Notes on Logic

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Propositional Calculus

Statements

Definition 1. Statement
A **statement** is an assertion that is true or false.

Let P be a propositional variable.
Then P represents a statement, so P is a **statement form**.
A statement form is neither true nor false.
A statement form becomes a statement when a specific statement is substituted for each of its components.

A statement form is exactly one of three possibilities.
1. tautology - true for all possible cases
2. contingency - sometimes true or sometimes false - depends on the specific truth values of its components
3. contradiction - false for all possible cases

Definition 2. Proposition
A **proposition** is a statement that can be proven to be true.

Definition 3. Axiom
An **axiom** is a proposition that is assumed to be true but cannot be proved (self-evident truth).

It is the starting point for logically deriving other propositions.

Definition 4. Theorem
A **theorem** is a proposition of major importance and can be proved.

Definition 5. Lemma
A **lemma** is a proposition that is used to help prove another proposition.

Definition 6. Corollary
A **corollary** is a proposition that can be proven directly from a proposition just proved.

Definition 7. Conjecture

A **conjecture** is a proposition that is probably true, but not yet proven true.

Definition 8. Propositional Constant

A **propositional constant** is a literal value that represents true or false.

$$\begin{aligned} T &= \text{true} \\ F &= \text{false} \end{aligned}$$

Let $\{T, F\}$ be a set of propositional constants(truth values).

Definition 9. Propositional Variable

A **propositional variable** is a variable that represents a proposition.

Let P represent a proposition.
Then P may take a value from $\{T, F\}$.
$P = T$ means the proposition is true.
$P = F$ means the proposition is false.

If P is a tautology then $P = T$ in all cases. Therefore $P \Leftrightarrow T$.

If P is a contradiction then $P = F$ in all cases. Therefore $P \Leftrightarrow F$.

Definition 10. Tautology

A **tautology** is a statement form that is always true.

Let P represent a proposition.
Then P **is a tautology** iff P is true under all possible truth conditions of its components.

Example 11. Let P be a statement form.
Then $P \vee \neg P$ is a statement form.
When P is true, $P \vee \neg P$ is true, so $(P \vee \neg P) = T$.
When P is false, $P \vee \neg P$ is true, so $(P \vee \neg P) = T$.
Therefore, $P \vee \neg P$ is a tautology.
Hence, $P \vee \neg P \Leftrightarrow T$.

The negation of a tautology is a contradiction.

Example 12. Observe that

$$\begin{aligned} \neg(P \vee \neg P) &\Leftrightarrow \neg P \wedge \neg\neg P \\ &\Leftrightarrow \neg P \wedge P \\ &\Leftrightarrow P \wedge \neg P \end{aligned}$$

Therefore $P \wedge \neg P$ is a contradiction.
Hence, $P \wedge \neg P \Leftrightarrow F$.

Suppose P is a tautology.
Then P is always true.
Hence, P is logically equivalent to the truth constant T.
Therefore $P \Leftrightarrow T$.

Definition 13. Contradiction
A **contradiction** is a statement form that is always false.

Let P represent a proposition.
P **is a contradiction** iff P is false under all possible truth conditions of its components.

Example 14. Let P be a statement form.
Then $P \wedge \neg P$ is a statement form.
When P is true, $P \wedge \neg P$ is false, so $P \wedge \neg P = F$.
When P is false, $P \wedge \neg P$ is false, so $P \wedge \neg P = F$.
Therefore, $P \wedge \neg P$ is a contradiction.
Hence, $P \wedge \neg P \Leftrightarrow F$.

The negation of a contradiction is a tautology.

Example 15. Observe that

$$\begin{aligned} \neg(P \wedge \neg P) &\Leftrightarrow \neg P \vee (\neg\neg P) \\ &\Leftrightarrow \neg P \vee P \\ &\Leftrightarrow P \vee \neg P \end{aligned}$$

Therefore $P \vee \neg P$ is a tautology.
Hence, $P \vee \neg P \Leftrightarrow T$.

Suppose P is a contradiction.
Then P is always false.
Hence, P is logically equivalent to the truth constant F.
Therefore $P \Leftrightarrow F$.
Law of Excluded Middle
Either a proposition is true or its negation is true (there is no third possibility).

$$P \vee \neg P \quad \Leftrightarrow \quad T$$

Logical Operators

Let $S = \{T, F\}$.
Each logical operator (exception negation) is a binary operation $S \times S \to S$.
Therefore each logical operator is truth functional.

Definition 16. Negation(NOT)
Let P represent a proposition.
The negation of P, denoted $\neg P$, means 'not P'.

Truth table
$\neg P$ is true iff P is false
$\neg P$ is false iff P is true

English translation:
It's not true that P.

Negation is a unary operation $S \mapsto S$ where $S = \{T, F\}$.

Definition 17. Conjunction(AND)
Let P and Q represent propositions.
The conjunction, denoted $P \wedge Q$, means 'P and Q'.

Truth table
$P \wedge Q$ is true iff both P and Q are true
$P \wedge Q$ is false iff P is false or Q is false (or both)

Definition 18. Inclusive Disjunction (OR)
Let P and Q represent propositions.
The inclusive disjunction, denoted $P \vee Q$, means 'either P or Q (or both)'.

Truth table
$P \vee Q$ is true iff P is true, or Q is true, or both P and Q are true
$P \vee Q$ is false iff both P and Q are false

Definition 19. Exclusive Disjunction (XOR)
Let P and Q represent propositions.
The exclusive disjunction, denoted $P \veebar Q$, means 'either P or Q, but not both'.

Truth table
$P \veebar Q$ is true iff both P is true and Q is false, or both P is false and Q is true
$P \veebar Q$ is false iff both P and Q are true or both P and Q are false

$$\begin{aligned} P \veebar Q &\Leftrightarrow (P \wedge \neg Q) \vee (Q \wedge \neg P) \\ &\Leftrightarrow (P \vee Q) \wedge \neg (P \wedge Q) \\ &\Leftrightarrow \neg (P \leftrightarrow Q) \end{aligned}$$

Let $S = \{T, F\}$.
Then $(S, \oplus)$ is isomorphic to $(\mathbb{Z}_2, +)$.

Definition 20. Material Implication (Conditional)
Let P and Q represent propositions.
The conditional, denoted $P \to Q$, means 'if P then Q'.

Equivalent English meanings:
P implies Q, so P is a **sufficient condition** for Q
P only if Q, so Q is a **necessary condition** for P and if not Q, then not P
Q if P
Q whenever P
P = **antecedent**
Q = **consequent**

Truth table
$P \to Q$ is true iff P is false or Q is true (or both)
$P \to Q$ is false iff P is true and Q is false

Whenever P is false then $P \to Q$ is **vacuously** true.

Definition 21. Let P and Q represent propositions.
Let $P \to Q$ be a conditional.
The **inverse** is $\neg P \to \neg Q$.
The **converse** is $Q \to P$.
The **contrapositive** is $\neg Q \to \neg P$.

The converse of a conditional $P \to Q$ is logically equivalent to the inverse of the conditional $P \to Q$.

$P \to Q \Leftrightarrow \neg P \vee Q \Leftrightarrow \neg Q \to \neg P$
$\neg(P \to Q) \Leftrightarrow P \wedge \neg Q$
$P \Rightarrow P$ (reflexive)

Definition 22. Biconditional (Material Equivalence)
Let P and Q represent propositions.
The biconditional, denoted $P \leftrightarrow Q$, means 'P if and only if Q'.
Therefore, $P \leftrightarrow Q$ means:
1. P if Q, so Q is a sufficient condition for P, so $Q \to P$.
2. P only if Q, so Q is a necessary condition for P, so $P \to Q$.

Equivalent English meanings:
P is equivalent to Q
P iff Q
P exactly when Q
P is a necessary and sufficient condition for Q
P implies Q and Q implies P

Truth table
$P \leftrightarrow Q$ is true iff P and Q are both true or P and Q are both false
$P \leftrightarrow Q$ is false iff either P is true and Q is false or Q is true and P is false

$P \leftrightarrow Q \Leftrightarrow (P \rightarrow Q) \wedge (Q \rightarrow P) \Leftrightarrow (P \wedge Q) \vee (\neg P \wedge \neg Q) \Leftrightarrow \neg P \leftrightarrow \neg Q$
$\neg(P \leftrightarrow Q) \Leftrightarrow P \leftrightarrow \neg Q \Leftrightarrow (P \wedge \neg Q) \vee (Q \wedge \neg P)$
$(P \leftrightarrow Q) \Rightarrow (P \rightarrow Q)$
$P \rightarrow (Q \rightarrow R) \Leftrightarrow (P \wedge Q) \rightarrow R$

The biconditional is an equivalence relation.
$P \Leftrightarrow P$ (reflexive)
$(P \leftrightarrow Q) \Rightarrow (Q \leftrightarrow P)$ (symmetric)
$(P \leftrightarrow Q) \wedge (Q \leftrightarrow R) \Rightarrow (P \leftrightarrow R)$ (transitive)

A definition is a universally quantified biconditional that is true.

