

a. Discuss $\forall m \forall n [d(m, n) \to d(n, m)]$.

b. Discuss $\forall a \forall b \forall c [(d(a, b) \wedge d(b, c)) \to d(a, c)]$.

Solution. For a)

Observe that variables m and n are quantified, so all variables are bound.

Hence, $\forall m \forall n [d(m, n) \to d(n, m)]$ is a statement.

Is it true or false?

The statement translates as: for any integers m and n, if m divides n, then n divides m.

Taking the negation of this statement, we obtain $\neg\forall m\forall n[d(m,n) \rightarrow d(n,m)] \Leftrightarrow \exists m\exists n\neg[d(m,n) \rightarrow d(n,m)] \Leftrightarrow \exists m\exists n[d(m,n) \wedge \neg d(n,m)]$.

Thus, the statement is false if and only if there exist integers m and n such that m divides n and n does not divide m.

Can we find such integers?

Let m be an arbitrary integer.

Let n be an integer such that m divides n and n does not divide m.

Since m divides n, then n must be a multiple of m.

Let's try $m = 6$ and $n = 18$.

Observe that 6 divides 18 because $18 = 6 * 3$, but 18 does not divide 6.

Therefore, this counter-example shows that the statement $\forall m\forall n[d(m,n) \rightarrow d(n,m)]$ is false.

For b)

Observe that variables a, b, c are each quantified, so all variables are bound.

Hence, $\forall a\forall b\forall c[(d(a,b) \wedge d(b,c)) \rightarrow d(a,c)]$ is a statement.

Is it true or false?

The statement translates as: for any integers a, b, and c, if a divides b and b divides c, then a divides c.

To determine its truth let's try some examples.

Let $a = 3, b = 6, c = 12$.

Consider the statement $(3|6 \wedge 6|12) \rightarrow 3|12$.

Since $6 = 3 * 2$ then $3|6$.

Since $12 = 6 * 2$ then $6|12$.

Since $12 = 3 * 4$ then $3|12$.

Thus the statement $(3|6 \wedge 6|12) \rightarrow 3|12$ is true.

Let $a = 4, b = 24, c = 48$.

Consider the statement $(4|24 \wedge 24|48) \rightarrow 4|48$.

Since $24 = 4 * 6$ then $4|24$.

Since $48 = 24 * 2$ then $24|48$.

Since $48 = 4 * 12$ then $4|48$.

Thus the statement $(4|24 \wedge 24|48) \rightarrow 4|48$ is true.

We can try more examples to observe that the statement appears true.

Thus, we conjecture that this statement appears true.

Let's try to prove this.

We prove $\forall a\forall b\forall c[(d(a,b) \wedge d(b,c)) \rightarrow d(a,c)]$ is true.

Let m, n, p be arbitrary integers.

Suppose m divides n and n divides p.

Can we deduce that m divides p?

Since $m|n$, then $n = mk_1$ for some integer k_1.

Since $n|p$, then $p = nk_2$ for some integer k_2.

Thus, $p = (mk_1)k_2 = m(k_1k_2)$.

Let $k = k_1k_2$.

Then $p = mk$.

Since the set of integers is closed under multiplication, then k is an integer.

By existential generalization, $p = mk$ for some integer k.

Thus, the statement $\exists(k \in \mathbb{Z})(p = mk)$ is true, so m divides p.

Therefore, the statement $(m|n \wedge n|p) \rightarrow m|p$ is true.

Since p is arbitrary, then by universal generalization, $(m|n \wedge n|c) \rightarrow m|c$ is true for every $c \in \mathbb{Z}$.

Thus, $\forall c.[(m|n \wedge n|c) \rightarrow m|c]$ is true.

Since n is arbitrary, then by universal generalization, $\forall c.[(m|b \wedge b|c) \rightarrow m|c]$ is true for every $b \in \mathbb{Z}$.

Thus, $\forall b \forall c.[(m|b \wedge b|c) \rightarrow m|c]$ is true.

Since m is arbitrary, then by universal generalization, $\forall b \forall c.[(a|b \wedge b|c) \rightarrow a|c]$ is true for every $a \in \mathbb{Z}$.

Thus, $\forall a \forall b \forall c.[(a|b \wedge b|c) \rightarrow a|c]$ is true.

This is equivalent to stating that $\forall a \forall b \forall c.[(d(a,b) \wedge d(b,c)) \rightarrow d(a,c)]$ is true.

Since $\forall a \forall b \forall c.[(d(a,b) \wedge d(b,c)) \rightarrow d(a,c)]$ is true, then $(d(a,b) \wedge d(b,c)) \rightarrow d(a,c)$ is true for all integers a, b, c. Thus, $(d(a,b) \wedge d(b,c)$ logically implies $d(a,c)$.

Therefore, $(d(a,b) \wedge d(b,c)) \Rightarrow d(a,c)$.

This is equivalent to $(a|b \wedge b|c) \Rightarrow a|c$. □

Exercise 35. Define predicate $d(a,b)$: a divides b over domain of discourse $\mathbb{N} \times \mathbb{N}$ by $(\exists k \in \mathbb{N})(b = ak)$.

a. Prove $\forall a \exists b(d(a,b))$.

b. Prove $(\forall b \in \mathbb{N})(\exists a \in \mathbb{N})(a|b)$

Solution. For a)

We can try various examples to demonstrate that a appears to be true.

We observe that each natural number appears to divide itself and every multiple of itself.

The statement translates to every natural number divides some natural number.

We prove $\forall a \exists b(d(a,b))$.

Let m be an arbitrary natural number.

