Number Theory Exercises 1

Jason Sass

July 17, 2023

Natural number system

Peano Axioms for natural number system

Exercise 1. The relation $<$ on $\mathbb{N}$ is not reflexive.

Proof. Since $1 \in \mathbb{N}$ and $1 \not< 1$, then the relation $<$ is not reflexive on $\mathbb{N}$. □

Exercise 2. The relation $<$ on $\mathbb{N}$ is not symmetric.

Proof. Since $1 < 2$ and $2 \not< 1$, then $1 < 2$ does not imply $2 < 1$.
Therefore, the relation $<$ is not symmetric on $\mathbb{N}$. □

Exercise 3. The relation $\leq$ on $\mathbb{N}$ is not symmetric.

Proof. Since $2 \not< 1$ and $2 \neq 1$, then neither $2 < 1$ nor $2 = 1$, so $2 \not\leq 1$.
Hence, $1 \leq 2$ does not imply $2 \leq 1$.
Therefore, $\leq$ is not symmetric. □

Proposition 4. *The set $\mathbb{N}$ is inductive.*

Solution. To prove $\mathbb{N}$ is inductive we must show that
1. $\mathbb{N} \subset \mathbb{N}$.
2. $(\forall m \in \mathbb{Z}^+)(m \in \mathbb{N} \to m + 1 \in \mathbb{N})$.

To prove 2:
We let $m \in \mathbb{N}$ be arbitrary.
We must prove $m + 1 \in \mathbb{N}$. □

Proof. Since every set is a subset of itself and $\mathbb{N}$ is a set, then $\mathbb{N} \subset \mathbb{N}$.
Since $1 \in \mathbb{N}$, then $\mathbb{N}$ is not empty.
Since $\mathbb{N}$ is not empty, then there exists an element in $\mathbb{N}$.
Therefore, let m be an arbitrary element of $\mathbb{N}$.
We must prove $m + 1 \in \mathbb{N}$.
Since $\mathbb{N}$ is closed under addition and $m, 1 \in \mathbb{N}$, then it follows that the element $m + 1$ is in $\mathbb{N}$.
Hence, $m + 1 \in \mathbb{N}$, as desired. □

Proposition 5. *The empty set is inductive.*

Solution. To prove $\emptyset$ is inductive we must show that

1. $\emptyset \subset \mathbb{N}$.
2. $(\forall n \in \mathbb{N})(n \in \mathbb{N} \to n+1 \in \mathbb{N})$.

To prove 2:
We let $m \in \emptyset$ be arbitrary.
We must prove $m+1 \in \emptyset$. ☐

Proof. Since the empty set is a subset of every set, then $\emptyset \subset \mathbb{N}$.
Let m be an arbitrary natural number.
Since $\emptyset$ is empty, then $m \notin \emptyset$, so $m \in \emptyset$ is false.
Hence, the conditional $m \in \emptyset \to m+1 \in \emptyset$ is vacuously true.
Since m is arbitrary, then $n \in \emptyset \to n+1 \in \emptyset$ is true for every $n \in \mathbb{N}$.
Therefore, $\emptyset$ is inductive. ☐

Proposition 6. *Let $n \in \mathbb{Z}^+$.*
The set $\{n, n+1, n+2, ...\}$ is inductive.

Solution.
Let $S = \{n, n+1, n+2, ...\}$.
To prove S is inductive we must show that

1. $S \subset \mathbb{N}$.
2. $(\forall m \in \mathbb{Z}^+)(m \in S \to m+1 \in S)$.

To prove 2:
We let $m \in \mathbb{N}$ be arbitrary.
We must prove $m+1 \in S$. ☐

Proof. Let $n \in \mathbb{Z}^+$.
Let $S = \{n, n+1, n+2, ...\}$.
Then $S = \{k : k \geq n\}$.
Since n is a natural number and $k \geq n$, then each k is a natural number.
Hence, S is a set of natural numbers, so $S \subset \mathbb{N}$.
Observe that $n \in S$.
Hence, S is not empty.
Since S is not empty, then there exists an element in S.
Therefore, let m be an arbitrary element of S.
To prove $m+1 \in S$, we must prove $m+1 \in \mathbb{N}$ and $m+1 \geq n$.
Since $m \in S$, then $m \in \mathbb{N}$ and $m \geq n$.
Since $m \in \mathbb{N}$ and $1 \in \mathbb{N}$, then by closure of $\mathbb{N}$ under addition, $m+1 \in \mathbb{N}$.
Since $m+1 > m$ and $m \geq n$, then $m+1 > n$.
Therefore, $m+1 \in \mathbb{N}$ and $m+1 > n$, as desired. ☐

Proposition 7. *Every nonempty inductive set has the form $\{m, m+1, m+2, ...\}$.*

Solution. The statement to prove is:
for all nonempty sets S, if S is inductive, then there exists $m \in \mathbb{N}$ such that $S = \{m, m+1, m+2, ...\}$.
We can use well ordering principle of $\mathbb{N}$ and previously proved theorems. ☐

Proof. Let S be a nonempty inductive set.

Since S is inductive, then S is a subset of $\mathbb{N}$.

Since S is not empty and $S \subset \mathbb{N}$, then S has a least element, by the well ordering principle of $\mathbb{N}$.

Let $m \in \mathbb{N}$ be the least element of S.

Since S is inductive and $m \in S$, then we know $\{m, m+1, m+2, ...\} \subset S$.

Since m is the least element contained in S, then there is no natural number smaller than m that is contained in S.

Hence, $S = \{m, m+1, m+2, ...\}$, as desired. □

Exercise 8. Show that $n^2 \geq n$ for every $n \in \mathbb{N}$.

Proof. Let $n \in \mathbb{N}$ be arbitrary.

Then $n \geq 1$.

Since $1 > 0$, then $n > 0$.

Since $n \geq 1$ and $n > 0$, then $n^2 = nn \geq 1n = n$, so $n^2 \geq n$, as desired. □

Exercise 9. The natural numbers are well ordered.

Solution. We know that the pair $\langle \mathbb{N}, \leq \rangle$ is a poset.

By definition of well ordering $\mathbb{N}$ is well ordered iff every nonempty subset of $\mathbb{N}$ has a least element.

Thus our proposition to prove is:

Every nonempty subset of natural numbers has a least element.

Let S be a nonempty subset of $\mathbb{N}$.

We must prove S has a least element.

Let's think of what we can deduce if we assume S is a nonempty subset of $\mathbb{N}$.

We can use mathematical induction to prove the proposition.

Our statement P_n is:

If $S \subset \mathbb{N}$ and S is nonempty, then S has a least element.

How do we describe an arbitrary subset S of $\mathbb{N}$ that has a least element in such a way so that it becomes a statement about $\mathbb{N}$?

We need to specify precisely what it means for a subset of $\mathbb{N}$ to have a least element.

Let $S \subset \mathbb{N}$ be nonempty.

Let $s \in S$ be the smallest element of S.

What can we deduce about s?

Well, we know 1 is the least positive natural number, so $s \geq 1$.

In order to make this into a statement about $\mathbb{N}$, we can let $s \leq n$ where $n \in \mathbb{N}$ is arbitrary.

Thus, our revised statement is:

P_n : If $S \subset \mathbb{N}$ is nonempty and $\exists s \in S$ such that $1 \leq s \leq n$, then S has a least element.

To summarize we must prove the proposition $\forall n \in \mathbb{N}, P_n$ where the statement P_n is If $S \subset \mathbb{N}$ is nonempty and $\exists s \in S$ such that $1 \leq s \leq n$, then S has a least element.

Our basis is $n_0 = 1$ and we must prove P_1.

For induction we must prove $P_k \to P_{k+1}$ for any $k \geq 1$.

We need to prove P_k implies P_{k+1}.

We use direct proof so we assume the statement: if $S \subset \mathbb{N}$ is nonempty and $\exists s \in S$ such that $1 \leq s \leq k$, then S has a least element is true.

We must prove:

if $\exists t \in S$ such that $1 \leq t \leq k+1$, then S has a least element.

In order to prove this we assume: $\exists t \in S$ such that $1 \leq t \leq k+1$. □

Proof. We know $\langle \mathbb{N}, \leq \rangle$ is a total order.

We must prove $\mathbb{N}$ is well ordered.

We prove the proposition $\forall n \in \mathbb{N}, P_n$ where the statement P_n is If $S \subset \mathbb{N}$ is nonempty and $\exists s \in S$ such that $1 \leq s \leq n$, then S has a least element.

We prove using mathematical induction(weak).

Basis:

For $n = 1$ the statement P_1 is:

If $S \subset \mathbb{N}$ is nonempty and $\exists s \in S$ such that $1 \leq s \leq 1$, then S has a least element.

Let $S \subset \mathbb{N}$ be nonempty.

Suppose there exists $s \in S$ such that $1 \leq s \leq 1$.

Then $s = 1$, so $1 \in S$.

Since 1 is the smallest natural number, then 1 is the least element of S, so S has a least element.

Therefore the statement P_1 is true.

Induction:

Suppose it is true that if there is some $s \in S$ such that $1 \leq s \leq k$ for any $k \geq 1$, then S has a least element.

We must prove: if there is some integer in S that is less than or equal to $k+1$, then S has a least element.

Let $k \geq 1$.

Observe that S either contains an element less than $k+1$ or it does not.

We consider these cases separately.

Case 1: Suppose S does not contain an element that is less than $k+1$.

Since S is not empty, let $s \in S$.

Since $\neg\exists s \in S$ such that $s < k+1$, then $\forall s \in S. s \geq k+1$.