Logical Equivalence

Definition 23. Logical Implication
Let P and Q represent statements.
The statement P **logically implies** Q, denoted $P \Rightarrow Q$, means the truth of P implies the truth of Q.

Suppose $P \Rightarrow Q$.
Then the truth of P implies the truth of Q.
Either P is true or false.
Suppose P is true.
Since the truth of P implies the truth of Q, and P is true, then by modus ponens, Q must also be true.
Hence, the conditional $P \rightarrow Q$ is true.
Suppose P is false.
Then the conditional $P \rightarrow Q$ is vacuously true.
Hence, in all cases, $P \rightarrow Q$ is always true, so $P \rightarrow Q$ is a tautology.
Thus, if $P \Rightarrow Q$, then $P \rightarrow Q$ is a tautology.

Suppose $P \rightarrow Q$ is a tautology.
Then $P \rightarrow Q$ is always true, so $P \rightarrow Q$ is true under all possible truth conditions of P and Q.
Thus, based on the truth table for $P \rightarrow Q$, either P is false or Q is true.
Furthermore it is never the case that P is true and Q is false.
Moreover, if P is true, then Q must also be true.
Thus, the truth of P implies the truth of Q, so $P \Rightarrow Q$.
Hence, if $P \rightarrow Q$ is a tautology, then $P \Rightarrow Q$.

Therefore, $P \Rightarrow Q$ iff $P \rightarrow Q$ is a tautology, so $P \rightarrow Q$ is always true.

Hence, whenever $P \Rightarrow Q$, then the truth of P forces Q to be true, so P is a stronger statement than Q and Q is a weaker statement than P.

Stronger and Weaker statements:
$(P \wedge Q) \Rightarrow P \Rightarrow (P \vee Q)$.
$\neg P \Rightarrow (P \to Q)$
$Q \Rightarrow (P \to Q)$
$(P \to R) \Rightarrow (P \wedge Q) \to R$

Definition 24. Logical Equivalence

Two statement forms that have the same logical content are logically equivalent.

Let P and Q represent statements.

The statement P **is logically equivalent to** Q, denoted $P \Leftrightarrow Q$, means P and Q have the same truth values under all possible truth conditions for their components.

Suppose $P \Leftrightarrow Q$.
Then P and Q have the same truth values under all possible truth conditions of their components.
Hence, the truth value of P matches line for line with the truth value of Q.
Thus, the truth table for P is identical to the truth table for Q.
Hence, P is true whenever Q is true and P is false whenever Q is false.
Therefore, either P and Q are both true, or P and Q are both false.
Hence, the biconditional $P \leftrightarrow Q$ is always true, so $P \leftrightarrow Q$ is a tautology.
The argument can be reversed to show if $P \leftrightarrow Q$ is a tautology, then $P \Leftrightarrow Q$.

Therefore, $P \Leftrightarrow Q$ iff $P \leftrightarrow Q$ is a tautology, so $P \leftrightarrow Q$ is always true.

Suppose $P \Leftrightarrow Q$.
Then $P \leftrightarrow Q$ is a tautology.
Since $P \leftrightarrow Q$ is logically equivalent to $(P \to Q) \wedge (Q \to P)$, then $(P \to Q) \wedge (Q \to P)$ is a tautology.
Hence, $P \to Q$ is a tautology and $Q \to P$ is a tautology.
Thus, $P \Rightarrow Q$ and $Q \Rightarrow P$.
Therefore, if $P \Leftrightarrow Q$, then $P \Rightarrow Q$ and $Q \Rightarrow P$.
The argument can be reversed to show if $P \Rightarrow Q$ and $Q \Rightarrow P$, then $P \Leftrightarrow Q$.

Therefore $P \Leftrightarrow Q$ iff $P \Rightarrow Q$ and $Q \Rightarrow P$.

Logic Translation Tips

$\neg P \wedge \neg Q$ means 'P and Q are both not true' because we're denying each of them

$\neg(P \wedge Q)$ means 'not both P and Q' because one of them may be true, just not both

$\neg(P \vee Q)$ means 'neither P nor Q', so P and Q are both not true which means $\neg P \wedge \neg Q$

$\neg P \vee \neg Q$ means 'either not P or not Q', so P and Q are not both true which means $\neg(P \wedge Q)$

P unless Q means not Q implies P, so $\neg Q \rightarrow P$

Logic Equivalence Laws

Let P, Q, R be statement forms (propositional variables).

Identity

$$P \wedge T \Leftrightarrow P$$
$$P \vee F \Leftrightarrow P$$

Double Negation

$$\neg(\neg P) \Leftrightarrow P$$

Domination

$$P \vee T \Leftrightarrow T$$
$$P \wedge F \Leftrightarrow F$$

Idempotent

$$P \wedge P \Leftrightarrow P$$
$$P \vee P \Leftrightarrow P$$

Associative

$$(P \wedge Q) \wedge R \Leftrightarrow P \wedge (Q \wedge R)$$
$$(P \vee Q) \vee R \Leftrightarrow P \vee (Q \vee R)$$

Commutative

$$P \wedge Q \Leftrightarrow Q \wedge P$$
$$P \vee Q \Leftrightarrow Q \vee P$$

Absorption

$$P \vee (P \wedge Q) \Leftrightarrow P$$
$$P \wedge (P \vee Q) \Leftrightarrow P$$

Distributive

$$P \wedge (Q \vee R) \Leftrightarrow (P \wedge Q) \vee (P \wedge R)$$
$$P \vee (Q \wedge R) \Leftrightarrow (P \vee Q) \wedge (P \vee R)$$

DeMorgan

$$\neg(P \wedge Q) \Leftrightarrow \neg P \vee \neg Q$$
$$\neg(P \vee Q) \Leftrightarrow \neg P \wedge \neg Q$$

Conjunction

$$P \rightarrow (Q \wedge R) \Leftrightarrow (P \rightarrow Q) \wedge (P \rightarrow R)$$
$$(P \vee Q) \rightarrow R \Leftrightarrow (P \rightarrow R) \wedge (Q \rightarrow R)$$

Disjunction

$$P \rightarrow (Q \vee R) \Leftrightarrow (P \rightarrow Q) \vee (P \rightarrow R) \Leftrightarrow (P \wedge \neg Q) \rightarrow R$$
$$(P \wedge Q) \rightarrow R \Leftrightarrow (P \rightarrow R) \vee (Q \rightarrow R) \Leftrightarrow (P \wedge \neg R) \rightarrow \neg Q$$

Rules of Inference

Conjunction Introduction (Conjunction)
P.
Q.
Therefore $P \wedge Q$.
$P, Q \Rightarrow (P \wedge Q)$

Conjunction Elimination (Simplification)
P and Q.
Therefore P.
$P \wedge Q \Rightarrow P$

Disjunction Introduction (Addition)
P.
Therefore P or Q.
$P \Rightarrow (P \vee Q)$

Disjunction Elimination
P or Q.
P implies R.
Q implies R.
Therefore R.
$[(P \vee Q) \wedge (P \rightarrow R) \wedge (Q \rightarrow R)] \Rightarrow R$

Conditional Introduction
P.
Q.
Therefore $P \rightarrow Q$.
$P, Q \Rightarrow (P \rightarrow Q)$

Conditional Elimination (Modus Ponens/ Affirming the Antecedent)
If P then Q.
P.
Therefore Q.
$[(P \to Q) \wedge P] \Rightarrow Q$

Modus Tollens (Denying the Consequent)
P implies Q.
not Q.
Therefore not P.
$[(P \to Q) \wedge \neg Q] \Rightarrow \neg P$

Biconditional Elimination
P iff Q.
P.
Therefore Q.
$(P \leftrightarrow Q) \wedge P \Rightarrow Q$

Q iff P.
P.
Therefore Q.
$(Q \leftrightarrow P) \wedge P \Rightarrow Q$

Biconditional Introduction
$P \to Q$.
$Q \to P$.
Therefore P iff Q.
$(P \to Q) \wedge (Q \to P) \Rightarrow (P \leftrightarrow Q)$

Reductio ad absurdum (Proof by Contradiction)
not P implies $Q \wedge \neg Q$.
Therefore P.
$[\neg P \to (Q \wedge \neg Q)] \Rightarrow P$

Hypothetical Syllogism (Transitive property of implication)
P implies Q.
Q implies R.
Therefore P implies R.
$[(P \to Q) \wedge (Q \to R)] \Rightarrow (P \to R)$

Disjunctive Syllogism
P or Q.
not P.
Therefore Q.
$(P \vee Q) \wedge \neg P \Rightarrow Q$

Resolution
P or Q.
not P or R.
Therefore Q or R.
$(P \vee Q) \wedge (\neg P \vee R) \Rightarrow (Q \vee R)$

Proofs

Definition 25. Logical Argument
Let $P_1, ..., P_n$ and Q represent propositions.
The statement form $(P_1 \wedge ... \wedge P_n) \rightarrow Q$ is an **argument**.
The **hypothesis** is $P_1 \wedge ... \wedge P_n$.
The **conclusion** is Q.
The **premises** are $P_1, ...P_n$.