We must prove there exists a natural number b such that m divides b.

Since $m \in \mathbb{N}$ and $\mathbb{N} \subset \mathbb{Z}$, then $m \in \mathbb{Z}$.

Since $m \in \mathbb{N}$, then $m \neq 0$.

Hence, m is a nonzero integer.

Since every nonzero integer divides itself, then m divides itself.

Thus, $m|m$ is true.

Let $b = m$.

Then b is a natural number and $m|b$.

By existential generalization, $m|b$ for some natural number b.

Thus, the statement $(\exists b \in \mathbb{N})(m|b)$ is true.

Since m is arbitrary, then by universal generalization, $(\exists b \in \mathbb{N})(a|b)$ is true for every natural number a.

Therefore, the statement $(\forall a \in \mathbb{N})(\exists b \in \mathbb{N})(a|b)$ is true.

For b)

We can try some examples.

We observe that a prime number is divisible by 1 and itself and each non-prime number is certainly divisible by 1. So, 1 divides each number.

The statement translates to each natural number is a multiple of some natural number.

We prove $(\forall b \in \mathbb{N})(\exists a \in \mathbb{N})(a|b)$.

Let n be an arbitrary natural number.

We must prove there exists a natural number a such that a divides n.

Let $a = 1$.

Then $a \in \mathbb{N}$.

Since $n \in \mathbb{N}$ and $\mathbb{N} \subset \mathbb{Z}$, then $n \in \mathbb{Z}$.

Since 1 divides every integer, then 1 divides n.

Thus, $1|n$ for the natural number 1.

By existential generalization, $a|n$ for some natural number a.

Thus, $(\exists a \in \mathbb{N})(a|n)$ is true.

Since n is arbitrary, then by universal generalization, $(\exists a \in \mathbb{N})(a|b)$ is true for every natural number b.

Therefore, $(\forall b \in \mathbb{N})(\exists a \in \mathbb{N})(a|b)$ is true. □

Exercise 36. If today is Tuesday, then I will go to work.

Today is Tuesday. Prove: I will go to work.

Solution. The conclusion is obvious, but we need to prove why it is 'obvious'.

We translate the argument into logical symbols.

A : Today is Tuesday.

B : I will go to work.

Our statements are:

$P_1 : A \to B$

$P_2 : A$

$P_3 : B$

We must prove $(A \to B) \wedge A$ implies B.

Our hypothesis is $(A \to B) \wedge A$.

Our conclusion is B.

To prove the conclusion follows from the premises, we assume all of the premises are true.

Thus, we assume $A \to B$ is true and A is true. □

Proof. Let A be 'Today is Tuesday' and B be 'I will go to work'.

Suppose $(A \to B) \wedge A$.

Then A is true.

Since A is true and $A \to B$, then by modus ponens, B is true.

Therefore, B. □

Exercise 37. Determine the negation of 'Either all athletes are young women or some men are athletes'.

Solution. Let our domain of discourse U be the set of all men and women.

We define predicates over U as follows:

$p(x) : x$ is an athlete

$q(x) : x$ is young

$r(x) : x$ is a woman

$s(x) : x$ is a man

The statement S has the form $P \vee Q$ where

P : All athletes are young women

Q : Some men are athletes

The statement P means every athlete is young and is a woman, so P is $\forall x.[p(x) \to (q(x) \wedge r(x))]$.

The statement Q means at least one man is an athlete, so Q is $\exists x.[s(x) \wedge p(x)]$.

Thus, S is the statement $\forall x.[p(x) \to (q(x) \wedge r(x))] \vee \exists x.[s(x) \wedge p(x)]$.

Its negation is $\neg S$.

Therefore,

$$\begin{aligned}
\neg S \quad & \Leftrightarrow \quad \neg[\forall x.[p(x) \to (q(x) \wedge r(x))] \vee \exists x.[s(x) \wedge p(x)]] \\
& \Leftrightarrow \quad \neg[\forall x.[p.x \to (q.x \wedge r.x)] \vee \exists x.(s.x \wedge p.x)] \\
& \Leftrightarrow \quad \neg\forall x.[p.x \to (q.x \wedge r.x)] \wedge \neg\exists x.(s.x \wedge p.x) \\
& \Leftrightarrow \quad \exists x.\neg[p.x \to (q.x \wedge r.x)] \wedge \neg\exists x.(s.x \wedge p.x) \\
& \Leftrightarrow \quad \exists x.[p.x \wedge \neg(q.x \wedge r.x)] \wedge \neg\exists x.(s.x \wedge p.x) \\
& \Leftrightarrow \quad \exists x.[p.x \wedge (\neg q.x \vee \neg r.x)] \wedge \neg\exists x.(s.x \wedge p.x)
\end{aligned}$$

Thus we find that the negation of S is 'some athletes are either not young or are not women, and no men are athletes'.

This is equivalent to 'some athletes are either not young or are men, and no men are athletes'. □

Exercise 38. Write the definition of subset in logic symbols.

Solution. Our domain of discourse is a collection of sets.

Let A and B be arbitrary sets in the domain of discourse.

$A \subseteq B$ means 'Every element of A is an element of B'.

We translate this statement into logic symbols.

This is $\forall x.[x \in A \to x \in B]$. □

Exercise 39. All apples are blue. A banana is an apple.

Prove: A banana is blue.

Solution. Let P_1 : All apples are blue.

Let P_2 : A banana is an apple.

Let P_3 : A banana is blue.

Our hypothesis is : $P_1 \wedge P_2$.

Our conclusion is : P_3.

