Book Number Theory: Concepts and Problems by Andreescu Exercises

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Chapter 2 Divisibility

Chapter 2.1 Basic Properties

Chapter 2.1.1 Divisibility and congruences

Example 1. Find the last digit of $9^{1003} - 7^{902} + 3^{801}$.

Proof. The last digit of a positive integer n is $n \pmod{10}$.

Observe that

$$\begin{aligned} 9^{1003} &\equiv (-1)^{1003} \\ &\equiv -1 \\ &\equiv 9 \pmod{10}. \end{aligned}$$

Hence, $9^{1003} \equiv 9 \pmod{10}$.

Observe that

$$\begin{aligned} 7^{902} &\equiv 7^{2\cdot 451} \\ &\equiv (7^2)^{451} \\ &\equiv 49^{451} \\ &\equiv (-1)^{451} \\ &\equiv -1 \\ &\equiv 9 \pmod{10}. \end{aligned}$$

Hence, $7^{902} \equiv 9 \pmod{10}$.

Observe that

$$\begin{aligned} 3^{801} &\equiv 3^{4\cdot 200+1} \\ &\equiv 3^{4\cdot 200}\cdot 3 \\ &\equiv (3^4)^{200}\cdot 3 \\ &\equiv 81^{200}\cdot 3 \\ &\equiv 1^{200}\cdot 3 \\ &\equiv 1\cdot 3 \\ &\equiv 3 \pmod{10}. \end{aligned}$$

Hence, $3^{801} \equiv 3 \pmod{10}$.

Observe that

$$\begin{aligned} 9^{1003} - 7^{902} + 3^{801} &\equiv 9 - 9 + 3 \\ &\equiv 3 \pmod{10}. \end{aligned}$$

Therefore, the last digit of $9^{1003} - 7^{902} + 3^{801}$ is 3. □

Example 2. For any natural number n, the number $11^{n+2} + 12^{2n+1}$ is divisible by 133.

Proof. Let n be any natural number.

Observe that $144 \equiv 11 \pmod{133}$ and

$$\begin{aligned} 11^{n+2} + 12^{2n+1} &= 11^{n+2} + 12^{2n}\cdot 12 \\ &= 11^{n+2} + (12^2)^n\cdot 12 \\ &= 11^{n+2} + 144^n\cdot 12 \\ &= 11^{n+2} + 12\cdot 144^n \\ &\equiv 11^{n+2} + 12\cdot 11^n \pmod{133} \\ &\equiv 11^n(11^2+12) \pmod{133} \\ &\equiv 11^n\cdot 133 \pmod{133} \\ &\equiv 0 \pmod{133} \end{aligned}$$

Hence, $11^{n+2} + 12^{2n+1} \equiv 0 \pmod{133}$.

Therefore, 133 divides $11^{n+2} + 12^{2n+1}$, so $11^{n+2} + 12^{2n+1}$ is divisible by 133. □

Example 3. Find the least number of the form $|11^k - 5^l|$, where k and l are positive integers.

Proof. TODO □