Definition 26. Valid Argument
An argument is **valid** if and only if the conclusion logically follows from the premises.

Let $(P_1 \wedge ... \wedge P_n) \rightarrow Q$ be a valid argument.
Then $P_1 \wedge ... \wedge P_n$ logically implies Q.
Hence $(P_1 \wedge ... \wedge P_n) \Rightarrow Q$, so $(P_1 \wedge ... \wedge P_n) \rightarrow Q$ is true.
Suppose $P_1 \wedge ... \wedge P_n$ is true.
Since $(P_1 \wedge ... \wedge P_n) \Rightarrow Q$ and $P_1 \wedge ... \wedge P_n$, then by modus ponens, Q.
Thus Q is true.

Therefore if an argument is valid:
1. the argument is true
2. if the hypothesis is true, then the conclusion is true.

Let $(P_1 \wedge ... \wedge P_n) \rightarrow Q$ be an argument.
Suppose $P_1 \wedge ... \wedge P_n$ is true.
If we can show that Q logically follows from the hypothesis, then $(P_1 \wedge ... \wedge P_n) \Rightarrow Q$.
Hence, we may conclude that the argument is valid.
Since $(P_1 \wedge ... \wedge P_n) \Rightarrow Q$, then the statement $(P_1 \wedge ... \wedge P_n) \rightarrow Q$ is true.

Definition 27. Invalid Argument
An argument is **invalid** if and only if the conclusion does not follow from the premises.

Let $(P_1 \wedge ... \wedge P_n) \rightarrow Q$ be an invalid argument.
Then $P_1 \wedge ... \wedge P_n$ does not logically imply Q.
Hence $(P_1 \wedge ... \wedge P_n) \not\Rightarrow Q$, so $(P_1 \wedge ... \wedge P_n) \rightarrow Q$ is false.
Suppose $P_1 \wedge ... \wedge P_n$ is true.
Then Q is false.

Therefore if an argument is invalid:

1. the argument is false
2. if the hypothesis is true, then the conclusion is false.

Let $(P_1 \wedge ... \wedge P_n) \rightarrow Q$ be an argument.

Suppose $P_1 \wedge ... \wedge P_n$ is true.

If Q does not logically follow from the hypothesis, then $(P_1 \wedge ... \wedge P_n) \not\Rightarrow Q$.

Hence, we may conclude that the argument is invalid.

Since $P_1 \wedge ... \wedge P_n$ is true and Q is false, then the statement $(P_1 \wedge ... \wedge P_n) \rightarrow Q$ is false.

Definition 28. Sound Argument

An argument is **sound** if and only if the argument is valid and all premises are true.

Let $(P_1 \wedge ... \wedge P_n) \rightarrow Q$ be a sound argument.

Then $P_1 \wedge ... \wedge P_n$ logically implies Q and $P_1 \wedge ... \wedge P_n$ is true.

Hence $(P_1 \wedge ... \wedge P_n) \Rightarrow Q$ and $P_1 \wedge ... \wedge P_n$.

By modus ponens, we conclude Q.

Since $P_1 \wedge ... \wedge P_n$ is true and Q is true, then $(P_1 \wedge ... \wedge P_n) \rightarrow Q$ is true.

Therefore if an argument is sound:

1. the argument is true
2. the conclusion is true.

Definition 29. Proof

A **proof** is a step by step demonstration that a conclusion logically follows from the hypothesis.

Let P represent a statement.

To prove P means to show that P is true.

Let $(P_1 \wedge ... \wedge P_n) \rightarrow Q$ be an argument.

To prove $(P_1 \wedge ... \wedge P_n) \rightarrow Q$ means to show that the argument $(P_1 \wedge ... \wedge P_n) \rightarrow Q$ is true.

To prove $(P_1 \wedge ... \wedge P_n) \rightarrow Q$ is true, we must show that $P_1 \wedge ... \wedge P_n$ logically implies Q; that is, we must show $(P_1 \wedge ... \wedge P_n) \Rightarrow Q$.

We prove a statement is true by using the hypothesis and applicable axioms, definitions, and proven theorems using inference rules.

A proof is a valid argument.

fallacy - form of incorrect reasoning

Example 30. fallacy of affirming the conclusion
$(P \Rightarrow Q) \wedge Q \vdash P$

Example 31. fallacy of denying the hypothesis
$(P \Rightarrow Q) \wedge \neg P \vdash \neg Q$

circular reasoning - reasoning based on the truth of the statement being proved

Methods of Proof

Direct proof
We establish the truth of a statement directly by assuming the truth of the hypothesis and using known theorems/axioms.

Indirect proof
We establish the truth of a statement by proving a logically equivalent statement.

I. **Direct Proof (Conditional Introduction)**
Let $P \to Q$ be an argument.
To Prove $P \to Q$
1. Assume P is true
2. Show that Q must also be true.

Template: Prove $P \to Q$.
Suppose P.
...
Therefore Q.

To prove $P \to Q$ is true means to prove $P \Rightarrow Q$; that is, to prove P logically implies Q.

This means to prove the truth of P implies the truth of Q.

Thus, if we assume P is true and show that Q is true, then P logically implies Q, and this proves that $P \to Q$ is true.

II. **Vacuous Proof**
To Prove $P \to Q$
1. Prove P is false.

$P \to Q$ is true when P is false, regardless of the truth value of Q.

III. **Trivial Proof**
To Prove $P \to Q$
1. Prove Q is true.

$P \to Q$ is true when Q is true, regardless of the truth value of P.

IV. **Indirect Proof (Proof by Contrapositive)**
Let $P \to Q$ be an argument.
To Prove $P \to Q$
1. Prove $\neg Q \to \neg P$

Template: Prove $P \to Q$.
Suppose $\neg Q$.
...
Therefore $\neg P$.

This method is valid because $P \to Q \Leftrightarrow \neg Q \to \neg P$.
Also, P is the hypothesis.
When we prove $\neg Q$ implies $\neg P$, then we have P and $\neg P$, a contradiction.
Therefore, the assumption of $\neg Q$ cannot be true, so its negation, Q must be true.
Hence, proof by contrapositive is a special type of proof by contradiction.

V. **Indirect Proof (Proof by Contradiction)**
Let P be a statement.
We prove by assuming the negation of a statement leads to a contradiction.

To Prove P
1. Assume $\neg P$
2. Show $\neg P \Rightarrow (Q \wedge \neg Q)$

Template: Prove P.
Suppose $\neg P$.
...
$Q \wedge \neg Q$
Therefore P.

This method is valid because $[\neg P \to (Q \wedge \neg Q)] \to P$ is a tautology.
We prove $[\neg P \to (Q \wedge \neg Q)] \Rightarrow P$.

Proof. Assume $\neg P \to (Q \wedge \neg Q)$.
Assume $\neg P$.
From $\neg P$ and $\neg P \to (Q \wedge \neg Q)$, we conclude $Q \wedge \neg Q$, by modus ponens.
From $\neg Q$ and Q, we derive a contradiction, so we conclude P.

From premise $\neg P \to (Q \wedge \neg Q)$ and P, we conclude $[\neg P \to (Q \wedge \neg Q)] \to P$, by conditional introduction. $\square$

If we show that $\neg P \to (Q \wedge \neg Q)$ is true, we apply the tautology and use modus ponens to conclude P.

Let $P \to Q$ be an argument.
To prove $P \to Q$
1. Assume $P \wedge \neg Q$. (the negation of $P \to Q$)
2. Show $(P \wedge \neg Q) \Rightarrow (R \wedge \neg R)$ for some statement R (leads to a contradiction).

VI. **Proof by Division into Cases**
Let $P \to Q$ be an argument.
To Prove $P \to Q$
1. Express hypothesis P as a disjunction of propositions $P \Leftrightarrow P_1 \vee ... \vee P_n$.
The cases are always exhaustive and typically mutually exclusive.
2. Prove each $P_i \to Q$ individually.

Example: Prove $P \to Q$ by cases.
We decompose $P \Leftrightarrow P_1 \vee P_2$.
To prove $P \to Q$ we must prove $(P_1 \vee P_2) \to Q$.
To prove $(P_1 \vee P_2) \to Q$ we can prove $P_1 \to Q$ and $P_2 \to Q$.