Since the statement P_1 has the universal quantifier 'all', we define predicates.

Let our domain of discourse be the set of all fruit.

Let A(x) : x is an apple

Let B(x) : x is blue

Then

$P_1 : \forall x, A(x) \to B(x)$

P_2 : A(banana)

P_3 : B(banana)

Our hypothesis H is : $P_1 \wedge P_2$.

Our conclusion C is : B(banana).

We must prove $P_1 \wedge P_2$ implies C.

To prove C is true, we assume the hypothesis is true.

Thus, we assume $P_1 \wedge P_2$ is true, so this means P_1 is true and P_2 is true.

Since P_2 is true, then A(banana) is true.

Since P_1 is true, then $A(banana) \to B(banana)$ is true.

Since $A(banana) \to B(banana)$ and A(banana) is true, then by modus ponens, B(banana) is true.

Therefore, a banana is blue.

Note:

We know that P_1 is false and P_2 is false.

Our reasoning is correct in this proof.

Thus, our argument is valid, but not sound. □

Proof. Define A(x) : x is an apple

and B(x) : x is blue.

Let $H_1 : \forall x, A(x) \to B(x)$ and

$H_2 : A(banana)$.

Suppose $H_1 \wedge H_2$.

Then H_1 and H_2.

Since H_2 is true, then A(banana).

Since H_1 is true, then A(banana) → B(banana) is true.

Since A(banana) → B(banana) and A(banana), then by modus ponens inference rule, B(banana).

Therefore, a banana is blue. □

Exercise 40. It is not sunny today and it is colder than yesterday. We will go swimming only if it is sunny.

If we do not go swimming, then we will have a barbecue. If we will have a barbecue, then we will be home by sunset.

Prove: We will be home before sunset.

Solution. Intuitively, the conclusion sounds true.

Thus, we need to prove that the conclusion really does follow from the premises.

We translate the set of premises and conclusion into logical symbols.

Let P : It is not sunny today and it is colder than yesterday.

Let Q : We will go swimming only if it is sunny.

Let R: If we do not go swimming, then we will have a barbecue.

Let S: If we will have a barbecue, then we will be home by sunset.

Let T : We will be home before sunset.
Our hypothesis is H : $P \wedge Q \wedge R \wedge S$.
Our conclusion is C : T
We must show H implies C.
Let us decompose the premises into the propositions shown below.
Let p : it is sunny today
Let q : it is colder than yesterday
Let r : we will go swimming
Let s : we will have a barbecue
Let t : we will be home before sunset
Then,
P : $\neg p \wedge q$
Q : $r \to p$
R : $\neg r \to s$
S : $s \to t$
T : t
Thus, we must prove : $(\neg p \wedge q) \wedge (r \to p) \wedge (\neg r \to s) \wedge (s \to t) \Rightarrow t$.

To prove this we suppose all of the premises are true and deduce that the conclusion follows necessarily from the truth of the premises.

Suppose $(\neg p \wedge q) \wedge (r \to p) \wedge (\neg r \to s) \wedge (s \to t)$.
Since $(\neg p \wedge q) \wedge (r \to p) \wedge (\neg r \to s) \wedge (s \to t)$ is true, then
$\neg p \wedge q$ is true, by conjunction elimination.
Since $\neg p \wedge q$ is true, then $\neg p$ is true, by conjunction elimination.
Since $r \to p$ and $\neg p$ is true, then $\neg r$ is true, by modus tollens.
Since $\neg r \to s$ and $\neg r$ is true, then s is true, by modus ponens.
Since $s \to t$ and s is true, then t is true, by modus ponens.
Therefore, $(\neg p \wedge q) \wedge (r \to p) \wedge (\neg r \to s) \wedge (s \to t)$ implies t.
Note: Our reasoning is valid.
Also, all of the premises are true.
Therefore, our argument is sound. □

Proof. Let p : it is sunny today
Let q : it is colder than yesterday
Let r : we will go swimming
Let s : we will have a barbecue
Let t : we will be home before sunset
Our hypothesis is : $\neg p \wedge q, r \to p, \neg r \to s, s \to t$.
Our conclusion is : t.
Suppose $\neg p \wedge q, r \to p, \neg r \to s, s \to t$.
Then $\neg p \wedge q$, by hypothesis.
Thus, $\neg p$, by conjunction elimination.
Since $(r \to p) \wedge \neg p$, then $\neg r$, by modus tollens.
Since $(\neg r \to s) \wedge \neg r$, then s, by modus ponens.
Since $(s \to t) \wedge s$, then t, by modus ponens. □

Exercise 41. $(P \vee Q) \wedge (P \to R) \wedge (Q \to R) \Rightarrow R$.

Solution. We must prove $(P \vee Q) \wedge (P \to R) \wedge (Q \to R)$ implies R.

Our hypothesis is $H : (P \vee Q) \wedge (P \to R) \wedge (Q \to R)$.

Our conclusion is $C : R$.

We note that this logical argument is the disjunction elimination logic inference rule, which we know is valid.

To prove the conclusion follows from the premises, we assume all of the premises are true.

Thus, we assume $P \vee Q$ is true and $P \to R$ is true and $Q \to R$ is true.

How do we show R is true?

We can use the algebraic properties of propositional logic. □

Proof. Suppose $(P \vee Q) \wedge (P \to R) \wedge (Q \to R)$.