Hence every element of S must be greater than or equal to $k+1$.

Since S is not empty, let $t \in S$.

Suppose $t \leq k+1$.

Since every element of S must be greater than or equal to $k+1$, then $t \geq k+1$.

Since $t \geq k+1$ then $k+1 \leq t$.

Since $t \leq k+1$ and $k+1 \leq t$ then by the antisymmetric property of $\leq$, $t = k+1$.

Since every element of S must be greater than or equal to $k+1$ and $t \in S$ is $k+1$, then t is the least element of S.

Hence, S has a least element.

Case 2: Suppose S does contain an element that is less than $k+1$.

Then $\exists t \in S$ such that $t < k+1$.

Since S is nonempty, choose $t \in S$ to be the element less than $k+1$.

Then $t < k+1$.

Since 1 is the least element of $\mathbb{N}$ then any element of $\mathbb{N}$ is greater than or equal to 1.

Since $S \subset \mathbb{N}$ and $t \in S$ then $t \in \mathbb{N}$.

Hence, $t \geq 1$.

Thus $1 \leq t$ and $t < k+1$, so $1 \leq t < k+1$.

Therefore, $1 \leq t \leq k$.

Thus, there is some element $t \in S$ such that $1 \leq t \leq k$.

By the induction hypothesis, this implies S has a least element.

Both cases show for any nonempty $S \subset \mathbb{N}$ if there is some $s \in S$ such that $1 \leq s \leq k$ for any $k \geq 1$, then S has a smallest element.

Hence, by induction, if $S \subset \mathbb{N}$ is nonempty and there is some $s \in S$ such that $1 \leq s \leq n$, then S has a least element for every $n \in \mathbb{N}$.

Therefore, every nonempty subset of $\mathbb{N}$ has a least element, so $\mathbb{N}$ is well ordered. □

Construction of $\mathbb{Z}$

Exercise 10. There is no integer between 0 and 1.

Proof. Suppose there exists an integer n between 0 and 1.

Let $S = \{n \in \mathbb{Z} : 0 < n < 1\}$.

Since $n \in S$, then $S \neq \emptyset$.

If $x \in S$, the $x \in \mathbb{Z}$ and $0 < x < 1$, so $0 < x$.

Since $x \in \mathbb{Z}$ and $x > 0$, then $x \in \mathbb{Z}^+$, so $S \subset \mathbb{Z}^+$.

Since $S \subset \mathbb{Z}^+$ and $S \neq \emptyset$, then by the well-ordering principle of $\mathbb{Z}^+$, S has a least element $m \in S$.

Thus, $m \leq x$ for all $x \in S$.

Since $m \in S$, then $m \in \mathbb{Z}$ and $0 < m < 1$, so $0 < m$ and $m < 1$.

Since $m > 0$ and $0 < m < 1$, then we multiply by m to obtain $0 < m^2 < m$, so $0 < m^2 < m < 1$.

Since $m \in \mathbb{Z}$, then $m^2 \in \mathbb{Z}$.

Since $m^2 \in \mathbb{Z}$ and $0 < m^2 < 1$, then $m^2 \in S$.

Thus, $m^2 \in S$ and $m^2 < m$.

But, this contradicts that m is the least element of S.

Hence, S does not have a least element, so S must be empty.

Therefore, there does not exist an integer between 0 and 1. □

Proposition 11. *Let S be an inductive subset of $\mathbb{N}$ containing a positive integer m_0.*

Then S contains m for every positive integer m greater than m_0; that is, $\{m_0, m_0+1, m_0+2, ...\} \subset S$.

Solution.

The hypothesis is:

1. $S \subset \mathbb{N}$
2. S is inductive.
3. $m \in \mathbb{Z}^+$ is arbitrary.
4. $m_0 \in S$.

The conclusion is:

1. $\{m_0, m_0 + 1, m_0 + 2, ...\} \subset S$

To prove the conclusion:

Let $R = \{m_0, m_0 + 1, m_0 + 2, ...\}$.

Assume $x \in R$.

We must prove $x \in S$.

Now, how do we prove $x \in S$?

Since S is inductive, then we know $(\forall n \in \mathbb{Z}^+)(n \in S \rightarrow n + 1 \in S)$.

Since $m_0 \in S$, then this implies any natural number greater than or equal to m_0 must be in S.

However, S could possibly contain some natural numbers less than m_0, so we need a precise way to completely describe S.

We should think about how S is related to m_0.

The key idea is to partition $\mathbb{N}$ into a set of numbers less than m_0.

Consider $T = \{1, 2, ...m_0 - 1\} \cup S$.

What can we deduce about T?

Well, intuitively, it appears $T = \mathbb{N}$.

So, let's prove that $T = \mathbb{N}$.

Observe that $T = \{t : t \in \{1, 2, ...m_0 - 1\} \vee t \in S\}$.

Since $\{1, 2, ...m_0 - 1\} \subset \mathbb{N}$ and $S \subset \mathbb{N}$, then $t \in \mathbb{N}$.

Hence, $T \subset \mathbb{N}$.

To prove $T = \mathbb{N}$, we can use the principle of mathematical induction.

Thus, we must prove:

1. $1 \in T$.
2. T is an inductive set.

To prove 2, we let $m \in \mathbb{Z}^+$ be arbitrary such that $m \in T$.

We must prove $m + 1 \in T$.

Basis:

Observe that $1 \in T$.

Induction:

Since T is not empty, then there exists an element in T.

Let $m \in T$ be arbitrary.

We must prove $m + 1 \in T$.

Since $m \in T$ and $T \subset \mathbb{N}$, then $m \in \mathbb{N}$.

Hence, either $m < m_0 - 1$ or $m = m_0 - 1$ or $m > m_0 - 1$, by trichotomy of $\mathbb{N}$.

We consider each case separately.

Case 1: Suppose $m < m_0 - 1$.

Then $m + 1 < m_0$, so $m + 1 \in \{1, 2, ..., m_0 - 1\}$.

Thus, $m + 1 \in T$.

Case 2: Suppose $m = m_0 - 1$.
Then $m + 1 = m_0$.
Since $m_0 \in S$, then $m_0 \in T$.
Hence, $m + 1 \in T$.
Case 3: Suppose $m > m_0 - 1$.
Then $m + 1 > m_0$.
Since S is inductive and $m_0 \in S$, then any natural number greater than m_0 is in S.
Thus, $m + 1 \in S$.
Hence, $m + 1 \in T$.
Therefore, in all cases, $m + 1 \in T$, so $m \in T$ implies $m + 1 \in T$.
Thus, T is inductive.
Consequently, by the principle of induction, it follows that $T = \mathbb{N}$.
To prove $x \in S$, let $x \in \mathbb{N}$ such that $x \in R$.
Then either $x = m_0$ or $x > m_0$.
We consider each case separately.
Case 1: Suppose $x = m_0$.
By hypothesis, we know $m_0 \in S$ and $m_0 \in R$, so the conditional $m_0 \in S \rightarrow m_0 \in R$ is true.
Hence, $m_0 \in S \Rightarrow m_0 \in R$.
Case 2: Suppose $x > m_0$.
Since $x \in \mathbb{N}$ and $\mathbb{N} = T$, then either $x \in \{1, 2, ..., m_0 - 1\}$ or $x \in S$.
Since $x > m_0$, then x cannot be in $\{1, 2, ..., m_0 - 1\}$.
Hence, $x \in S$.
Therefore, in all cases, $x \in S$, so that $x \in R$ implies $x \in S$.
Consequently, $R \subset S$, as desired. □

Exercise 12. Prove $(\forall n \in \mathbb{Z}^+)(n < 2^n)$.

Proof. We prove by induction.
Basis:
If $n = 1$, then $1 < 2 = 2^1$, so the statement is true for $n = 1$.
Induction:
Suppose $k \in \mathbb{Z}^+$ such that $k < 2^k$.
Since $k \in \mathbb{Z}^+$, then $k \in \mathbb{Z}$ and $k \geq 1$.
Since $1 \leq k$ and $k < 2^k$, then $1 < 2^k$.
Since $k < 2^k$ and $1 < 2^k$, then $k + 1 < 2^k + 2^k = 2 \cdot 2^k = 2^{k+1}$.
Hence, $k + 1 < 2^{k+1}$.
It follows by induction that $n \leq n^2$ for every positive integer n. □

Exercise 13. For all $n \in \mathbb{N}, 2n + 1 \leq 3n^2$.

Proof. Let $n \in \mathbb{N}$ be arbitrary.
Then $n \geq 1$, so $3n \geq 3$.
Thus, $3n - 2 \geq 1$.
Since $n \geq 1$ and $3n - 2 \geq 1$, then $n(3n - 2) \geq 1$.
Hence, $3n^2 - 2n \geq 1$, so $3n^2 \geq 2n + 1$.
Therefore, $2n + 1 \leq 3n^2$. □

Exercise 14. For all integers $n \geq 3, n^2 + 5 < n^3$.

Proof. Let $n \in \mathbb{N}$ with $n \geq 3$.
Then $n^2 \geq 9$ and $n - 1 \geq 2$.
Since $n^2 \geq 9 > 5$, then $n^2 > 5$.
Since $n - 1 \geq 2 > 1$, then $n - 1 > 1$.
Thus, $n^2(n-1) > 5$, so $n^3 - n^2 > 5$.
Therefore, $n^3 > n^2 + 5$, so $n^2 + 5 < n^3$, as desired. □

Exercise 15. For all $n \in \mathbb{N}, 2n^2 - 1 \leq n^3$.