This method is valid because

$$\begin{aligned}
(P_1 \to Q) \wedge (P_2 \to Q) &\Leftrightarrow (\neg P_1 \vee Q) \wedge (\neg P_2 \vee Q) \\
&\Leftrightarrow (\neg P_1 \wedge \neg P_2) \vee Q \\
&\Leftrightarrow \neg(P_1 \vee P_2) \vee Q \\
&\Leftrightarrow (P_1 \vee P_2) \to Q
\end{aligned}$$

Template: Prove $P \to Q$.
$P \Leftrightarrow P_1 \vee P_2$.
We consider these cases separately.
Case 1: Suppose P_1.
...
Therefore Q.
Case 2: Suppose P_2.
...
Therefore Q.

VII. **Proof of Biconditional**
To Prove $P \leftrightarrow Q$
1. $\Rightarrow$ Prove $P \rightarrow Q$
2. $\Leftarrow$ Prove $Q \rightarrow P$
This method is valid because $P \leftrightarrow Q$ is logically equivalent to $(P \rightarrow Q) \wedge (Q \rightarrow P)$.

VIII. **Disproof**
To disprove a statement means to prove the statement is false.

Let P represent a statement.
To disprove P
1. Prove $\neg P$.

Let $(P_1 \wedge P_2 \wedge ... \wedge P_n) \rightarrow Q$ be an argument.
To disprove $(P_1 \wedge P_2 \wedge ... \wedge P_n) \rightarrow Q$
1. Prove $(P_1 \wedge P_2 \wedge ... \wedge P_n) \rightarrow Q$ is false.
We can do this by proving $(P_1 \wedge P_2 \wedge ... \wedge P_n) \wedge \neg Q$.
Thus we find a possible circumstance in which all the premises are true and the conclusion is false.

First Order/Predicate Calculus

extends propositional logic(zero order) by adding predicates and quantifiers

Definition 32. Propositional Function(Predicate)
A **predicate** is a declarative sentence containing one or more variables(unknowns).
It is a **propositional function(open sentence)** whose truth value depends on the value of its variables.
It is not a statement so it is neither true nor false.

Let U be a universal set.
Define function $p : U \rightarrow \{T, F\}$ over domain of discourse U.
Then p is a **propositional function(predicate)**.
Let $x \in U$.
Then $p(x)$ is true or false.

Let $P = \{x \in U : p(x)\}$.
Then P is the **truth set** of p.
Therefore P is the set of all $x \in U$ such that $p(x)$ is true.
Thus $P \subset U$.

A propositional function (predicate) becomes a proposition (statement) when
1. a specific value is substituted for each of its variable(s)
2. the predicate is quantified

A variable is **bound** if it is assigned a value or if it is quantified.

A variable is **free** if it is not bound.

A predicate describes a property of objects or a relationship among objects.
It is assigned a semantic meaning (interpretation).

Let $p : U \to \{T, F\}$ be a propositional function.
Let $P = \{x \in U : p(x)\}$ be the truth set of p.

Then the following are true:
Truth set of $\neg p(x)$ is $\overline{P}$.
Truth set of $p(x) \wedge q(x)$ is $P \cap Q$.
Truth set of $p(x) \vee q(x)$ is $P \cup Q$.
Truth set of $p(x) \to q(x)$ is $\overline{P} \cup Q$.
Truth set of $p(x) \leftrightarrow q(x)$ is $(P \cap Q) \cup (\overline{P} \cap \overline{Q})$.

Negations:
$\neg\neg p(x) \Leftrightarrow p(x)$
$\neg[p(x) \wedge q(x)] \Leftrightarrow \neg p(x) \vee \neg q(x)$
$\neg[p(x) \vee q(x)] \Leftrightarrow \neg p(x) \wedge \neg q(x)$
$\neg[p(x) \to q(x)] \Leftrightarrow p(x) \wedge \neg q(x)$
$\neg[p(x) \leftrightarrow q(x)] \Leftrightarrow [p(x) \wedge \neg q(x)] \vee [q(x) \wedge \neg p(x)]$

Let $p : U \to \{T, F\}$ be a function defined by

$$p(x) = \begin{cases} T & \text{iff } x \in P \\ F & \text{iff } x \notin P \end{cases}$$

Then p is a propositional function defined over U.
The truth set of p is $P = \{x \in U : p(x) \text{ is a true statement }\}$.

Suppose a is an arbitrary element in U.
Then $a \in U$.
Suppose $p(a)$ is true.
Then $a \in P$.
Thus, the conditional $p(a) \to a \in P$ is true.
Suppose $a \in P$.
Then $p(a)$ is true.
Thus, the conditional $a \in P \to p(a)$ is true.
Since $p(a) \to a \in P$ and $a \in P \to p(a)$, then the biconditional $p(a) \leftrightarrow a \in P$ is true.
Since $p(a) \leftrightarrow a \in P$ is true, then $p(a) \Leftrightarrow a \in P$
Since a is arbitrary, then by universal generalization, $p(x) \Leftrightarrow x \in P$ for every $x \in U$.

Hence, $(\forall x \in U)(p(x) \leftrightarrow a \in P)$ is true.
Therefore $p(x) \Leftrightarrow x \in P$.

Definition 33. Logical Implication
Let $p(x)$ and $q(x)$ be predicates defined over U.
Let P = truth set of $p(x)$.
Let Q = truth set of $q(x)$.

Define statement $p(x)$ **logically implies** $q(x)$, denoted $p(x) \Rightarrow q(x)$, to mean '$p(x) \to q(x)$ is true for every $x \in U$'.

Suppose $p(x) \Rightarrow q(x)$.
Then $p(x) \to q(x)$ is true for every $x \in U$.
Hence the statement $(\forall x \in U)[p(x) \to q(x)]$ is true.
Since $p(x) \Leftrightarrow x \in P$ and $q(x) \Leftrightarrow x \in Q$, then the statement $(\forall x \in U)(x \in P \to x \in Q)$ is true.
Thus $P \subset Q$.

Suppose $P \subset Q$.
Then the statement $(\forall x \in U)[x \in P \to x \in Q]$ is true.
Since $x \in P \Leftrightarrow p(x)$ and $x \in Q \Leftrightarrow q(x)$, then the statement $(\forall x \in U)[p(x) \to q(x)]$ is true.
Hence $p(x) \to q(x)$ is true for every $x \in U$.
Thus $p(x) \Rightarrow q(x)$.

Therefore $p(x) \Rightarrow q(x)$ iff $P \subset Q$.

Definition 34. Logical Equivalence
Let $p(x)$ and $q(x)$ be predicates defined over U.
Let P = truth set of $p(x)$.
Let Q = truth set of $q(x)$.

Define statement $p(x)$ **is logically equivalent to** $q(x)$, denoted $p(x) \Leftrightarrow q(x)$, to mean '$p(x) \leftrightarrow q(x)$ is true for every $x \in U$'.

Suppose $p(x) \Leftrightarrow q(x)$.
Then $p(x) \leftrightarrow q(x)$ is true for every $x \in U$.
Hence the statement $(\forall x \in U)[p(x) \leftrightarrow q(x)]$ is true.
Since $p(x) \Leftrightarrow x \in P$ and $q(x) \Leftrightarrow x \in Q$, then the statement $(\forall x \in U)(x \in P \leftrightarrow x \in Q)$ is true.
Thus $P = Q$.

Suppose $P = Q$.
Then the statement $(\forall x \in U)[x \in P \leftrightarrow x \in Q]$ is true.
Since $x \in P \Leftrightarrow p(x)$ and $x \in Q \Leftrightarrow q(x)$, then the statement $(\forall x \in U)[p(x) \leftrightarrow q(x)]$ is true.
Hence $p(x) \leftrightarrow q(x)$ is true for every $x \in U$.
Thus $p(x) \Leftrightarrow q(x)$.

Therefore $p(x) \Leftrightarrow q(x)$ iff $P = Q$.

Quantified Statements

Universal Quantifier $\forall$ means 'for all' or 'for every'

Existential Quantifier $\exists$ means 'there exists' or 'there is some' or 'there is at least one'

Uniqueness Quantifier $\exists!$ means 'there exists exactly one'

Let U be a universal set (**domain of discourse**) for variable x.
Let $p(x)$ be a propositional function defined over U.
Let $P = \{x \in U : p(x) \text{ is true } \}$ be the truth set of p.
Then $P \subset U$.

Universally Quantified Statement

Let $p(x)$ be a predicate defined over domain of discourse U.
Let P be the truth set of $p(x)$.
Then $P = \{x \in U : p(x) \text{ is true } \}$.
Hence $P \subset U$.

Consider $\forall x.p(x)$.
Since x is bound, then $\forall x.p(x)$ is a statement.
Therefore $\forall x.p(x)$ is true or false.