Then

$$\begin{aligned}
(P \vee Q) \wedge (P \to R) \wedge (Q \to R) &\equiv (P \vee Q) \wedge (\neg P \vee R) \wedge (\neg Q \vee R) \\
&= (P \vee Q) \wedge [(\neg P \wedge \neg Q) \vee R] \\
&= (P \vee Q) \wedge [(\neg (P \vee Q) \vee R] \\
&= [(P \vee Q) \wedge \neg (P \vee Q)] \vee [(P \vee Q) \wedge R] \\
&= F \vee [(P \vee Q) \wedge R] \\
&= (P \vee Q) \wedge R \\
&= R
\end{aligned}$$

Hence, if all of the premises are true, then it follows that R is true. □

Exercise 42. $(P \vee Q) \wedge (P \to R) \wedge (Q \to S) \Rightarrow (R \vee S)$.

Proof. Suppose $P \vee Q$ and $P \to R$ and $Q \to S$.

To prove $R \vee S$, we assume $\neg R$.

We must prove S.

Since $P \to R \Leftrightarrow \neg R \to \neg P$, then $\neg R \to \neg P$.

Since $\neg R$, then by modus ponens, $\neg P$.

Since either P or Q, then by disjunctive syllogism, Q.

Since $Q \to S$, then by modus ponens, S, as desired. □

Exercise 43. Let $A \subseteq U$.

Then $(\forall x \in A).p(x) \Leftrightarrow \forall x.[x \in A \to p(x)]$.

Solution. We must prove

1. $(\forall x \in A).p(x) \Rightarrow \forall x.[x \in A \to p(x)]$
2. $\forall x.[x \in A \to p(x)] \Rightarrow (\forall x \in A).p(x)$

To prove 1:

We must prove $\forall x.[x \in A \to p(x)]$.

To prove this we must prove $a \in A \to p(a)$ is true for every $a \in U$.

To prove this we assume a is an arbitrary object in U.

We must prove $a \in A \to p(a)$ is true.

To prove this we assume $a \in A$ and prove $p(a)$ is true.

To prove $p(a)$ is true we use the hypothesis $(\forall x \in A).p(x)$ by assuming the hypothesis is true.

To prove 2:

We must prove $(\forall x \in A).p(x)$.

To prove this we must prove $p(a)$ is true for every $a \in A$, so we assume a is an arbitrary element of A.

To prove $p(a)$ is true we assume the hypothesis $\forall x.[x \in A \to p(x)]$ is true. □

Proof. We first prove $(\forall x \in A).p(x) \Rightarrow \forall x.[x \in A \to p(x)]$.

Let a be an arbitrary element of U.

Then $a \in U$.

Suppose $a \in A$.

Suppose the hypothesis $(\forall x \in A).p(x)$ is true.

Then $p(x)$ is true for each $x \in A$.

Since $a \in A$ then it follows that $p(a)$ must be true.

Hence, $a \in A \to p(a)$ is true.

Since a is arbitrary then $a \in A \to p(a)$ is true for every $a \in U$.

Therefore, the statement $\forall x.[x \in A \to p(x)]$ is true.

Conversely, we prove $\forall x.[x \in A \to p(x)] \Rightarrow (\forall x \in A).p(x)$.

Let a be an arbitrary element of A.

Then $a \in A$.

Suppose the hypothesis $\forall x.[x \in A \to p(x)]$ is true.

Then $x \in A \to p(x)$ is true for every $x \in U$.

Since $a \in A$ and $A \subseteq U$ then $a \in U$.

Hence, it follows that $a \in A \to p(a)$ is true.

Since $a \in A$ and $a \in A \to p(a)$, then $p(a)$ is true, by modus ponens.

Since a is arbitrary, then $p(a)$ is true for every $a \in A$.

Therefore, the statement $(\forall x \in A).p(x)$ is true. □

Exercise 44. Let $A \subseteq U$.

Then $(\exists x \in A).p(x) \Leftrightarrow \exists x.[x \in A \wedge p(x)]$.

Solution. We must prove

1. $(\exists x \in A).p(x) \Rightarrow \exists x.[x \in A \wedge p(x)]$.
2. $\exists x.[x \in A \wedge p(x)] \Rightarrow (\exists x \in A).p(x)$.

To prove 1:

We must prove $\exists x.[x \in A \wedge p(x)]$.

To prove this we must prove there exists an object a such that $a \in A$ and $p(a)$ is true.

To prove this we assume the hypothesis $(\exists x \in A).p(x)$ is true.

Thus, we assume there exists a specific object a in A such that $p(a)$ is true.

To prove 2:

We must prove $(\exists x \in A).p(x)$.

To prove this we must prove there exists an object a in A such that $p(a)$ is true.

To prove this we assume the hypothesis $\exists x.[x \in A \wedge p(x)]$ is true.
Thus, we assume there exists a specific object a such that $a \in A$ and $p(a)$ is true. $\square$

Proof. We first prove $(\exists x \in A).p(x) \Rightarrow \exists x.[x \in A \wedge p(x)]$.
Suppose a is a specific element of A such that $p(a)$ is true.
Then $a \in A$ and $p(a)$ is true.
Since $a \in A$ and $A \subseteq U$ then $a \in U$.
Thus, there is some element a in U such that $a \in A$ and $p(a)$ is true.
Hence, the statement $\exists x.[x \in A \wedge p(x)]$ is true.
Conversely, we prove $\exists x.[x \in A \wedge p(x)] \Rightarrow (\exists x \in A).p(x)$.
Suppose a is a specific element of U such that $a \in A$ and $p(a)$ is true.
Then there exists some object $a \in A$ such that $p(a)$ is true.
Hence, the statement $(\exists x \in A).p(x)$ is true. $\square$

Exercise 45. Let $p(x, y)$ be the propositional function $x + y = y$.
Discuss the logical implication of $\exists x \forall y.p(x, y)$.