Proof. Let $n \in \mathbb{N}$ be arbitrary.
Then either $n = 1$ or $n = 2$ or $n > 2$.
We consider these cases separately.
Case 1: Suppose $n = 1$.
Then $2n^2 - 1 = 2(1)^2 - 1 = 2 - 1 = 1 = 1^3 = n^3$.
Case 2: Suppose $n = 2$.
Then $2n^2 - 1 = 2(2^2) - 1 = 7 < 8 = 2^3 = n^3$.
Case 3: Suppose $n > 2$.
Then $n - 2 > 0$.
Since $n > 2 > 0$, then $n > 0$, so $n^2 > 0$.
Since $n^2 > 0$ and $n - 2 > 0$, then $n^2(n-2) > 0$, so $n^3 - 2n^2 > 0$.
Since $n^3 - 2n^2 > 0 > -1$, then $n^3 - 2n^2 > -1$, so $n^3 > 2n^2 - 1$.
Thus, $2n^2 - 1 < n^3$.
Therefore, in all cases, either $2n^2 - 1 < n^3$ or $2n^2 - 1 = n^3$, so $2n^2 - 1 \leq n^3$. □

Exercise 16. For all integers $n \geq 4, n! > 2^n$.

Proof. Let $p(n)$ be the predicate $n! > 2^n$ defined over $\mathbb{N}$.
We prove $p(n)$ is true for all $n \geq 4$ by induction on n.
Basis:
Since $4! = 24 > 16 = 2^4$, then $p(4)$ is true.
Induction:
Suppose $p(k)$ is true for any natural number $k \geq 4$.
Then $k! > 2^k$.
Since $k + 1 > k \geq 4 > 2$, then $k + 1 > 2$.
Since $k + 1 > 2 > 0$, then $k + 1 > 0$.
Since $k \geq 4 > 0$, then $k > 0$, so $2^k > 0$.
Observe that

$$\begin{aligned} (k+1)! &= (k+1)k! \\ &> (k+1)2^k \\ &> 2 \cdot 2^k \\ &= 2^{k+1}. \end{aligned}$$

Since $(k+1)! > 2^{k+1}$, then $p(k+1)$ is true.

Thus, $p(k)$ implies $p(k+1)$ for any natural number $k \geq 4$.

Since $p(4)$ is true and $p(k)$ implies $p(k+1)$ for any natural number $k \geq 4$, then by PMI, $p(n)$ is true for any natural number $n \geq 4$. □

Exercise 17. For all integers $n \geq 3, 2^n > n + 4$.

Proof. Let $p(n)$ be the predicate $2^n > n + 4$ defined over $\mathbb{N}$.

We prove $p(n)$ is true for all $n \geq 3$ by induction on n.

Basis:

Since $2^3 = 8 > 7 = 3 + 4$, then $p(3)$ is true.

Induction:

Let $k \in \mathbb{N}$ with $k \geq 3$ such that $p(k)$ is true.

Then $2^k > k + 4$.

To prove $p(k+1)$ is true, we must prove $2^{k+1} > (k+1) + 4$.

Since $k \geq 3$, then $k + 4 \geq 7 > 1$, so $k + 4 > 1$.

Since $2^k > k + 4 > 1$, then $2^k > 1$.

Observe that

$$\begin{aligned} 2^{k+1} &= 2^k \cdot 2 \\ &= 2^k + 2^k \\ &> (k+4) + 1 \\ &= (k+1) + 4. \end{aligned}$$

Therefore, $2^{k+1} > (k+1) + 4$, as desired. □

Exercise 18. For all integers $n \geq 3, 2n + 1 < 2^n$.

Proof. Let $p(n)$ be the predicate $2n + 1 < 2^n$ defined over $\mathbb{N}$.

We prove $p(n)$ is true for all natural numbers $n \geq 3$ by induction on n.

Basis:

Since $2 \cdot 3 + 1 = 7 < 8 = 2^3$, then $p(3)$ is true.

Induction:

Let $k \in \mathbb{N}$ with $k \geq 3$ such that $p(k)$ is true.

Then $2k + 1 < 2^k$.

Since $k \geq 3 > 1$, then $k > 1$, so $2^k > 2$.

Thus, $2 < 2^k$.

Observe that

$$\begin{aligned} 2(k+1) + 1 &= 2k + 2 + 1 \\ &= (2k+1) + 2 \\ &< 2^k + 2^k \\ &= 2 \cdot 2^k \\ &= 2^{k+1}. \end{aligned}$$

Since $2(k+1) + 1 < 2^{k+1}$, then $p(k+1)$ is true.

Thus, $p(k)$ implies $p(k+1)$ for $k \geq 3$.

Therefore, by PMI, $p(n)$ is true for all $n \geq 3$. □

Exercise 19. For all natural numbers $n \geq 5$, $n^2 < 2^n$.

Proof. Let $p(n)$ be the predicate $n^2 < 2^n$ defined over $\mathbb{N}$.

We prove $p(n)$ is true for all natural numbers $n \geq 5$ by induction on n.

Basis:

Since $5^2 = 25 < 32 = 2^5$, then $p(5)$ is true.

Induction:

Let $k \in \mathbb{N}$ with $k \geq 5$ such that $p(k)$ is true.

Since $k \geq 5 > 1$, then $k > 1$.

Since $k \geq 5$, then $k - 2 \geq 3 > 1$, so $k - 2 > 1$.

Since $k > 1$ and $k - 2 > 1$, then $k(k-2) > 1$, so $k^2 - 2k > 1$.

Thus, $k^2 > 2k + 1$, so $2k + 1 < k^2$.

Since $p(k)$ is true, then $k^2 < 2^k$.

Observe that

$$\begin{aligned}(k+1)^2 &= k^2 + 2k + 1 \\ &< k^2 + k^2 \\ &= 2k^2 \\ &< 2 \cdot 2^k \\ &= 2^{k+1}.\end{aligned}$$

Hence, $(k+1)^2 < 2^{k+1}$, so $p(k+1)$ is true.

Thus, $p(k)$ implies $p(k+1)$ for all natural numbers $k \geq 5$.

Since $p(5)$ is true and $p(k)$ implies $p(k+1)$ for all natural numbers $k \geq 5$, then by PMI, $p(n)$ is true for all natural numbers $n \geq 5$, as desired. □

Exercise 20. Prove that $2^n > (n+1)^2$ for all integers $n \geq 6$.

Proof. We prove the statement $2^n > (n+1)^2$ for all integers $n \geq 6$ by induction on n.

Let $S = \{n \in \mathbb{N} : 2^n > (n+1)^2, n \geq 6\}$.

Basis:

Since $2^6 = 64 > 49 = (6+1)^2$, then $6 \in S$.

Induction:

Suppose $k \in S$.

Then $k \in \mathbb{Z}$ and $k \geq 6$ and $2^k > (k+1)^2$.

Since $k \geq 6$, then $k^2 \geq 36 > 2$, so $k^2 > 2$.

Thus, $(k^2 + 4k + 2) + k^2 > (k^2 + 4k + 2) + 2$, so $2k^2 + 4k + 2 > k^2 + 4k + 4$.

Hence, $2(k+1)^2 = 2(k^2 + 2k + 1) = 2k^2 + 4k + 2 > k^2 + 4k + 4 = (k+2)^2$, so $2(k+1)^2 > (k+2)^2$.

Since $2^{k+1} = 2^{1+k} = 2 \cdot 2^k > 2(k+1)^2 > (k+2)^2$, then $2^{k+1} > (k+2)^2$, so $k + 1 \in S$.

Hence, $k \in S$ implies $k + 1 \in S$.

Therefore, by PMI, $2^n > (n+1)^2$ for all integers $n \geq 6$. □

Exercise 21. For all $n \in \mathbb{N}$, if $n \geq 17$, then $n^4 < 2^n$.

Proof. Let $p(n)$ be the predicate $n^4 < 2^n$ defined over the domain of discourse, the set $\{n \in \mathbb{N} : n \geq 17\}$.

To prove $p(n)$ is true for all $n \geq 17$, we prove by induction on n.

Basis:

Since $17^4 = 83521 < 131072 = 2^{17}$, then $p(17)$ is true.

Induction:

Let $k \in \mathbb{N}$ with $k \geq 17$ such that $p(k)$ is true.
Then $k^4 < 2^k$.
To prove $p(k+1)$ is true, we must prove $(k+1)^4 < 2^{k+1}$.

Since $k \geq 17 > 16$, then $k > 16$, so $\frac{k}{4} > 4$.
Since $k \geq 17 > 0$, then $k > 0$, so $k^3 > 0$.
Thus, $\frac{k^4}{4} > 4k^3$.
Since $k \geq 17$, then $k^2 \geq 17^2 = 289 > 24$, so $k^2 > 24$.
Thus, $\frac{k^2}{4} > 6$.
Since $k > 0$, then $k^2 > 0$, so $\frac{k^4}{4} > 6k^2$.
Since $k \geq 17$, then $k^3 \geq 17^3 = 4913 > 16$, so $k^3 > 16$.
Thus, $\frac{k^3}{4} > 4$.
Since $k > 0$, then $\frac{k^4}{4} > 4k$.
Since $k \geq 17$, then $k^4 \geq 17^4 = 83521 > 4$, so $k^4 > 4$.
Thus, $\frac{k^4}{4} > 1$.
Hence, $4k^3 < \frac{k^4}{4}$ and $6k^2 < \frac{k^4}{4}$ and $4k < \frac{k^4}{4}$ and $1 < \frac{k^4}{4}$.
Observe that

$$\begin{aligned}(k+1)^4 &= k^4 + 4k^3 + 6k^2 + 4k + 1 \\ &< k^4 + \frac{k^4}{4} + \frac{k^4}{4} + \frac{k^4}{4} + \frac{k^4}{4} \\ &= 2k^4 \\ &< 2 \cdot 2^k \\ &= 2^{k+1}.\end{aligned}$$

Therefore, $(k+1)^4 < 2^{k+1}$, as desired. □

Elementary Aspects of Integers

Exercise 22. The square of an odd integer is odd.