Suppose $\forall x.p(x)$ is true.
Then $p(c)$ is true for every specific object $c \in U$.
Let a be an arbitrary element of U.
Then $a \in U$, so $p(a)$ is true.
Thus, $a \in P$.
Hence, $a \in U$ and $a \in P$, so the conditional $a \in U \to a \in P$ is true.
Since a is arbitrary, then by universal generalization, the conditional $x \in U \to x \in P$ is true for every $x \in U$.
Thus, the statement $(\forall x \in U)(x \in U \to x \in P)$ is true, so $U \subset P$.
Since $P \subset U$ and $U \subset P$, then $P = U$.
Therefore $\forall x.p(x)$ implies $P = U$.

Suppose $P = U$.
Then $P \subset U$ and $U \subset P$.
Hence $U \subset P$.
Let a be an arbitrary element of U.
Then $a \in U$.
Since $a \in U$ and $U \subset P$, then $a \in P$.
Thus $p(a)$ is true.
Since a is arbitrary, then by universal generalization, $p(x)$ is true for every $x \in U$.
Hence $\forall x.p(x)$ is true.
Therefore $P = U$ implies $\forall x.p(x)$.
Since $\forall x.p(x) \Rightarrow P = U$ and $P = U \Rightarrow \forall x.p(x)$, then $\forall x.p(x) \Leftrightarrow P = U$.

Therefore $\forall x.p(x) \Leftrightarrow (P = U)$.

Suppose $\forall x.p(x)$ is false.
Then $p(c)$ is false for at least one specific object $c \in U$.
Hence, by existential generalization, there exists $x \in U$ such that $p(x)$ is false.
Thus, $\exists x.\neg p(x)$.
Since $\forall x.p(x)$ is false, then $\neg\forall x.p(x)$ is true.
Observe that $\neg\forall x.p(x) \Leftrightarrow \exists x.\neg p(x)$.
Since $\forall x.p(x)$ iff $P = U$, then $\neg\forall x.p(x)$ iff $P \neq U$.
Observe that $\exists x.\neg p(x)$ is true if and only if $\overline{P}$ is not empty.

Therefore $\neg\forall x.p(x) \Leftrightarrow \exists x.\neg p(x) \Leftrightarrow P \neq U \Leftrightarrow \overline{P} \neq \emptyset$.

Let U be a nonempty finite set.
Then $|U| = n$ for some $n \in \mathbb{Z}^+$.
Therefore $(\forall x \in U)p(x) \Leftrightarrow p(x_1) \wedge ... \wedge p(x_n)$.

Existentially Quantified Statement

Let $p(x)$ be a predicate defined over domain of discourse U.
Let P be the truth set of $p(x)$.
Then $P = \{x \in U : p(x) \text{ is true } \}$.
Hence $P \subset U$.

Consider $\exists x.p(x)$.
Since x is bound then $\exists x.p(x)$ is a statement.
Therefore $\exists x.p(x)$ is true or false.

Suppose $\exists x.p(x)$ is true.
Then

$$\begin{aligned} \exists x.p(x) &\Rightarrow p(c) \text{ existential instantiation} \\ &\Rightarrow c \in P \text{ defn of } P \\ &\Rightarrow (\exists x)(x \in P) \text{ existential generalization} \\ &\Rightarrow P \text{ is not empty (defn empty set)} \\ &\Rightarrow P \neq \emptyset \end{aligned}$$

Therefore $\exists x.p(x)$ implies $P \neq \emptyset$.

Suppose $P \neq \emptyset$.
Then

$$\begin{aligned} P \neq \emptyset &\Rightarrow P \text{ is not empty (defn empty set)} \\ &\Rightarrow (\exists x)(x \in P) \text{ defn empty set} \\ &\Rightarrow c \in P \text{ existential instantiation} \\ &\Rightarrow p(c) \text{ defn of } P \\ &\Rightarrow \exists x.p(x) \text{ existential generalization} \end{aligned}$$

Since $\exists x.p(x) \Rightarrow P \neq \emptyset$ and $P \neq \emptyset \Rightarrow \exists x.p(x)$, then $\exists x.p(x) \Leftrightarrow P \neq \emptyset$.

Therefore $\exists x.p(x) \Leftrightarrow P \neq \emptyset$.

Suppose $\exists x.p(x)$ is false.
Then there is no object $c \in U$ such that $p(c)$ is true.
Thus $p(c)$ is false for every $c \in U$.
Hence, $p(x)$ is false for every $x \in U$.
Thus, $\forall x.\neg p(x)$ is true.
Since $\exists x.p(x)$ is false, then $\neg\exists x.p(x)$ is true.
Observe that $\neg\exists x.p(x) \Leftrightarrow \forall x.\neg p(x)$.
Since $\exists x.p(x)$ iff $P \neq \emptyset$, then $\neg\exists x.p(x)$ iff $P = \emptyset$.
Observe that $\forall x.\neg p(x)$ is true if and only if $\overline{P} = U$.

Therefore $\neg\exists x.p(x) \Leftrightarrow \forall x.\neg p(x) \Leftrightarrow \overline{P} = U \Leftrightarrow P = \emptyset$.

Let U be a non empty finite set.
Then $|U| = n$ for some $n \in \mathbb{Z}^+$.
Therefore $(\exists x \in U)p(x) \Leftrightarrow p(x_1) \vee ... \vee p(x_n)$.

Quantifiers with Predicates

$\exists x.[p(x) \rightarrow q(x)]$ means 'there is at least one x such that $p(x)$ implies $q(x)$'

Distributing $\forall$ through $\wedge$

$\forall x.[p(x) \wedge q(x)] \Leftrightarrow \forall x.p(x) \wedge \forall x.q(x)$ same as $P \cap Q = U$ iff ($P = U$ and $Q = U$)

This means 'All are P and Q if and only if all are P and all are Q'.

Distributing $\exists$ through $\vee$

$\exists x.[p(x) \vee q(x)]$ means 'either $p(x)$ or $q(x)$ for at least one x'

$\exists x.[p(x) \vee q(x)] \Leftrightarrow \exists x.p(x) \vee \exists x.q(x)$ same as $P \cup Q \neq \emptyset$ iff ($P \neq \emptyset$ or $Q \neq \emptyset$)

This means 'Some are either P or Q if and only if either some are P or some are Q'.

$\forall x.p(x) \vee \forall x.q(x)$ means 'either $p(x)$ for all x, or $q(x)$ for all x'

$\forall x.p(x) \vee \forall x.q(x) \Rightarrow \forall x.[p(x) \vee q(x)]$ same as ($P = U$ or $Q = U$) $\Rightarrow$ ($P \cup Q = U$)

This means 'if either all are P or all are Q, then all are either P or Q'.

Suppose $P = U$ or $Q = U$.

Then $P \cup Q = U \cup U = U$.

$\exists x.[p(x) \wedge q(x)]$ means '$p(x)$ and $q(x)$ for at least one x'

$\exists x.[p(x) \wedge q(x)] \Rightarrow \exists x.p(x) \wedge \exists x.q(x)$ same as ($P \cap Q \neq \emptyset$) $\Rightarrow$ ($P \neq \emptyset$ and $Q \neq \emptyset$)

This means 'if some are P and Q, then some are P and some are Q'.

Suppose $P \cap Q \neq \emptyset$.

Then $P \cap Q$ is not empty so $P \cap Q$ contains at least one element.

Let $a \in P \cap Q$.

Then $a \in P$ and $a \in Q$.

Since $a \in P$ then P is not empty, so $P \neq \emptyset$.

Since $a \in Q$ then Q is not empty, so $Q \neq \emptyset$.

Therefore $P \neq \emptyset$ and $Q \neq \emptyset$.

Proposition 35. *Let $p(x)$ and $q(x)$ be predicates defined over domain of discourse U.*

Let r be a proposition.

Then the following are true:

1. $(\forall x)r \Leftrightarrow r$.

2. $(\exists x)r \Leftrightarrow r$.

3. $(\forall x)(r \vee q(x)) \Leftrightarrow r \vee (\forall x)(q(x))$.

4. $(\exists x)(r \wedge q(x)) \Leftrightarrow r \wedge (\exists x)(q(x))$.

5. $(\forall x)[r \to q(x)] \Leftrightarrow r \to (\forall x)(q(x))$.

6. $(\exists x)[r \to q(x)] \Leftrightarrow r \to (\exists x)(q(x))$.

7. $(\forall x)[q(x) \to r] \Leftrightarrow (\exists x)q(x) \to r$.

8. $(\exists x)[q(x) \to r] \Leftrightarrow (\forall x)q(x) \to r$.

9. $(\forall x)(p(x)) \Leftrightarrow (\forall y)(p(y))$.

10. $(\exists x)(p(x)) \Leftrightarrow (\exists y)(p(y))$.

11. $[(\forall x)(p(x) \to r)] \Rightarrow [((\forall x)(p(x))) \to r]$.