Solution. The predicate $p(x, y) : x+y = y$ has domain of discourse $\mathbb{R}\times\mathbb{R} = \mathbb{R}^2$.
Thus, $x, y \in \mathbb{R}$.
Observe that x and y are bound in $\exists x \forall y.p(x, y)$.
Thus, $\exists x \forall y.p(x, y)$ is a statement.
We know $\exists y \forall x.p(x, y) \Rightarrow \forall x \exists y.p(x, y)$, so we conclude $\exists x \forall y.p(x, y) \Rightarrow \forall y \exists x.p(x, y)$.
The statement $\exists x \forall y.p(x, y)$ is $\exists x \forall y.(x + y = y)$.
Is $\exists x \forall y.(x + y = y)$ true or false?
Consider the equation $x + y = y$.
Observe that $x + y = y \Rightarrow x = 0$.
Let $x = 0$.
Observe that $0 + y = y$ for every $y \in \mathbb{R}$.
This means zero is an additive identity in the real number system.
Thus, for every $y \in \mathbb{R}$, $x + y = y$ when $x = 0$.
Hence, for every $y \in \mathbb{R}$, $x + y = y$ for at least one $x \in \mathbb{R}$.
Therefore the statement $\forall y \exists x(x + y = y)$ is true, as expected. $\square$

Exercise 46. Discuss the truth of $\forall x \exists y(x + y = 0)$ and $\exists y \forall x(x + y = 0)$.

Solution. Let $p(x, y) : x + y = 0$ be a propositional function.
Let x and y be real numbers.
Then the domain of discourse for $p(x, y)$ is $\mathbb{R} \times \mathbb{R} = \mathbb{R}^2$ and $x, y \in \mathbb{R}$.
We consider $\forall x \exists y.p(x, y)$ which is $\forall x \exists y.(x + y = 0)$.
Observe that x and y are bound, so $\forall x \exists y.p(x, y)$ is a statement.
Is this statement true or false?
Consider the equation $x + y = 0$.
Observe that $x + y = 0 \Rightarrow y = -x$.
Let x be an arbitrary real number.
Then $x \in \mathbb{R}$.
Let $y = -x$.

Then $y \in \mathbb{R}$.

Since $-x$ is the additive inverse of x, then $x + (-x) = 0 = x + y$.

Hence, $x + y = 0$ when $y = -x$, so $p(x, y)$ is true for some $y \in \mathbb{R}$.

Since x is arbitrary, then for every $x \in \mathbb{R}$, $p(x, y)$ is true for some $y \in \mathbb{R}$.

Thus, the statement $\forall x \exists y.p(x, y)$ is true, so $\forall x \exists y.(x + y = 0)$ is true.

This means every real number has an additive inverse.

We consider $\exists y \forall x.p(x, y)$ which is $\exists y \forall x.(x + y = 0)$.

Observe that x and y are bound, so $\exists y \forall x.p(x, y)$ is a statement.

Is this statement true or false?

The statement $\exists y \forall x.(x + y = 0)$ is true whenever there exists y such that $x + y = 0$ for every x; that is, whenever y is an additive inverse for every x.

Thus, the statement $\exists y \forall x.(x+y = 0)$ is true if and only if there exists a real number y that is the additive inverse of every real number x.

However, no such real number y exists; ie, there is no real number that is the additive inverse of every real number.

Therefore, the statement $\exists y \forall x.(x + y = 0)$ is false, so $\exists y \forall x.p(x, y)$ is false.

We conclude that $\forall x \exists y.(x + y = 0)$ is true while $\exists y \forall x.(x + y = 0)$ is false.

Hence, $\forall x \exists y.p(x, y)$ is true but $\exists y \forall x.p(x, y)$ is false.

This means $\forall x \exists y.p(x, y) \not\Rightarrow \exists y \forall x.p(x, y)$, so in general, $\forall x \exists y.p(x, y)$ does not necessarily imply $\exists y \forall x.p(x, y)$.

Contrast with $\exists x \forall y.p(x, y) \Rightarrow \forall y \exists x.p(x, y)$, which is always true. $\square$

Exercise 47. Let U be the collection of all finite subsets of $\mathbb{N}$.

Define the propositional function $s(X, Y)$ on the domain of discourse $U \times U = U^2$ by $s(X, Y) : X \subseteq Y$.

Discuss the truth of the statements

a) $\forall X \exists Y.s(X, Y)$.

b) $\forall Y \exists X.s(X, Y)$.

c) $\exists Y \forall X.s(X, Y)$

d) $\exists X \forall Y.s(X, Y)$

Solution. For $a)$ we know that the statement $\forall X \exists Y.s(X, Y)$ is true if and only if for every $X \in U$, there is at least one Y such that $s(X, Y)$ is true; i.e., the statement means for every set X in the collection U, there corresponds some set Y in U that contains X.

Can we find such a Y?

Yes, because we can let $Y = X$ and observe that $X \subseteq X$.

This is because we know that every set is a subset of itself. The proof is below.

Let X be an arbitrary finite subset of $\mathbb{N}$.

Then $X \in U$.

Let $Y = X$.

Then $Y \in U$.

Since every set is a subset of itself, then $X \subseteq X$.

Hence, $X \subseteq Y$.

Thus, there exists $Y \in U$ such that $X \subseteq Y$, so there exists $Y \in U$ such that $s(X, Y)$ is true.

Since X is arbitrary, then the statement $\forall X \exists Y.s(X,Y)$ is true.