Proof. Let $n \in \mathbb{Z}$.
Suppose n is odd.
Since the product of two odd integers is odd, then $n^2 = n \cdot n$ is odd. □

Exercise 23. For every natural number n, $2^n + 1 \leq 3^n$.

Proof. Let $p(n)$ be the predicate $2^n + 1 \leq 3^n$ defined over $\mathbb{N}$.
We prove $p(n)$ is true for all $n \in \mathbb{N}$ by induction on n.
Basis:
Since $2^1 + 1 = 3 = 3^1 \leq 3^1$, then $p(1)$ is true.
Induction:
Let $k \in \mathbb{N}$ such that $p(k)$ is true.
Then $k \geq 1$ and $2^k + 1 \leq 3^k$.
To prove $p(k+1)$, we must prove $2^{k+1} + 1 \leq 3^{k+1}$.
Since $k \geq 1 > 0$, then $k > 0$, so $3^k > 0$.
Observe that

$$\begin{aligned} 2^{k+1} + 1 &= 2 \cdot 2^k + 1 \\ &< 2 \cdot 2^k + 2 \\ &= 2(2^k + 1) \\ &\leq 2 \cdot 3^k \\ &< 3 \cdot 3^k \\ &= 3^{k+1}. \end{aligned}$$

Thus, $2^{k+1} + 1 < 3^{k+1}$, so $2^{k+1} + 1 \leq 3^{k+1}$.
It follows by induction that $2^n + 1 \leq 3^n$ for every $n \in \mathbb{N}$. □

Exercise 24. For each $n \in \mathbb{N}, 2^n \leq 2^{n+1} - 2^{n-1} - 1$.

Proof. Let $p(n)$ be the predicate $2^n \leq 2^{n+1} - 2^{n-1} - 1$ defined over $\mathbb{N}$.
We prove $p(n)$ is true for all $n \in \mathbb{N}$ by induction on n.
Basis:
Since $2^1 = 2 \leq 2 = 4 - 1 - 1 = 2^{1+1} - 2^{1-1} - 1$, then the statement $p(1)$ is true.
Induction:
Let $k \in \mathbb{N}$ such that $p(k)$ is true.
Then $k \geq 1$ and $2^k \leq 2^{k+1} - 2^{k-1} - 1$.
To prove $p(k+1)$, we must prove $2^{k+1} \leq 2^{k+2} - 2^k - 1$.
Observe that

$$\begin{aligned} 2^{k+1} &= 2 \cdot 2^k \\ &\leq 2(2^{k+1} - 2^{k-1} - 1) \\ &= 2^{k+2} - 2^k - 2 \\ &< 2^{k+2} - 2^k - 1. \end{aligned}$$

Thus, $2^{k+1} < 2^{k+2} - 2^k - 1$, so $2^{k+1} \leq 2^{k+2} - 2^k - 1$.
Hence, $p(k)$ implies $p(k+1)$ for any $k \geq 1$.
It follows by induction that $2^n \leq 2^{n+1} - 2^{n-1} - 1$ for each $n \in \mathbb{N}$. □

Exercise 25. For all $n \in \mathbb{N}$, $\sum_{k=1}^{n} k(k+1) = \frac{n(n+1)(n+2)}{3}$.

Proof. We prove by induction on n.

Let $S = \{n \in \mathbb{N} : \sum_{k=1}^{n} k(k+1) = \frac{n(n+1)(n+2)}{3}\}$.

Basis:

Since $1 \in \mathbb{N}$ and $1 \cdot 2 = 2 = \frac{1 \cdot 2 \cdot 3}{3}$, then $1 \in S$.

Induction:

Suppose $m \in S$.

Then $m \in \mathbb{N}$ and $\sum_{k=1}^{m} k(k+1) = \frac{m(m+1)(m+2)}{3}$.

Thus,

$$\begin{aligned}\sum_{k=1}^{m+1} k(k+1) &= \sum_{k=1}^{m} k(k+1) + (m+1)(m+2) \\ &= \frac{m(m+1)(m+2)}{3} + (m+1)(m+2) \\ &= (m+1)(m+2)(\frac{m}{3} + 1) \\ &= \frac{(m+1)(m+2)(m+3)}{3}.\end{aligned}$$

Since $m+1 \in \mathbb{N}$ and $\sum_{k=1}^{m+1} k(k+1) = \frac{(m+1)(m+2)(m+3)}{3}$, then $m+1 \in S$.

Therefore, $m \in S$ implies $m+1 \in S$, so by PMI, $\sum_{k=1}^{n} k(k+1) = \frac{n(n+1)(n+2)}{3}$ for all $n \in \mathbb{N}$, as desired. $\square$

Exercise 26. For all $n \in \mathbb{N}$, $\sum_{k=1}^{n} \frac{1}{k(k+1)} = \frac{n}{n+1}$.

Proof. We prove by induction on n.

Let $S = \{n \in \mathbb{N} : \sum_{k=1}^{n} \frac{1}{k(k+1)} = \frac{n}{n+1}\}$.

Basis:

Since $1 \in \mathbb{N}$ and $\sum_{k=1}^{1} \frac{1}{k(k+1)} = \frac{1}{1 \cdot (1+1)} = \frac{1}{1 \cdot 2} = \frac{1}{2} = \frac{1}{1+1}$, then $1 \in S$.

Induction:

Suppose $m \in S$.

Then $m \in \mathbb{N}$ and $\sum_{k=1}^{m} \frac{1}{k(k+1)} = \frac{m}{m+1}$.

Thus,

$$\begin{aligned}
\sum_{k=1}^{m+1} \frac{1}{k(k+1)} &= \sum_{k=1}^{m} \frac{1}{k(k+1)} + \frac{1}{(m+1)(m+2)} \\
&= \frac{m}{m+1} + \frac{1}{(m+1)(m+2)} \\
&= \frac{1}{m+1} \cdot (m + \frac{1}{m+2}) \\
&= \frac{1}{m+1} \cdot \frac{m^2+2m+1}{m+2} \\
&= \frac{(m+1)^2}{(m+1)(m+2)} \\
&= \frac{m+1}{m+2} \\
&= \frac{m+1}{(m+1)+1}.
\end{aligned}$$

Since $m+1 \in \mathbb{N}$ and $\sum_{k=1}^{m+1} \frac{1}{k(k+1)} = \frac{m+1}{(m+1)+1}$, then $m+1 \in S$.

Therefore, $m \in S$ implies $m+1 \in S$, so by PMI, $\sum_{k=1}^{n} \frac{1}{k(k+1)} = \frac{n}{n+1}$ for all $n \in \mathbb{N}$, as desired. □

Exercise 27. The sum of the squares of the first n odd natural numbers is $\frac{n(4n^2-1)}{3}$.

Proof. We must prove $\sum_{k=1}^{n}(2k-1)^2 = \frac{n(4n^2-1)}{3}$ for all $n \in \mathbb{N}$.

Let $n \in \mathbb{N}$.

Then

$$\begin{aligned}
\sum_{k=1}^{n}(2k-1)^2 &= \sum_{k=1}^{n}(4k^2-4k+1) \\
&= \sum_{k=1}^{n}4k^2 - \sum_{k=1}^{n}4k + \sum_{k=1}^{n}1 \\
&= 4\sum_{k=1}^{n}k^2 - 4\sum_{k=1}^{n}k + \sum_{k=1}^{n}1 \\
&= 4\cdot\frac{n(n+1)(2n+1)}{6} - 4\cdot\frac{n(n+1)}{2} + n \\
&= \frac{2n(n+1)(2n+1)}{3} - 2n(n+1) + n \\
&= \frac{n}{3}[2(n+1)(2n+1) - 6(n+1) + 3] \\
&= \frac{n}{3}[2(2n^2+3n+1) - 6(n+1) + 3] \\
&= \frac{n}{3}(4n^2+6n+2-6n-6+3) \\
&= \frac{n}{3}(4n^2-1).
\end{aligned}$$

Therefore, $\sum_{k=1}^{n}(2k-1)^2 = \frac{4n^3-n}{3}$, as desired. ∎

Exercise 28. The cube of any positive integer can be written as the difference of two squares.

Proof. We prove for every $n \in \mathbb{Z}^+$, there exist integers k and m such that $n^3 = k^2 - m^2$.

Let $n \in \mathbb{Z}^+$.

Let $k = \frac{n(n+1)}{2}$.

Let $m = \frac{(n-1)n}{2}$.

Since n and $n+1$ are consecutive integers, then the product $n(n+1)$ is even, so k is an integer.

Since $n-1$ and n are consecutive integers, then the product $(n-1)n$ is even, so m is an integer.