12. $[(\forall x)(p(x) \to q(x))] \Rightarrow [(\forall x)(p(x)) \to (\forall x)(q(x))]$.

More Quantified Statements

Let $U \neq \emptyset$. Then

$\forall x.p(x) \Rightarrow \exists x.p(x)$ same as $P = U \Rightarrow P \neq \emptyset$.
This means 'if all are P, then some are P'.
Let $U \neq \emptyset$.
Suppose $P = U$.
Then $P \neq \emptyset$.

Let c be a specific element in U. Then

1. $\forall x.p(x) \Rightarrow p(c)$ same as $P = U \Rightarrow c \in P$.
Suppose $P = U$.
Since $c \in U$ and $U = P$, then $c \in P$.

2. $p(c) \Rightarrow \exists x.p(x)$ same as $c \in P \Rightarrow P \neq \emptyset$.
Suppose $c \in P$.
Then P contains at least one element, so P is not empty.
Therefore $P \neq \emptyset$.
Since $(\forall x)(p(x)) \Rightarrow p(c)$ and $p(c) \Rightarrow (\exists x)(p(x))$, then $(\forall x)(p(x)) \Rightarrow (\exists x)(p(x))$.

Proposition 36. *Let A be a subset of U.*
Then
1. $(\forall x \in A).p(x) \Leftrightarrow \forall x.[x \in A \rightarrow p(x)]$
2. $(\exists x \in A).p(x) \Leftrightarrow \exists x.[x \in A \wedge p(x)]$

Observe that

$$\begin{aligned} \neg(\forall x \in A).p(x) &\Leftrightarrow \neg(\forall x)[x \in A \rightarrow p(x)] \\ &\Leftrightarrow \exists x.[\neg(x \in A \rightarrow p(x))] \\ &\Leftrightarrow \exists x.[x \in A \wedge \neg p(x)] \\ &\Leftrightarrow (\exists x \in A).\neg p(x) \end{aligned}$$

Therefore
$\neg(\forall x \in A).p(x) \Leftrightarrow (\exists x \in A).\neg p(x)$.
This means 'not every x in A is P' same as 'some x in A is not P'.

Observe that

$$\begin{aligned} \neg(\exists x \in A).p(x) &\Leftrightarrow \neg\exists x.[x \in A \wedge p(x)] \\ &\Leftrightarrow \forall x.\neg[x \in A \wedge p(x)] \\ &\Leftrightarrow \forall x.[x \notin A \vee \neg p(x)] \\ &\Leftrightarrow \forall x.[x \in A \rightarrow \neg p(x)] \\ &\Leftrightarrow (\forall x \in A).\neg p(x) \end{aligned}$$

Therefore

$\neg(\exists x \in A).p(x) \Leftrightarrow (\forall x \in A).\neg p(x)$.

This means 'no x in A is P' same as 'all x in A are not P'.

Let x be an element of U. Then

$(\forall x \in U).p(x) \Leftrightarrow \forall x.p(x)$

$(\exists x \in U).p(x) \Leftrightarrow \exists x.p(x)$

Uniqueness

Let $p(x)$ be a predicate defined over a universal set U.

Define statement 'there exists a unique x such that $p(x)$' by

$(\exists! x)((p(x)) \Leftrightarrow [(\exists x)(p(x))] \wedge [(\forall x)(\forall y)((p(x) \wedge p(y)) \rightarrow (x = y))]$.

Therefore, the statement 'there exists a unique x such that $p(x)$' means

1. **Existence** 'there exists at least one x such that $p(x)$: $(\exists x)(p(x))$.

2. **Uniqueness** 'there exists at most one x such that $p(x)$: $(\forall x)(\forall y)((p(x) \wedge p(y)) \rightarrow (x = y))]$.

Aristotelian Forms

All P's are Q's means $\forall x.[p(x) \rightarrow q(x)]$ which is same as $P \subset Q$

Some P's are Q's means $\exists x.[p(x) \wedge q(x)]$ which is same as $P \cap Q \neq \emptyset$

No P's are Q's means $\neg\exists x.[p(x) \wedge q(x)]$ which is same as $P \cap Q = \emptyset$

Negation of 'All P's are Q's' is 'Not all P's are Q's'
which means
'some P's are not Q's'
which is same as
$\neg\forall x.[p(x) \rightarrow q(x)] \Leftrightarrow \exists x.[p(x) \wedge \neg q(x)]$

Negation of 'Some P's are Q's' is 'No P is Q'
which means
'All P's are not Q's'
which is same as
$\neg\exists x.[p(x) \wedge q(x)] \Leftrightarrow (\forall x)[\neg p(x) \vee \neg q(x)] \Leftrightarrow (\forall x)[p(x) \rightarrow \neg q(x)]$

Negation of 'No P's are Q's' is 'Some P's are Q's'
which means
'there exists a P that is Q'
which is same as
$\neg\neg\exists x.[p(x) \wedge q(x)] \Leftrightarrow \exists x.[p(x) \wedge q(x)]$

Propositional Functions of Several Variables

Let $p(x_1, x_2, ..., x_n)$ be a propositional function of n variables.
Let the set U_i be the domain of discourse for the i^{th} variable, x_i.
Then $U_1 \times U_2 \times ... \times U_n$ = domain of discourse for p.
If U = common domain of discourse for all n variables,
then $U^n = U \times U \times ... \times U$ = domain of discourse for p.

The **truth set** of $p(x_1, x_2, ..., x_n)$ is
$P = \{(x_1, x_2, ..., x_n) \in U_1 \times U_2 \times ... \times U_n : p(x_1, x_2, ..., x_n)$ is a true statement $\}$.
Thus $P \subset U_1 \times U_2 \times ... \times U_n$.

Multiple Quantifiers

To translate mixed quantifiers from symbolic logic to English:

1. Insert 'such that' or 'having the property that' after any occurrence of $\exists$ that is followed directly by $\forall$ or by a predicate.
2. Insert 'and' between any two occurrences of the same quantifier.
3. $\forall x \exists y$ translates as 'to every x, there corresponds at least one y'.

Let U_1, U_2 be nonempty sets.
Let $p(x, y)$ be a propositional function defined over domain of discourse $U_1 \times U_2$.
Then $x \in U_1$ and $y \in U_2$ and truth set of p is $P = \{(x, y) \in U_1 \times U_2 : p(x, y)$ is true $\}$.
Thus $P \subset U_1 \times U_2$.

Quantifiers of the Same Type

Order of quantifiers does not matter when quantifiers are of the same kind(both universal or both existential).
Thus all permutations of quantifiers of the same kind are logically equivalent.

$\forall x \forall y . p(x, y)$ means $p(x, y)$ for all x and y
is true whenever $p(x, y)$ is true for every x and every y
is false whenever there is at least one x and at least one y such that $p(x, y)$ is false
$\forall x \forall y . p(x, y) \Leftrightarrow \forall y \forall x . p(x, y)$
$\neg[\forall x \forall y . p(x, y)] \Leftrightarrow \exists x . \neg[\forall y . p(x, y)] \Leftrightarrow \exists x \exists y . \neg p(x, y)$

$\exists x \exists y . p(x, y)$ means there exist x and y such that $p(x, y)$
is true whenever there is at least one x and at least one y such that $p(x, y)$ is true
is false whenever there is no x and y such that $p(x, y)$ is true

which is same as
$p(x, y)$ is false for every x and every y

$\exists x \exists y . p(x, y) \Leftrightarrow \exists y \exists x . p(x, y)$
$\neg[\exists x \exists y . p(x, y)] \Leftrightarrow \forall x . \neg[\exists y . p(x, y)] \Leftrightarrow \forall x \forall y . \neg p(x, y)$

$\forall x \forall y . p(x, y) \Rightarrow \exists x \exists y . p(x, y)$.

Quantifiers of Different Types

Order of quantifiers matters when quantifiers are of different kinds.