For b) we know that the statement $\forall Y \exists X.s(X,Y)$ is true if and only if to each $Y \in U$, there corresponds at least one $X \in U$ such that $s(X,Y)$ is true; i.e., the statement means for every Y in U, there is at least one $X \in U$ such that $X \subseteq Y$.

We can devise concrete examples and conjecture that this statement seems true.We must also consider the empty set because $\emptyset$ is a finite subset of $\mathbb{N}$, so $\emptyset$ is in U too. Hence we must consider two cases, one in which Y is non-empty and another in which Y is empty. The proof is below.

Let Y be an arbitrary finite subset of $\mathbb{N}$.

Either Y is empty or non-empty.

We consider these cases separately.

Case 1: Suppose $Y \neq \emptyset$.

Since Y is an arbitrary finite subset of $\mathbb{N}$, then $Y \in U$.

Since Y is not empty, then Y contains at least one natural number.

Let a be a natural number in Y.

Then $a \in \mathbb{N}$ and $a \in Y$.

Let $X = \{a\}$.

Then X is a finite set consisting of a natural number.

Thus, $X \in U$.

Observe that $X \subseteq Y$.

Hence, there exists some $X \in U$ such that $s(X,Y)$ is true.

Case 2: Suppose $Y = \emptyset$.

The empty set is finite, so Y is finite.

Since the empty set is a subset of every set, then $\emptyset \subseteq \mathbb{N}$.

Hence, Y is a finite subset of $\mathbb{N}$, so $Y \in U$.

Let $X = \emptyset$.

Then $X = Y$, so $X \in U$.

Every set is a subset of itself, so $\emptyset \subseteq \emptyset$.

Hence, $X \subseteq Y$.

Thus, there exists some $X \in U$ such that $s(X,Y)$ is true.

Both cases together show that for every $Y \in U$ there exists some $X \in U$ such that $s(X,Y)$ is true.

Therefore, the statement $\forall Y \exists X.s(X,Y)$ is true.

For c) we know the statement $\exists Y \forall X.s(X,Y)$ is true if and only if there exists a $Y \in U$ such that $s(X,Y)$ is true for every $X \in U$; that is, if there exists a finite subset of natural numbers Y such that $X \subset Y$ for every $X \in U$. This means if there exists a finite subset of natural numbers that contains every finite subset of natural numbers. Intuitively, there does not seem to be a single finite subset of natural numbers that contains every finite subset of natural numbers. We prove by contradiction below.

Suppose Y is a finite subset of natural numbers that contains every finite subset of natural numbers.

Then $Y \in U$ and $X \subset Y$ for every $X \in U$.
Since $\{1\} \in U$ then $\{1\} \subset Y$, so $1 \in Y$.
Hence, Y cannot be empty, so $Y \neq \emptyset$.
Since Y is finite then let $n \in N$ be the cardinality of Y.(Note that $n \neq 0$ because Y is not empty.)
The elements in Y can be arranged in a sequence in which the i^{th} element corresponds to the natural number denoted by a_i.
Thus, $Y = \{a_1, ..., a_n\}$.
Since $n \in \mathbb{N}$ then $n + 1 \in \mathbb{N}$.
Since Y is finite and $\mathbb{N}$ is infinite, then there must exist some natural number that is not in Y.
Let a_{n+1} denote this natural number that is not in Y.
Thus, $a_{n+1} \notin Y$.
Let $Y' = Y \cup \{a_{n+1}\}$.
Then $Y' = \{a_1, ..., a_n, a_{n+1}\}$.
Thus, Y' is a finite subset of natural numbers, so $Y' \in U$.
By assumption Y contains every finite subset of natural numbers, so Y must contain Y'.
Hence, $Y' \subset Y$, so every element in Y' must be an element of Y.
But, $a_{n+1} \in Y'$ and $a_{n+1} \notin Y$.
Therefore, no such set Y exists.
Hence, the statement $\exists Y \forall X.s(X, Y)$ is false.

For d) we know the statement $\exists X \forall Y.s(X, Y)$ is true if and only if there exists $X \in U$ such that for every $Y \in U$, $s(X, Y)$; i.e., there is a finite subset of natural numbers X such that $X \subseteq Y$ for every $Y \in U$. This means there is a finite subset of natural numbers that is contained in every finite subset of natural numbers.
Is this statement true or false?
Well, we know $\emptyset$ is a finite subset of $\mathbb{N}$ and $\emptyset \subseteq Y$ for all sets Y.
The proof is below.
Let $X = \emptyset$.
Since $\emptyset$ is finite and $\emptyset$ is a subset of every set, then $\emptyset$ is a finite subset of $\mathbb{N}$.
Hence, $\emptyset \in U$, so $X \in U$.
Let Y be an arbitrary finite subset of $\mathbb{N}$.
Then $Y \in U$.
Since $\emptyset$ is a subset of every set, then $\emptyset \subset Y$.
Hence, $X \subset Y$.
Since Y is arbitrary then $X \subset Y$ for all $Y \in U$ when $X \in U$.
Thus, there exists $X \in U$ such that $X \subset Y$ for all $Y \in U$.
Therefore, the statement $\exists X \forall Y.s(X, Y)$ is true. □

Exercise 48. Let $m, n \in \mathbb{Z}$.
Then m divides n if and only if there exists $p \in \mathbb{Z}$ such that $n = mp$.
Let $r(m, n, p) : n = mp$ be a propositional function of 3 variables.
The domain of discourse of $r(m, n, p)$ is $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} = \mathbb{Z}^3$.