Observe that

$$\begin{aligned}
n^3 &= n^3 + 0 \\
&= n^3 + [1^3 + 2^3 + ... + (n-1)^3] - [1^3 + 2^3 + ... + (n-1)^3] \\
&= [1^3 + 2^3 + ... + (n-1)^3] + n^3 - [1^3 + 2^3 + ... + (n-1)^3] \\
&= [1^3 + 2^3 + ... + (n-1)^3 + n^3] - [1^3 + 2^3 + ... + (n-1)^3] \\
&= \sum_{k=1}^{n} k^3 - \sum_{k=1}^{n-1} k^3 \\
&= (\frac{n(n+1)}{2})^2 - (\frac{(n-1)n}{2})^2 \\
&= k^2 - m^2.
\end{aligned}$$

□

Exercise 29. For all $n \in \mathbb{N}$, $\sum_{k=1}^{n} \frac{1}{(2k-1)(2k+1)} = \frac{n}{2n+1}$.

Proof. We prove by induction on n.

Let $S = \{n \in \mathbb{N} : \sum_{k=1}^{n} \frac{1}{(2k-1)(2k+1)} = \frac{n}{2n+1}\}$.

Basis:

Since $1 \in \mathbb{N}$ and $\sum_{k=1}^{1} \frac{1}{(2k-1)(2k+1)} = \frac{1}{(2\cdot 1-1)(2\cdot 1+1)} = \frac{1}{1\cdot 3} = \frac{1}{3} = \frac{1}{2\cdot 1+1}$, then $1 \in S$.

Induction:

Suppose $m \in S$.

Then $m \in \mathbb{N}$ and $\sum_{k=1}^{m} \frac{1}{(2k-1)(2k+1)} = \frac{m}{2m+1}$.

Thus,

$$\begin{aligned}
\sum_{k=1}^{m+1} \frac{1}{(2k-1)(2k+1)} &= \sum_{k=1}^{m} \frac{1}{(2k-1)(2k+1)} + \frac{1}{[2(m+1)-1][2(m+1)+1]} \\
&= \frac{m}{2m+1} + \frac{1}{(2m+1)(2m+3)} \\
&= \frac{1}{2m+1}(m + \frac{1}{2m+3}) \\
&= \frac{1}{2m+1} \cdot \frac{m(2m+3)+1}{2m+3} \\
&= \frac{2m^2+3m+1}{(2m+1)(2m+3)} \\
&= \frac{(2m+1)(m+1)}{(2m+1)(2m+3)} \\
&= \frac{m+1}{2m+3} \\
&= \frac{m+1}{2(m+1)+1}.
\end{aligned}$$

Since $m+1 \in \mathbb{N}$ and $\sum_{k=1}^{m+1} \frac{1}{(2k-1)(2k+1)} = \frac{m+1}{2(m+1)+1}$, then $m+1 \in S$.

Therefore, $m \in S$ implies $m+1 \in S$, so by PMI, $\sum_{k=1}^{n} \frac{1}{(2k-1)(2k+1)} = \frac{n}{2n+1}$ for all $n \in \mathbb{N}$, as desired. $\square$

Exercise 30. For all $n \in \mathbb{Z}^+$, $\sum_{k=0}^{n-1} 2^k = 2^n - 1$.

Proof. We prove by induction on n.

Let $S = \{n \in \mathbb{Z}^+ : \sum_{k=0}^{n-1} 2^k = 2^n - 1\}$.

Basis:

Since $1 \in \mathbb{Z}^+$ and $\sum_{k=0}^{1-1} 2^k = \sum_{k=0}^{0} 2^k = 1 = 2^1 - 1$, then $1 \in S$.

Induction:

Let $m \in S$.

Then $m \in \mathbb{Z}^+$ and $\sum_{k=0}^{m-1} 2^k = 2^m - 1$.

Since $m \in \mathbb{Z}^+$, then $m+1 \in \mathbb{Z}^+$.

Observe that

$$\begin{aligned}
\sum_{k=0}^{(m+1)-1} 2^k &= \sum_{k=0}^{m} 2^k \\
&= \sum_{k=0}^{m-1} 2^k + 2^m \\
&= (2^m - 1) + 2^m \\
&= 2 \cdot 2^m - 1 \\
&= 2^{m+1} - 1.
\end{aligned}$$

Since $m+1 \in \mathbb{Z}^+$ and $\sum_{k=0}^{(m+1)-1} 2^k = 2^{m+1} - 1$, then $m+1 \in S$.

Therefore, $m \in S$ implies $m+1 \in S$, so by PMI, $\sum_{k=0}^{n-1} 2^k = 2^n - 1$ for all $n \in \mathbb{Z}^+$. $\square$

Exercise 31. If $n \in \mathbb{N}$, then $2^0 + 2^1 + 2^2 + 2^3 + ... + 2^n = 2^{n+1} - 1$.

Proof. Suppose $n \in \mathbb{N}$.

Let S_n be the number

$$S_n = 2^0 + 2^1 + 2^2 + \cdots + 2^{n-1} + 2^n \tag{1}$$

We must show that $S_n = 2^{n+1} - 1$.

Multiply both sides of Equation 1 by 2 to get

$$2S_n = 2^1 + 2^2 + 2^3 + \cdots + 2^n + 2^{n+1} \tag{2}$$

Now subtract 1 from both sides of Equation 2 to get

$$2S_n - 1 = 2^1 + 2^2 + 2^3 + \cdots + 2^n + 2^{n+1} - 1 \tag{3}$$

From Equation 1 we know that $S_n - 1 = 2^1 + 2^2 + 2^3 + \cdots + 2^n$ so we can substitute this fact into Equation 3 to get

$$2S_n - 1 = (S_n - 1) + 2^{n+1} - 1 \tag{4}$$

Now add 1 to both sides of Equation 4 to get

$$2S_n = S_n + 2^{n+1} - 1 \tag{5}$$

Now subtract S_n from both sides of Equation 5 to get

$$S_n = 2^{n+1} - 1$$

□

Exercise 32. For all integers $n \geq 4$, $n! > n^2$.

Proof. We prove $n! > n^2$ for all $n \in \mathbb{Z}^+$ with $n \geq 4$ by induction on n.
Let $S = \{n \in \mathbb{Z}^+, n \geq 4 : n! > n^2\}$.
Basis:
Since $4 \in \mathbb{Z}^+$ and $4! = 24 > 16 = 4^2$, then $4 \in S$.
Induction:
Suppose $k \in S$.
Then $k \in \mathbb{Z}^+$ and $k \geq 4$ and $k! > k^2$.
Since $k \in \mathbb{Z}^+$, then $k + 1 \in \mathbb{Z}^+$.
Since $k + 1 > k$ and $k \geq 4$ and $4 > 0$, then $k + 1 > 4$ and $k + 1 > 0$.
Since $k \geq 4 > 1$, then $k > 1$.
Since $k \geq 4$, then $k - 1 \geq 3$.
Since $k - 1 \geq 3 > 1$, then $k - 1 > 1$.
Since $k > 1$ and $k - 1 > 1$, then $k(k - 1) > 1$, so $k^2 - k > 1$.
Hence, $k^2 > k + 1$.
Observe that

$$\begin{aligned} (k+1)! &= (k+1)k! \\ &> (k+1)k^2 \\ &> (k+1)(k+1) \\ &= (k+1)^2. \end{aligned}$$

Since $k + 1 \in \mathbb{Z}^+$ and $k + 1 > 4$ and $(k + 1)! > (k + 1)^2$, then $k + 1 \in S$.
Therefore, $k \in S$ implies $k + 1 \in S$, so by PMI, $n! > n^2$ for all $n \in \mathbb{Z}^+$ with $n \geq 4$, as desired. □

Exercise 33. For all integers $n \geq 6$, $n! > n^3$.

Proof. We prove $n! > n^3$ for all $n \in \mathbb{Z}^+$ with $n \geq 6$ by induction on n.
Let $S = \{n \in \mathbb{Z}^+, n \geq 6 : n! > n^3\}$.
Basis:
Since $6 \in \mathbb{Z}^+$ and $6! = 720 > 216 = 6^3$, then $6 \in S$.
Induction:
Suppose $k \in S$.
Then $k \in \mathbb{Z}^+$ and $k \geq 6$ and $k! > k^3$.
Since $k \in \mathbb{Z}^+$, then $k > 0$ and $k + 1 \in \mathbb{Z}^+$, so $k + 1 > 0$.

Since $k+1 > k \geq 6$, then $k+1 > 6$.
Since $k \geq 6$, then $k^3 \geq 6^3 = 216 > 3$, so $k^3 > 3$.
Hence, $\frac{k^3}{3} > 1$.
Since $k \geq 6$, then $k^2 \geq 6^2 = 36 > 6$, so $k^2 > 6$.
Since $k > 0$, then $k^3 > 6k$, so $\frac{k^3}{3} > 2k$.
Since $k \geq 6 > 3$, then $k > 3$.
Since $k > 0$, then $k^2 > 0$, so $k^3 > 3k^2$.
Hence, $\frac{k^3}{3} > k^2$.
Since $\frac{k^3}{3} > k^2$ and $\frac{k^3}{3} > 2k$ and $\frac{k^3}{3} > 1$, then $k^3 = \frac{k^3}{3}+\frac{k^3}{3}+\frac{k^3}{3} > k^2+2k+1 = (k+1)^2$, so $k^3 > (k+1)^2$.
Observe that

$$\begin{aligned}(k+1)! &= (k+1)k! \\ &> (k+1)k^3 \\ &> (k+1)\cdot(k+1)^2 \\ &= (k+1)^3.\end{aligned}$$

Since $k+1 \in \mathbb{Z}^+$ and $k+1 > 6$ and $(k+1)! > (k+1)^3$, then $k+1 \in S$.
Therefore, $k \in S$ implies $k+1 \in S$, so by PMI, $n! > n^3$ for all $n \in \mathbb{Z}^+$ with $n \geq 6$, as desired. □

Exercise 34. For each $n \in \mathbb{N}, 2^n \leq 2^{n+1} - 2^{n-1} - 1$.