$\forall x \exists y . p(x, y)$ means for every x, there exists y such that $p(x, y)$
is true whenever for every x, there is at least one y such that $p(x, y)$ is true
is false whenever there is an x such that $p(x, y)$ is false for all y
$\neg[\forall x \exists y . p(x, y)] \Leftrightarrow \exists x . \neg[\exists y . p(x, y)] \Leftrightarrow \exists x \forall y . \neg p(x, y)$

$\exists x \forall y . p(x, y)$ means there exists x such that $p(x, y)$ for all y
is true whenever there is an x such that $p(x, y)$ is true for every y
is false whenever for every x, there is at least one y such that $p(x, y)$ is false
$\neg[\exists x \forall y . p(x, y)] \Leftrightarrow \forall x . \neg[\forall y . p(x, y)] \Leftrightarrow \forall x \exists y . \neg p(x, y)$

Consider $\exists x \forall y . p(x, y)$.
Observe that variables x and y are each quantified, so all variables are bound.
Hence $\exists x \forall y . p(x, y)$ is a statement, so $\exists x \forall y . p(x, y)$ is true or false.
Suppose $\exists x \forall y . p(x, y)$ is true.
Consider the innermost predicate $\forall y . p(x, y)$.
Observe that y is quantified, so y is bound, and x is free.
Hence $\forall y . p(x, y)$ is a function of x.
Let $r(x) = \forall y . p(x, y)$.
Then $\exists x . r(x)$ is true.
Let R be the truth set of r.
Then $R = \{x \in U_1 : r(x) \text{ is true } \} = \{x \in U_1 : p(x, y) \text{ is true for every } y \in U_2\} = \{x \in U_1 : (x, y) \in P \text{ for every } y \in U_2\}$.
Since $\exists x . r(x)$ is true, then by existential instantiation, let u_1 be an arbitrary element of U_1 such that $r(u_1)$ is true.
Then $u_1 \in U_1$ and $r(u_1)$ is true, so $u_1 \in R$.
Therefore, $u_1 \in U_1$ and $(u_1, y) \in P$ for every $y \in U_2$.
This means P must contain the vertical line $x = u_1$.
Since u_1 is a specific element, then by existential generalization, P must contain at least one vertical line.
Therefore, the truth of $\exists x \forall y . p(x, y)$ implies that the truth set of p must contain at least one vertical line.
Let $y = u_2$ be an arbitrary horizontal line.
Then the line $y = u_2$ must intersect P in at least one point, namely (u_1, u_2).

Thus $(u_1, u_2) \in (U_1 \times \{u_2\}) \cap P$.
Hence $(u_1, u_2) \in P$, so $p(u_1, u_2)$ is true.
Since $x = u_1$ is a particular element of U_1, then by existential generalization, $p(x, u_2)$ is true for some $x \in U_1$.
Thus, $\exists x.p(x, u_2)$ is true.
Since $y = u_2$ is arbitrary, then by universal generalization, $\exists x.p(x, y)$ is true for every $y \in U_2$.
Thus, $\forall y \exists x.p(x, y)$ is true.
Hence $\exists x \forall y.p(x, y)$ logically implies $\forall y \exists x.p(x, y)$.

Therefore $\exists x \forall y.p(x, y) \Rightarrow \forall y \exists x.p(x, y)$.

To negate a statement involving the predicate $p(x_1, x_2, ..., x_n)$ preceded by n quantifiers,
change each universal quantifier $\forall$ to $\exists$,
change each existential quantifier $\exists$ to $\forall$, and negate the predicate.

Proposition 37. *Let $p(x, y)$ be a predicate defined over domain of discourse $U_1 \times U_2$.*
Let $A \subseteq U_1$ and $B \subseteq U_2$.
Then the following are true:
1. $\neg(\forall x \in A)(\exists y \in B).p(x, y) \Leftrightarrow (\exists x \in A)(\forall y \in B).\neg p(x, y)$.
2. $\neg(\exists x \in A)(\forall y \in B).p(x, y) \Leftrightarrow (\forall x \in A)(\exists y \in B).\neg p(x, y)$.
3. $(\exists x \in A)(\forall y \in B).p(x, y) \Rightarrow (\forall y \in B)(\exists x \in A).p(x, y)$.
4. $(\forall x)(\forall y)p(x, y) \Rightarrow (\exists x)(\forall y)p(x, y) \Rightarrow (\forall y)(\exists x)p(x, y) \Rightarrow (\exists x)(\exists y)p(x, y)$.

Rules of Inference for Quantified Statements

Let $p : U \mapsto \{T, F\}$ be a predicate defined over a domain of discourse U.

existential introduction (existential generalization)
$p(c) \Rightarrow \exists x.p(x)$, where c is a particular (specific) element of U.

universal elimination (universal instantiation)
$\forall x.p(x) \Rightarrow p(c)$, where c is a particular (specific) element of U.

universal introduction (universal generalization)
$p(a) \Rightarrow \forall x.p(x)$, where a is an arbitrary element of U.

existential elimination (existential instantiation)
$\exists x.p(x) \Rightarrow p(c)$, where c is some element of U.

Methods of Proof for Quantified Statements

I. **Universally Quantified Statement**

Let $p(x)$ be a predicate defined over domain of discourse U.
Let $(\forall x)(p(x))$ be a statement.

To prove $(\forall x)(p(x))$

1. Assume $a \in U$ is arbitrary.
2. Prove $p(a)$.

This method is valid because if a is an arbitrary element of U, then $p(a) \Rightarrow (\forall x)p(x)$ (universal generalization).

To disprove $(\forall x)(p(x))$

1. Prove $(\exists x)\neg p(x)$.

Thus we find a specific $c \in U$ such that $p(c)$ is false, a **counter example**.

This method is valid because $\neg(\forall x)p(x) \Leftrightarrow (\exists x)\neg p(x)$.

To prove $(\forall x)(p(x) \rightarrow q(x))$

1. Assume $a \in U$ is arbitrary such that $p(a)$.
2. Prove $q(a)$.

Suppose $a \in U$ is arbitrary and $p(a)$ is true.
If $p(a) \Rightarrow q(a)$, then the conditional $p(a) \rightarrow q(a)$ is true.
By universal generalization, the conditional $p(a) \rightarrow q(a)$ is true for all $x \in U$.
Therefore the statement $(\forall x)(p(x) \rightarrow q(x))$ is true.

To disprove $(\forall x)[p(x) \rightarrow q(x)]$

1. Prove $(\exists x)[p(x) \wedge \neg q(x)]$.

Thus we find a specific counter example $c \in U$ such that $p(c)$ is true and $q(c)$ false.

This method is valid because $\neg(\forall x)[p(x) \rightarrow q(x)] \Leftrightarrow (\exists x)[p(x) \wedge \neg q(x)]$.

II. **Existentially Quantified Statement**

Let $p(x)$ be a predicate defined over domain of discourse U.
Let $(\exists x)(p(x))$ be a statement.

To prove $(\exists x)(p(x))$

1. Find a specific $c \in U$ such that $p(c)$ is true. (constructive proof)

This is method is valid because if c is a specific element of U, then $p(c) \Rightarrow (\exists x)p(x)$ (existential generalization).

Alternate approach:

Establish the existence of $c \in U$ such that $p(c)$ is true without actually finding c. (non-constructive proof).

To disprove $(\exists x)(p(x))$

1. Prove $(\forall x)\neg p(x)$.

This method is valid because $\neg(\exists x)p(x) \Leftrightarrow (\forall x)\neg p(x)$.

III. **Disproof**

Let $p(x)$ be a predicate defined over domain of discourse U.

We find a **counterexample**.

Example: Prove $\forall x.[p(x) \to q(x)]$ is false.

To prove $\forall x.[p(x) \to q(x)]$ is false, we take its negation which is $\exists x.[p(x) \wedge \neg q(x)]$.

Thus we find a specific element c such that $p(c) \wedge \neg q(c)$ is true (counter example).

IV. **Uniqueness Proof**

Let $p(x)$ be a predicate defined over domain of discourse U.

Let $(\exists! x)[p(x)]$ be a statement.

To prove $(\exists! x)[p(x)]$

1. Existence: Prove $(\exists x)[p(x)]$. (at least one x exists)
2. Uniqueness: Prove $(\forall x)(\forall y)[(p(x) \wedge p(y)) \to (x = y)]$. (at most one x exists)

To prove uniqueness, we assume two objects x and y exist such that $p(x)$ and $p(y)$ are true.

We must prove $x = y$.

Template: Prove $(\exists! x)(p(x))$.

Suppose $(\exists x)(p(x))$ and $(\exists y)(p(y))$.

...

Therefore $x = y$.

V. **Mathematical Induction**

Prove $\forall(n \geq n_0).p(n)$ where $n \in \mathbb{Z}^+$.

Mathematical Induction

Principle of Mathematical Induction

Let S be a subset of $\mathbb{Z}^+$ such that

1. $1 \in S$
2. $(\forall k \in \mathbb{Z}^+)(k \in S \to k + 1 \in S)$

Then $S = \mathbb{Z}^+$.

Observe that $\mathbb{Z}^+$ satisfies both of the above conditions.
No proper subset of $\mathbb{Z}^+$ satisfies both of the above conditions.

To prove $(\forall n \in \mathbb{Z}^+)[p(n)]$ by induction:
1. Define predicate $p(n)$ where domain of discourse is $\mathbb{Z}^+$.
2. Let S be the truth set of $p(n)$.
Then $S = \{n \in \mathbb{Z}^+ : p(n) \text{ is true }\}$, so $S \subset \mathbb{Z}^+$.
3. Show that $S = \mathbb{Z}^+$ by using the principle of mathematical induction.
a. **Basis**: To prove $1 \in S$, we prove $p(1)$ is true.
b. **Induction**: To prove $(\forall k \in \mathbb{Z}^+)(k \in S \to k + 1 \in S)$,
we assume arbitrary $k \in S$.
Thus, we assume $p(k)$ is true for arbitrary $k \in \mathbb{Z}^+$. **(Induction hypothesis)**
To prove $k + 1 \in S$, we must prove $p(k + 1)$ is true.