Consider $\exists p \in \mathbb{Z}.r(m, n, p)$.

Observe that variable p is quantified, so p is bound and variables m, n are free.

Hence, $\exists p \in \mathbb{Z}.r(m, n, p)$ is a propositional function of 2 variables, namely, m and n.

Define propositional function d on $\mathbb{Z} \times \mathbb{Z} = \mathbb{Z}^2$ by $d(m, n) : m$ divides n.

Then $d(m, n) : \exists p \in \mathbb{Z}.r(m, n, p)$ which means $d(m, n) : \exists p \in \mathbb{Z}.(n = mp)$.

Discuss

a) $d(5, 7)$.

b) $d(4, 16)$.

c) $d(16, 4)$.

Solution. For a) we observe that the variables m and n are assigned specific values in $d(5, 7)$.

Thus, m and n are bound, so all variables in d are bound.

Hence, $d(5, 7)$ is a proposition (statement).

Is $d(5, 7)$ true or false? That is, does 5 divide 7 or not?

The statement $d(5, 7)$ is $\exists p \in \mathbb{Z}.(7 = 5p)$.

Consider the equation $7 = 5p$. Observe that $7 = 5p$ implies $p = 7/5$, so $p \notin \mathbb{Z}$.

Thus, there is no integer p such that $7 = 5p$.

Hence, the statement $\exists p \in \mathbb{Z}.(7 = 5p)$ is false, so $d(5, 7)$ is false.

Therefore, 5 does not divide 7.

For b) we observe that m and n are assigned specific values in $d(4, 16)$, so all variables in d are bound.

Hence, $d(4, 16)$ is a proposition(statement).

Is $d(4, 16)$ true or false? That is, does 4 divide 16 or not?

The statement $d(4, 16)$ is $\exists p \in \mathbb{Z}(16 = 4p)$.

Consider the equation $16 = 4p$.

Observe that $16 = 4p$ implies $p = 4$.

Let $p = 4$.

Then $p \in \mathbb{Z}$.

Observe that $4p = 4 * 4 = 16$, so $16 = 4p$.

Hence, the proposition $\exists p \in \mathbb{Z}(16 = 4p)$ is true, so $d(4, 16)$ is true.

This means 4 divides 16.

For c) we observe that $d(16, 4)$ is a proposition because variables m, n are bound(assigned specific values).

Is $d(16, 4)$ true or false?

The statement $d(16, 4)$ is $\exists p \in \mathbb{Z}(4 = 16p)$.

Consider the equation $4 = 16p$.

Observe that $4 = 16p$ implies $p = 1/4$, so $p \notin \mathbb{Z}$.

Thus, there is no integer p such that $4 = 16p$.

Hence, the statement $\exists p \in \mathbb{Z}.(4 = 16p)$ is false, so $d(4, 16)$ is false.

Therefore, 16 does not divide 4. $\square$

Exercise 49. Is the following argument valid?

All pessimists are unhappy.

Some happy people are healthy. Therefore some healthy people are not pessimists.

Solution. We define predicates and decompose this argument.

Let $p(x) : x$ is a pessimist.

Let $q(x) : x$ is happy.

Let $r(x) : x$ is healthy.

Then

$P_1 : \forall x.[p(x) \to \neg q(x)]$.

$P_2 : \exists x.[q(x) \wedge r(x)]$.

$C : \exists x.[r(x) \wedge \neg p(x)]$.

We must prove or disprove the argument $P_1 \wedge P_2 \to C$.

To determine if this argument is valid, we assume the truth of the premise $P_1 \wedge P_2$.

Thus, we assume P_1 is true and P_2 is true.

Let P, Q, R be the truth set of $p(x), q(x), r(x)$, respectively.

Since P_1 is true, then $P \subseteq \overline{Q}$.

Since P_2 is true, then $Q \cap R \neq \emptyset$.

We need to show that $(R \cap \overline{P}) \neq \emptyset$.

We can try specific examples for sets P, Q, R and conjecture that the conclusion is true.

We now prove this argument to be valid.

By P_2 we know there exists x in domain of discourse such that $q(x) \wedge r(x)$ is true.

By existential instantiation, let a be an arbitrary element in the domain of discourse such that $q(a) \wedge r(a)$ is true.

Then $q(a)$ is true and $r(a)$ is true.

Suppose $p(a)$ is true.

By P_1 we know that $p(a) \to \neg q(a)$ is true.

Thus, by modus ponens, $\neg q(a)$ is true.

Hence, we have $q(a)$ is true and $\neg q(a)$ is true, a contradiction.

Therefore, $p(a)$ cannot be true.

Hence, $p(a)$ is false, so $\neg p(a)$ is true.

Thus, we have $r(a)$ is true and $\neg p(a)$ is true, so $r(a) \wedge \neg p(a)$ is true.

Therefore, there is an x in domain of discourse such that $r(x) \wedge \neg p(x)$ is true.

Hence, by existential generalization, $\exists x.[r(x) \wedge \neg p(x)]$ is true.

The same argument can be made by using set theory.

We know that C is true iff $(R \cap \overline{P}) \neq \emptyset$.

To prove C follows from truth of $P_1 \wedge P_2$ we assume that $P \subseteq \overline{Q}$ and $Q \cap R \neq \emptyset$. We must prove $(R \cap \overline{P}) \neq \emptyset$.

Since $Q \cap R \neq \emptyset$, then there exists an element x in domain of discourse such that $x \in (Q \cap R)$.

Thus, $x \in Q$ and $x \in R$.