Solution. We prove by induction(weak).
The statement S_n is $2^n \leq 2^{n+1} - 2^{n-1} - 1$.
The statement S_k is $2^k \leq 2^{k+1} - 2^{k-1} - 1$.
The statement S_{k+1} is $2^{k+1} \leq 2^{(k+1)+1} - 2^{(k+1)-1} - 1$.
The trick here is to add inequalities so if for example $a \leq b$ and $0 < 1$, then $a \leq b+1$. □

Proof. We prove by induction.
Basis:
If $n = 1$, then the statement is $2^1 \leq 2^{1+1} - 2^{1-1} - 1$, which simplifies to $2 \leq 4 - 1 - 1$, which is obviously true.
Induction:
Suppose $k \geq 1$.
We must prove $2^k \leq 2^{k+1} - 2^{k-1} - 1$ implies $2^{k+1} \leq 2^{(k+1)+1} - 2^{(k+1)-1} - 1$.
We use direct proof.
Suppose $2^k \leq 2^{k+1} - 2^{k-1} - 1$.
Then we have:

$$\begin{aligned}2^k &\leq 2^{k+1} - 2^{k-1} - 1 \\ 2^{k+1} &\leq 2^{k+2} - 2^k - 2 \text{ multiply both sides by 2} \\ 2^{k+1} &\leq 2^{k+2} - 2^k - 1 \text{ add 1 to the larger side} \\ 2^{k+1} &\leq 2^{(k+1)+1} - 2^{(k+1)-1} - 1\end{aligned}$$

It follows by induction that $2^n \leq 2^{n+1} - 2^{n-1} - 1$ for each $n \in \mathbb{N}$. $\square$

Exercise 35. Let (a_n) be the sequence defined by $a_1 = 1$ and $a_n = a_{n-1} + nn!$ for all positive integers $n > 1$.

Then $a_n = (n+1)! - 1$ for all $n \in \mathbb{Z}^+$.

Proof. We prove $(\forall n \in \mathbb{Z}^+)(a_n = (n+1)! - 1)$ by induction on n.

Let $S = \{n \in \mathbb{Z}^+ : a_n = (n+1)! - 1\}$.

Basis:

Since $1 \in \mathbb{Z}^+$ and $a_1 = 1 = 2 - 1 = (1+1)! - 1$, then $1 \in S$.

Induction:

Suppose $k \in S$.

Then $k \in \mathbb{Z}^+$ and $a_k = (k+1)! - 1$.

Since $k \in \mathbb{Z}^+$, then $k > 0$ and $k + 1 \in \mathbb{Z}^+$.

Since $k > 0$, then $k + 1 > 1$.

Observe that

$$\begin{aligned}
a_{k+1} &= a_k + (k+1)(k+1)! \\
&= [(k+1)! - 1] + (k+1)(k+1)! \\
&= (k+1)! - 1 + (k+1)(k+1)! \\
&= (k+2)(k+1)! - 1 \\
&= (k+2)! - 1 \\
&= [(k+1)+1]! - 1.
\end{aligned}$$

Since $k + 1 \in \mathbb{Z}^+$ and $a_{k+1} = [(k+1)+1]! - 1$, then $k + 1 \in S$.

Therefore, $k \in S$ implies $k + 1 \in S$, so by PMI, $a_n = (n+1)! - 1$ for all $n \in \mathbb{Z}^+$, as desired. $\square$

Exercise 36. Let (a_n) be the Lucas sequence defined by $a_1 = 1$ and $a_2 = 3$ and $a_n = a_{n-1} + a_{n-2}$ for $n \geq 3$.

Then $a_n < (\frac{7}{4})^n$ for all $n \in \mathbb{Z}^+$.

Proof. Let $p(n)$ be the predicate $a_n < (\frac{7}{4})^n$ defined over $\mathbb{Z}^+$.

We prove $p(n)$ is true for all positive integers n by strong induction on n.

Basis:

Since $a_1 = 1 < \frac{7}{4} = (\frac{7}{4})^1$, then $p(1)$ is true.

Since $a_2 = 3 < \frac{49}{16} = (\frac{7}{4})^2$, then $p(2)$ is true.

Induction:

For an integer $k \geq 3$, assume $p(n)$ is true for $n = 1, 2, ..., k-1$.

In particular, $p(k-2)$ and $p(k-1)$ are true, so $a_{k-2} < (\frac{7}{4})^{k-2}$ and $a_{k-1} < (\frac{7}{4})^{k-1}$.

Observe that

$$\begin{aligned}
a_k &= a_{k-1} + a_{k-2} \\
&< (\frac{7}{4})^{k-1} + (\frac{7}{4})^{k-2} \\
&= (\frac{7}{4})^{k-2}(\frac{7}{4} + 1) \\
&= (\frac{7}{4})^{k-2}(\frac{11}{4}) \\
&< (\frac{7}{4})^{k-2}(\frac{49}{16}) \\
&= (\frac{7}{4})^{k-2}(\frac{7}{4})^2 \\
&= (\frac{7}{4})^k.
\end{aligned}$$

Thus, $a_k < (\frac{7}{4})^k$, so $p(k)$ is true.

Therefore, by strong PMI, $p(n)$ is true for any positive integer n, so $a_n < (\frac{7}{4})^n$ for all $n \in \mathbb{Z}^+$. □

Exercise 37. For every natural number n, it follows that $2^n + 1 \leq 3^n$.

Solution. We prove by induction.

The statement S_n is $2^n + 1 \leq 3^n$.

The statement S_k is $2^k + 1 \leq 3^k$.

The statement S_{k+1} is $2^{k+1} + 1 \leq 3^{k+1}$. □

Proof. We prove by induction(weak).

Basis:

If $n = 1$ then the statement is $2^1 + 1 \leq 3^1$. This simplifies to $3 \leq 3$, which is true.

Induction: Suppose $k \geq 1$.

We must prove $2^k + 1 \leq 3^k$ implies $2^{k+1} + 1 \leq 3^{k+1}$.

We use direct proof.

Suppose $2^k + 1 \leq 3^k$ for any integer $k \geq 1$.

Observe the following inequalities:

$$\begin{aligned}
2^k + 1 &\leq 3^k \text{ induction hypothesis} \\
2^{k+1} + 2 &\leq 2 * 3^k \text{ multiply by 2} \\
2^{k+1} + 1 &\leq 2 * 3^k - 1 \text{ subtract 1} \\
2^{k+1} + 1 &\leq 2 * 3^k \text{ add the inequality } 0 \leq 1 \\
2^{k+1} + 1 &\leq 3 * 3^k \text{ transitive property of inequalities since } 2 * 3^k \leq 3 * 3^k \\
2^{k+1} + 1 &\leq 3^{k+1}
\end{aligned}$$

It follows by induction that $2^n + 1 \leq 3^n$ for every $n \in \mathbb{N}$. □

Exercise 38. For every natural number n, $\sum_{i=1}^{n} \frac{1}{i^2} \leq 2 - \frac{1}{n}$.

Proof. Let $p(n)$ be the predicate $\sum_{i=1}^{n} \frac{1}{i^2} \leq 2 - \frac{1}{n}$ defined over $\mathbb{N}$.
We prove $p(n)$ is true for all $n \in \mathbb{N}$ by induction on n.
Basis:
Since $\sum_{i=1}^{n} \frac{1}{i^2} = \frac{1}{1^2} = 1 = 2 - \frac{1}{1} \leq 2 - \frac{1}{1}$, then $p(1)$ is true.
Induction:
Let $k \in \mathbb{N}$ such that $p(k)$ is true.
Then $k \geq 1$ and $\sum_{i=1}^{k} \frac{1}{i^2} \leq 2 - \frac{1}{k}$.
To prove $p(k+1)$ is true, we must prove $\sum_{i=1}^{k+1} \frac{1}{i^2} \leq 2 - \frac{1}{k+1}$.
Since $k \geq 1 > 0$, then $k > 0$.
Since $0 < k < k+1$, then $0 < \frac{1}{k+1} < \frac{1}{k}$, so $\frac{1}{(k+1)^2} < \frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}$.
Thus, $\frac{1}{(k+1)^2} < \frac{1}{k} - \frac{1}{k+1}$, so $\frac{-1}{k} + \frac{1}{(k+1)^2} < \frac{-1}{k+1}$.
Observe that

$$\begin{aligned}
\sum_{i=1}^{k+1} \frac{1}{i^2} &= \sum_{i=1}^{k} \frac{1}{i^2} + \frac{1}{(k+1)^2} \\
&\leq (2 - \frac{1}{k}) + \frac{1}{(k+1)^2} \\
&< 2 - \frac{1}{k+1}.
\end{aligned}$$

Hence, $\sum_{i=1}^{k+1} \frac{1}{i^2} < 2 - \frac{1}{k+1}$, so $\sum_{i=1}^{k+1} \frac{1}{i^2} \leq 2 - \frac{1}{k+1}$, as desired. □

Exercise 39. If $n \in \mathbb{N}$, then $n^2 = 2\binom{n}{2} + \binom{n}{1}$.