This method of proof is valid because to prove a statement of the form $(\forall n \in \mathbb{Z}^+)(p(n))$ is to show that the truth set of $p(n)$ is the universal set $\mathbb{Z}^+$.
In this method, $U = \mathbb{Z}^+$ and S is the truth set of $p(n)$, so we must prove $S = \mathbb{Z}^+$.

Definition 38. Inductive Set
Let $S \subset \mathbb{N}$.
The statement S **is inductive** means $m \in S$ implies $m + 1 \in S$ for all positive integers m.
Therefore S is inductive iff $(\forall m \in \mathbb{Z}^+)(m \in S \to m + 1 \in S)$.

$\mathbb{N}$ is an inductive set.
$\emptyset$ is an inductive set.
Let $n \in \mathbb{N}$.
The set $\{n, n + 1, n + 2, ...\}$ is inductive.
Every nonempty inductive set has the form $\{n, n + 1, n + 2, ...\}$.

Let $T \subset \mathbb{N}$.
To prove $T = \mathbb{N}$ by induction:
1. Prove $1 \in T$.
2. Prove T is an inductive set.

Let S be an inductive subset of $\mathbb{N}$ containing a positive integer m_0.
Then S contains m for every positive integer m greater than m_0; that is, $\{m_0, m_0 + 1, m_0 + 2, ...\} \subset S$.

To prove $(\forall n \geq n_0)[p(n)]$:
Let $n_0 \in \mathbb{N}$.
Let $D = \{n \in \mathbb{N} : n \geq n_0\} = \{n_0, n_0 + 1, n_0 + 2, ...\}$.
Then $D \subset \mathbb{N}$ and D is an inductive set.
Let $p(n)$ be a predicate defined over domain of discourse D.
Let S be the truth set of p.
Then $S = \{n \in D : p(n) \text{ is true }\}$.
Thus, $S \subset D$.
To prove $S = D$, we use induction.
Thus, we must prove:
1. $n_0 \in S$.
2. S is an inductive set.
To prove S is an inductive set, we must prove $(\forall m \in \mathbb{N})(m \in S \to m{+}1 \in S)$
Or, we can prove the weaker statement: $(\forall m \in \mathbb{N})(m \in S \wedge m \geq n_0 \to m + 1 \in S)$.

Strong Induction

Prove that if S_n is true for all cases up to some arbitrary fixed point n, then S_n is true for all cases at point $n + 1$.
A. Prove: S_n for all $n \in \mathbb{N}$ (basis is 1)
Proof:
I. Basis step
S_1 = basis
Prove S_1 is true
II. Inductive step
Assume $S_1 \wedge S_2 \wedge ... \wedge S_k$ is true for fixed $k \geq 1$. (inductive hypothesis)
Show that $S_1 \wedge S_2 \wedge ... \wedge S_k \to S_{k+1}$ is true.
Conclude that S_n is true $\forall n \in \mathbb{N}$.

B. Prove: S_n for all $n \geq n_0 \in \mathbb{Z}$ (using more than one basis)
Proof:
I. Basis step
Consider multiple bases: $n_0, n_0 + 1, ..., j_0$
Prove each base case: $S_{(n_0)}, S_{(n_0+1)}, ..., S_{(j_0)}$
II. Inductive step
Assume $S_{(n_0)} \wedge S_{(n_0+1)} \wedge ... \wedge S_{(j_0)} \wedge S_k$ is true for arbitrary $k \geq j_0$. (inductive hypothesis)
Show that $S_{(n_0)} \wedge S_{(n_0+1)} \wedge ... \wedge S_{(j_0)} \wedge S_k \to S_{k+1}$ is true.
Conclude that S_n is true $\forall (n \geq n_0) \in \mathbb{Z}$.

Formal Language

Notes:
A = alpha set = set of propositional variables(atomic formulae/terminal elements)

Ω = omega set = set of logical connectives/operators = $\{\neg, \wedge, \vee, \rightarrow, \leftrightarrow\}$
Z = zeta set = set of inference rules(transformation)
I = iota set = set of initial points(axioms/laws)

1. alpha set consists of syntactic expressions(well formed formulae)
2. omega set consists of operator symbols (logical operators)
3. syntax
grammar recursively defines expressions and wffs
determines which collection of symbols are legal expressions
4. semantics - determines meaning behind the expressions(true or false)

L = language = set of wffs defined recursively
1. **Basis**: Any element of A is a formula(wff)
2. **Induction**: if $p_1, p_2, ..., p_k$ are formulae and f is in Ω_k, then $f(p_1, p_2, ..., p_k)$ is a formula
3. **Closure**: Nothing else is a formula

Definition 39. Well Formed Formula (wff)

A **well formed formula** is a formula(proposition) that is built up from atomic formulae using logical operators according to the rules of grammar.
1. propositional constant
Example: T,F
2. propositional variable
Example: P,Q,R
3. predicate
Example: p(x), p(x,y), p(x,y,z)
4. Let x be a variable in the domain of discourse.
Let A be a wff.
Then the following are wff:
a. $\forall x.A$
b. $\exists x.A$

Example:
Let $p(x)$ be a predicate defined over domain of discourse U.
Then $(\forall x)(p(x))$ is a wff and $(\exists x)(p(x))$ is a wff.

Definition 40. Interpretation of wff

Define the universe(domain of discourse) for each variable.
Define the predicate over a domain of discourse.
Assign a value to each free variable.
Then this specification is an **interpretation** for the wff.

A wff becomes a proposition when it is given an interpretation.

Example:
Let U be a domain of discourse for variable x.
Let $p(x)$ be a predicate defined over U.
Then $p(x)$ is a well-formed formula.
Since $p(x)$ is an open sentence, then it is not a statement.
Therefore the truth of $p(x)$ is not determined unless a specific value is assigned to x.
Let a be a specific element of U.
Then $p(a)$ is true or false, so $p(a)$ is a proposition(statement).

Definition 41. satisfiable wff
A wff is **satisfiable** iff there exists an interpretation that makes it true.

Therefore wff is **unsatisfiable** iff there does not exist an interpretation that makes it true.

Let U be a domain of discourse for variable x.
Let $p(x)$ be a predicate defined over U.
Then $p(x)$ is a well-formed formula.
Let c be a specific element of U such that $p(c)$ is true.
Then $p(x)$ is satisfiable.

Let U be a domain of discourse for variable x.
Let $p(x)$ be a predicate defined over U.
Then $p(x)$ is a well-formed formula.
Suppose there is no element of U that makes $p(x)$ true.
Then $p(x)$ is unsatisfiable.

Definition 42. Validity of wff
A wff is **valid** if it is true for every interpretation that makes sense.

Example:
Let U be a universal set.
Then $U = \{x : x \in U\}$.
Let U be a domain of discourse for variable x.
Let $p(x)$ be a predicate defined over U.
Let P be the truth set of p.
Then $P = \{x \in U : p(x) \text{ is true }\}$.
Therefore, $P \subset U$.
Since $p(x)$ is a predicate, then $p(x)$ is a well-formed formula.
Suppose $p(x)$ is true for every $x \in U$.
Then $P = U$.
Therefore, $p(x)$ is valid.

Definition 43. Equivalence of wff
Let W_1 and W_2 be wffs.
Then $W_1 \Leftrightarrow W_2$ iff $W_1 \leftrightarrow W_2$ is valid.

Example:

Let U be a domain of discourse for variable x.
Let $p(x)$ and $q(x)$ be predicates defined over U.
Let P and Q be the truth sets of p and q.
Then $P = \{x \in U : p(x) \text{ is true }\}$ and $Q = \{x \in U : q(x) \text{ is true }\}$.
Therefore, $P \subset U$ and $Q \subset U$.
Since $p(x)$ and $q(x)$ are predicates, then each is a well-formed formula.
Suppose $P = Q$.
Then $(\forall x)(x \in P \leftrightarrow x \in Q)$ is true.
Let $a \in U$ be arbitrary.
Then the statement $a \in P \leftrightarrow a \in Q$ is true.
Hence, $p(a) \leftrightarrow q(a)$ is true.
Thus, $p(a) \Leftrightarrow q(a)$.
Since a is arbitrary, then $p(a) \Leftrightarrow q(a)$ for all $a \in U$.
Therefore, $p(x) \Leftrightarrow q(x)$, so $p(x)$ and $q(x)$ are logically equivalent.