Suppose $x \in P$.
Since $P \subseteq \overline{Q}$ and $x \in P$, then $x \in \overline{Q}$.
We know that $x \in \overline{Q}$ iff $x \notin Q$.
Hence, $x \notin Q$.
But we now have $x \in Q$ and $x \notin Q$, a contradiction.
Therefore, $x \notin P$.
Thus, we have $x \in R$ and $x \notin P$.
We know $x \notin P$ iff $x \in \overline{P}$.
Hence, $x \in \overline{P}$.
Therefore, $x \in R$ and $x \in \overline{P}$, so $x \in (R \cap \overline{P})$.
Hence, $R \cap \overline{P}$ is not empty.
Thus, $(R \cap \overline{P}) \neq \emptyset$. □

Exercise 50. Every even integer that is the square of an integer is an integral multiple of 4.

Solution.
We must first translate the English statement into logical symbols.
Let P : Every even integer that is the square of an integer is an integral multiple of 4.
The statement P is of the form: $\forall x \ni R(x), Q(x)$
where
R(x) : x is an even integer that is the square of an integer
$Q(x)$: x is an integral multiple of 4
Let P(x) : $\forall x \ni R(x), Q(x)$
We analyze R(x).
R(x) means x is an even integer and x is the square of some integer.
We define the predicates below.
E(x) : x is even
$F(x, a) : x = a^2$
Thus,
$R(x, y) : S(x) \wedge T(x, y)$
where
S(x) : x an integer and x is even
T(x,y) : there is some integer y such that $F(x, y)$
So,
$S(x) : x \in \mathbb{Z}, E(x)$
and
$T(x, y) : \exists y \in \mathbb{Z}, x = y^2$
So,
$R(x, y) : x \in \mathbb{Z}, E(x) \wedge (\exists y \in \mathbb{Z}, x = y^2)$
So R(x,y) means $x \in \mathbb{Z}$ such that E(x) and there is some integer y such that $x = y^2$.
Thus,
P(x,y) means for all x such that $R(x, y), Q(x)$.
Thus,

P(x,y) means for all x such that $(x \in \mathbb{Z}, E(x) \wedge (\exists y \in \mathbb{Z}, x = y^2)), Q(x)$.
Thus,
P(x,y) means for all $x \in \mathbb{Z}$ such that $(E(x) \wedge (\exists y \in \mathbb{Z}, x = y^2)), Q(x)$.
Thus,
$P(x,y) : \forall x \in \mathbb{Z} \ni E(x) \wedge (\exists y \in \mathbb{Z}, x = y^2), Q(x)$
Let our domain of discourse be the universal set $\mathbb{Z}$.
Then
$P(x,y) : \forall x \ni E(x) \wedge (\exists y \in \mathbb{Z}, x = y^2), Q(x)$
Now we analyze Q(x).
We define the predicate below.
M(a,b) : a is a multiple of b
By definition of divisibility, 'a is a multiple of b' is equivalent to 'b divides a' which is equivalent to
'there exists integer k such that a = bk'.
Thus, M(a,b) means there exists an integer k such that a = bk.
Thus,
M(a,b) : $\exists k \in \mathbb{Z}, a = bk$
Thus,
Q(x) : M(x,4) : $\exists k \in \mathbb{Z}, x = 4k$
Then
$P(x,y) : \forall x \ni E(x) \wedge (\exists y \in \mathbb{Z}, x = y^2), (\exists k \in \mathbb{Z}, x = 4k)$
We must prove P(x,y) is true.
Since $\mathbb{Z}$ is our universal set, we let $n \in \mathbb{Z}$ be arbitrary.
Then, by universal instantiation, $E(n) \wedge (\exists y \in \mathbb{Z}, n = y^2) \to (\exists k \in \mathbb{Z}, n = 4k)$.
To prove this material conditional, we use direct proof and assume the antecedent is true.
Then we try to show that the consequent logically follows from the truth of the antecedent.
Thus, our hypothesis is : $E(n) \wedge (\exists y \in \mathbb{Z}, n = y^2)$
Our conclusion is : $\exists k \in \mathbb{Z}, n = 4k$
To prove that the conclusion is true, we must find a concrete $k \in \mathbb{Z}$ such that $n = 4k$.
Suppose $E(n) \wedge (\exists y \in \mathbb{Z}, n = y^2)$ is true.
Then $n = 2a$ for some integer a and $n = y^2$ for some integer y.
Thus, $y^2 = 2a$.
Since $a \in \mathbb{Z}$, then y^2 is even, by definition of even integer.
To prove $n = 4k$ for some $k \in \mathbb{Z}$, we can work backwards.
Suppose $\exists k \in \mathbb{Z}, n = 4k$.
Then $n = 4k = 2(2k)$.
Since $n = 2a$ by hypothesis, we need to show $2(2k) = 2a$.
This means we need to show $a = 2k$.
Thus, we need to show a is even.
Somehow we must show y^2 is even $\to$ a is even is true. □

Proof. Let n and y be integers.

Suppose n is even and $n = y^2$.

Then $n = 2a$ for some integer a.

Thus, $y^2 = 2a$.

Since $a \in \mathbb{Z}$, then y^2 is even.

Since y^2 is even $\leftrightarrow y$ is even and y^2 is even, then y is even.

Hence $y = 2b$ for some integer b.

Thus $(2b)^2 = 2a$, so $a = 2b^2$.

Since $b \in \mathbb{Z}$, then $b^2 \in \mathbb{Z}$, so a is even.

Therefore, $a = 2k$ for some integer k, so $n = 2a = 2(2k) = 4k$.

Therefore, n is a multiple of 4. $\square$