Solution. By definition of binomial coefficient we know $\binom{n}{k} = \frac{n!}{k!(n-k)!}$.
In particular, for $n > 1$, $\binom{n}{1} = n$ and $\binom{n}{2} = \frac{n(n-1)}{2}$. □

Proof. Since $n \in \mathbb{N}$, then $n \geq 1$, so either $n > 1$ or $n = 1$.
We consider each case separately.
Case 1: Suppose $n = 1$.
Then $2\binom{1}{2} + \binom{1}{1} = 2 \cdot 0 + 1 = 1 = 1^2$.
Case 2: Suppose $n > 1$.
Then $2\binom{n}{2} + \binom{n}{1} = 2\frac{n(n-1)}{2} + n = n(n-1) + n = n^2$.
Both cases show $n^2 = 2\binom{n}{2} + \binom{n}{1}$. □

Exercise 40. Let $m, n \in \mathbb{Z}^+$.
Is $(mn)! = m!n!$?
Is $(m+n)! = m! + n!$?

Proof. Let $m = 4$ and $n = 5$.
Then $(mn)! = (4 * 5)! = 20! = 2432902008176640000 \neq 22880 = 24 * 120 = (4!)(5!) = m!n!$, so $(mn)! \neq m!n!$.

Let $m = 3$ and $n = 7$.

Then $(m+n)! = (3+7)! = 10! = 3628800 \neq 5046 = 6 + 5040 = 3! + 7! = m! + n!$, so $(m+n)! \neq m! + n!$. ☐

Exercise 41. Why is the argument below not valid?

Old MacDonald claims that all cows have the same color.

First, if you have just one cow, it certainly has the same color as itself.

Now, using PMI, assume that all cows in any collection of k cows have the same color.

Consider a collection of $k+1$ cows.

Removing one cow leaves a collection of k cows, which must therefore all have the same color.

Put back the removed cow.

Remove a different cow and this leaves another collection of k cows which all have the same color.

Certainly then, the original collection of $k+1$ cows must all have the same color.

By PMI, all cows in any finite collection of cows have the same color.

Solution. Let k represent the number of cows in the collection.

The error results from assuming $k > 1$ only.

However, to use PMI properly, $k \in \mathbb{N}$ can be any natural number, including 1.

Suppose that $k = 1$.

By PMI, the argument claims that a collection of $k + 1 = 2$ cows must all have the same color.

But, if you remove one cow from a collection of 2 cows, then you end up with two collections, each containing one cow.

Thus, there are two collections of 1 cow each.

So, the cows in each collection have the same color.

However, that does not imply that the color of the cow in one collection is the same as the color of the cow in the other collection.

Thus, we can not deduce that if a collection of $k = 1$ cows have the same color, then a collection of $k + 1 = 2$ cows also have the same color. ☐

Exercise 42. Let $f : \mathbb{N} \to \mathbb{N}$ be a function defined by $f(1) = 1$ and $f(2) = 2$ and $f(3) = 3$ and $f(n) = f(n-1) + f(n-2) + f(n-3)$ for all $n \geq 4$.

Then $f(n) < 2^n$ for all $n \in \mathbb{N}$.

Proof. To prove $f(n) < 2^n$ for all $n \in \mathbb{N}$, let $p(n)$ be the predicate $f(n) < 2^n$ defined over $\mathbb{N}$.

Since $f(1) = 1 < 2 = 2^1$, then $p(1)$ is true.

Since $f(2) = 2 < 4 = 2^2$, then $p(2)$ is true.

Since $f(3) = 3 < 8 = 2^3$, then $p(3)$ is true.

We prove $p(n)$ for all natural numbers $n \geq 4$ by strong induction on n.

Basis:

Since $f(4) = f(3) + f(2) + f(1) = 3 + 2 + 1 = 6 < 16 = 2^4$, then $p(4)$ is true.

Induction:
Let $k \in \mathbb{N}$ with $k \geq 4$.
Suppose $p(i)$ is true for each natural number i with $1 \leq i \leq k$.
To prove $p(k+1)$ is true, we must prove $f(k+1) < 2^{k+1}$.
Since $k+1 > k \geq 4$, then $k+1 > 4$, so $f(k+1) = f(k) + f(k-1) + f(k-2)$.
Since $k = k$, then $p(k)$ is true, so $f(k) < 2^k$.
Since $k \geq 4$, then $k - 1 \geq 3$.
Since $1 < 3 \leq k-1 < k$, then $1 < k-1 < k$, so $p(k-1)$ is true.
Hence, $f(k-1) < 2^{k-1}$.
Since $k \geq 4$, then $k - 2 \geq 2$.
Since $1 < 2 \leq k-2 < k$, then $1 < k-2 < k$, so $p(k-2)$ is true.
Hence, $f(k-2) < 2^{k-2}$.
Since $k \geq 4 > 0$, then $k > 0$, so $2^k > 0$.
Observe that

$$\begin{aligned} f(k+1) &= f(k) + f(k-1) + f(k-2) \\ &< 2^k + 2^{k-1} + 2^{k-2} \\ &= (1 + 2^{-1} + 2^{-2})2^k \\ &< \frac{7}{4} \cdot 2^k \\ &< 2 \cdot 2^k \\ &= 2^{k+1}. \end{aligned}$$

Thus, $f(k+1) < 2^{k+1}$, so $p(k+1)$ is true.
Hence, by strong induction, $p(n)$ is true for all natural numbers $n \geq 4$, as desired. □

Exercise 43. Let $f : \mathbb{N} \to \mathbb{N}$ be the function defined by $f(1) = 2$ and $f(2) = -8$ and $f(n) = 8f(n-1) - 15f(n-2) + 6 \cdot 2^n$ for $n \geq 3$.
Then $f(n) = -5 \cdot 3^n + 5^{n-1} + 2^{n+3}$ for all $n \in \mathbb{N}$.

Proof. Let $p(n)$ be the predicate $f(n) = -5 \cdot 3^n + 5^{n-1} + 2^{n+3}$ defined over $\mathbb{N}$.
We must prove $p(n)$ is true for all $n \in \mathbb{N}$.

Since $f(1) = 2 = -15 + 1 + 16 = -5 \cdot 3^1 + 5^{1-1} + 2^{1+3}$, then $p(1)$ is true.

We prove $p(n)$ is true for all $n \geq 2$ by strong induction on n.
Basis:
Since $f(2) = -8 = -45 + 5 + 32 = -5 \cdot 3^2 + 5^{2-1} + 2^{2+3}$, then $p(2)$ is true.
Induction:
Let $k \in \mathbb{N}$ with $k \geq 2$.
Suppose $p(i)$ is true for each natural number i with $1 \leq i \leq k$.
To prove $p(k+1)$ is true, we must prove $f(k+1) = -5 \cdot 3^{k+1} + 5^k + 2^{k+4}$.
Since $k \geq 2$, then $k+1 \geq 3$, so $f(k+1) = 8f(k) - 15f(k-1) + 6 \cdot 2^{k+1}$.
Since $k \geq 2 > 1$, then $k > 1$.

Since $1 < k = k$, then $p(k)$ is true, so $f(k) = -5 \cdot 3^k + 5^{k-1} + 2^{k+3}$.
Since $k \geq 2$, then $k - 1 \geq 1$.
Since $1 \leq k-1 < k$, then $p(k-1)$ is true, so $f(k-1) = -5 \cdot 3^{k-1} + 5^{k-2} + 2^{k+2}$.
Observe that

$$\begin{aligned}
f(k+1) &= 8f(k) - 15f(k-1) + 6 \cdot 2^{k+1} \\
&= 8(-5 \cdot 3^k + 5^{k-1} + 2^{k+3}) - 15(-5 \cdot 3^{k-1} + 5^{k-2} + 2^{k+2}) + 6 \cdot 2^{k+1} \\
&= -40 \cdot 3^k + 8 \cdot 5^{k-1} + 8 \cdot 2^{k+3} + 75 \cdot 3^{k-1} - 15 \cdot 5^{k-2} - 15 \cdot 2^{k+2} + 6 \cdot 2^{k+1} \\
&= -40 \cdot 3^k + 75 \cdot 3^{k-1} + 8 \cdot 5^{k-1} - 15 \cdot 5^{k-2} + 8 \cdot 2^{k+3} - 15 \cdot 2^{k+2} + 6 \cdot 2^{k+1} \\
&= 3^{k+1}(-40 \cdot 3^{-1} + 75 \cdot 3^{-2}) + 5^k(8 \cdot 5^{-1} - 15 \cdot 5^{-2}) + 2^{k+4}(8 \cdot 2^{-1} - 15 \cdot 2^{-2} + 6 \cdot 2^{-3}) \\
&= -5 \cdot 3^{k+1} + 5^k + 2^{k+4}.
\end{aligned}$$

Thus, $f(k+1) = -5 \cdot 3^{k+1} + 5^k + 2^{k+4}$, so $p(k+1)$ is true.
Therefore, by strong induction, $p(n)$ is true for all natural numbers $n \geq 4$. □

Exercise 44. Let $f : \mathbb{Z}^+ \cup \{0\} \to \mathbb{N}$ be the function defined by $f(0) = 7$ and $f(1) = 4$ and $f(n) = 6f(n-2) - f(n-1)$ for $n \geq 2$.
Then $f(n) = 5 \cdot 2^n + 2(-3)^n$ for all $n \geq 0$.

Proof. Let $p(n)$ be the predicate $f(n) = 5 \cdot 2^n + 2 \cdot (-3)^n$ defined over $\mathbb{Z}^+ \cup \{0\}$.
We must prove $p(n)$ is true for all $n \geq 0$.

Since $f(0) = 7 = 5 \cdot 1 + 2 \cdot 1 = 5 \cdot 2^0 + 2 \cdot (-3)^0$, then $p(0)$ is true.
Since $f(1) = 4 = 5 \cdot 2 + 2 \cdot (-3) = 5 \cdot 2^1 + 2(-3)^1$, then $p(1)$ is true.

We prove $p(n)$ is true for all $n \geq 2$ by strong induction on n.
Basis:
Since $f(2) = 6f(0) - f(1) = 6 \cdot 7 - 4 = 38 = 20 + 18 = 5 \cdot 2^2 + 2(-3)^2$, then $p(2)$ is true.
Induction:
Let $k \in \mathbb{N}$ with $k \geq 2$.
Suppose $p(i)$ is true for each natural number i with $1 \leq i \leq k$.
To prove $p(k+1)$ is true, we must prove $f(k+1) = 5 \cdot 2^{k+1} + 2(-3)^{k+1}$.
Since $k \geq 2$, then $k + 1 \geq 3 > 2$, so $k + 1 > 2$.
Hence, $f(k+1) = 6f(k-1) - f(k)$.
Since $k \geq 2 > 1$, then $k > 1$.
Since $1 < k = k$, then $p(k)$ is true, so $f(k) = 5 \cdot 2^k + 2(-3)^k$.
Since $k \geq 2$, then $k - 1 \geq 1$.
Since $1 \leq k - 1 < k$, then $p(k-1)$ is true, so $f(k-1) = 5 \cdot 2^{k-1} + 2(-3)^{k-1}$.

Observe that

$$\begin{aligned}
f(k+1) &= 6f(k-1) - f(k) \\
&= 6[5 \cdot 2^{k-1} + 2(-3)^{k-1}] - [5 \cdot 2^k + 2(-3)^k] \\
&= 30 \cdot 2^{k-1} + 12(-3)^{k-1} - 5 \cdot 2^k - 2(-3)^k \\
&= 30 \cdot 2^{k-1} - 5 \cdot 2^k + 12(-3)^{k-1} - 2(-3)^k \\
&= 2^{k+1}(30 \cdot 2^{-2} - 5 \cdot 2^{-1}) + (-3)^{k+1}[12(-3)^{-2} - 2(-3)^{-1}] \\
&= 2^{k+1}(\frac{30}{4} - \frac{5}{2}) + (-3)^{k+1}(\frac{12}{9} + \frac{2}{3}) \\
&= 2^{k+1}(\frac{15}{2} - \frac{5}{2}) + (-3)^{k+1}(\frac{4}{3} + \frac{2}{3}) \\
&= 5 \cdot 2^{k+1} + 2(-3)^{k+1}.
\end{aligned}$$

Thus, $f(k+1) = 5 \cdot 2^{k+1} + 2(-3)^{k+1}$, so $p(k+1)$ is true.
Therefore, by strong induction, $p(n)$ is true for all integers $n \geq 2$. $\square$

Exercise 45. There is no $n \in \mathbb{N}$ such that $0 < n < 1$.

Proof. Suppose there is $n \in \mathbb{N}$ such that $0 < n < 1$.
Let $S = \{n \in \mathbb{N} : 0 < n < 1\}$.
Then $n \in S$, so $S \neq \emptyset$.
Since $S \subset \mathbb{N}$ and $S \neq \emptyset$, then by WOP, S has a least element m.
Thus, $m \in S$ and $m \leq s$ for all $s \in S$.
Since $m \in S$, then $m \in \mathbb{N}$ and $0 < m < 1$.
Since $0 < m < 1$, then $0 < m$ and $m < 1$ and $0 < m^2 < 1$.
Since $m \in \mathbb{N}$, then $m^2 \in \mathbb{N}$.
Since $m^2 \in \mathbb{N}$ and $0 < m^2 < 1$, then $m^2 \in S$.
Since $m < 1$ and $m > 0$, then $m^2 = m \cdot m < m \cdot 1 = m$, so $m^2 < m$.
Thus, there is $m^2 \in S$ such that $m > m^2$.
This contradicts the fact that m is the least element of S.
Therefore, there is no $n \in \mathbb{N}$ such that $0 < n < 1$. $\square$

Exercise 46. Let $k \in \mathbb{Z}$.
Then $\binom{n}{k} < \binom{n}{k+1}$ iff $0 \leq k < \frac{n-1}{2}$ for all $n \in \mathbb{Z}^+$.

Proof. Let $n \in \mathbb{Z}^+$.
We must prove $\binom{n}{k} < \binom{n}{k+1}$ iff $0 \leq k < \frac{n-1}{2}$.
We first prove if $\binom{n}{k} < \binom{n}{k+1}$, then $0 \leq k < \frac{n-1}{2}$.
Suppose $\binom{n}{k} < \binom{n}{k+1}$.
Then $\frac{n!}{(n-k)!k!} < \frac{n!}{(n-k-1)!(k+1)!}$.
Since $n \in \mathbb{Z}^+$, then $n > 0$, so $\frac{1}{(n-k)!k!} < \frac{1}{(n-k-1)!(k+1)!}$.
By definition of factorial, $(k+1)! > 0$ and $(n-k)! > 0$, so $\frac{(k+1)!}{k!} < \frac{(n-k)!}{(n-k-1)!}$.
Since $k!$ exists, then $k \geq 0$, by definition of factorial.
Since $k+1 > k \geq 0$, then $k+1 > 0$.
Thus, $\frac{(k+1)k!}{k!} < \frac{(n-k)!}{(n-k-1)!}$, so $k+1 < \frac{(n-k)!}{(n-k-1)!}$.

Since $(n-k-1)!$ exists, then $n-k-1 \geq 0$, by definition of factorial.
Hence, $n-k \geq 1 > 0$, so $n-k > 0$.
Thus, $k+1 < \frac{(n-k)(n-k-1)!}{(n-k-1)!}$, so $k+1 < n-k$.
Therefore, $2k < n-1$, so $k < \frac{n-1}{2}$.
Since $0 \leq k$ and $k < \frac{n-1}{2}$, then $0 \leq k < \frac{n-1}{2}$. □

Proof. Conversely, we prove if $0 \leq k < \frac{n-1}{2}$, then $\binom{n}{k} < \binom{n}{k+1}$.
Suppose $0 \leq k < \frac{n-1}{2}$.
Then $0 \leq k$ and $k < \frac{n-1}{2}$.
Since $k < \frac{n-1}{2}$, then $2k < n-1$, so $k+k < n-1$.
Thus, $k+1 < n-k$ and $k < n-k-1$.
Since $n-k-1 > k$ and $k \geq 0$, then $n-k-1 > k \geq 0$, so $n-k-1 > 0$.
Hence, $k+1 < \frac{(n-k)(n-k-1)!}{(n-k-1)!}$, so $k+1 < \frac{(n-k)!}{(n-k-1)!}$.
Since $k \geq 0$, then $\frac{(k+1)k!}{k!} < \frac{(n-k)!}{(n-k-1)!}$, so $\frac{(k+1)!}{k!} < \frac{(n-k)!}{(n-k-1)!}$.
Thus, $\frac{1}{(n-k)!k!} < \frac{1}{(n-k-1)!(k+1)!}$.
Since $n \in \mathbb{Z}^+$, then $n > 0$, so $\frac{n!}{(n-k)!k!} < \frac{n!}{(n-k-1)!(k+1)!}$.
Therefore, $\binom{n}{k} < \binom{n}{k+1}$. □

Exercise 47. Let $n, k \in \mathbb{Z}$ and $0 \leq k \leq n$.
Then $\binom{n}{k} = \binom{n}{k+1}$ iff n is odd and $k = \frac{n-1}{2}$.

Proof. We prove if $\binom{n}{k} = \binom{n}{k+1}$, then n is odd and $k = \frac{n-1}{2}$.
Suppose $\binom{n}{k} = \binom{n}{k+1}$.
Since $\binom{n}{k+1}$ exists, then $0 \leq k+1 \leq n$.
Since $0 \leq k < k+1$, then $0 < k+1$, so $k+1 > 0$.
Since $k < k+1 \leq n$, then $k < n$, so $n > k$.
Hence, $n-k > 0$.
Observe that

$$\begin{aligned}
\binom{n}{k} &= \binom{n}{k+1} \\
\frac{n!}{(n-k)!k!} &= \frac{n!}{(n-k-1)1(k+1)!} \\
\frac{1}{(n-k)!k!} &= \frac{1}{(n-k-1)!(k+1)!} \\
\frac{(k+1)!}{k!} &= \frac{(n-k)!}{(n-k-1)!} \\
\frac{(k+1)k!}{k!} &= \frac{(n-k)(n-k-1)!}{(n-k-1)!} \\
k+1 &= n-k \\
2k &= n-1 \\
k &= \frac{n-1}{2}.
\end{aligned}$$

Since k is an integer and $k = \frac{n-1}{2}$, then $2k = n - 1$, so $2k + 1 = n$.
Hence, n is odd. $\square$

Exercise 48. Let $k, n \in \mathbb{Z}$.
If $2 \leq k \leq n-2$, then $\binom{n}{k} = \binom{n-2}{k-2} + 2\binom{n-2}{k-1} + \binom{n-2}{k}$ for all $n \geq 4$.

Proof. Suppose $2 \leq k \leq n-2$.
We prove the statement $\binom{n}{k} = \binom{n-2}{k-2} + 2\binom{n-2}{k-1} + \binom{n-2}{k}$ for all $n \geq 4$ by induction on n.
Let $S = \{n \in \mathbb{Z}^+, n \geq 4 : \binom{n}{k} = \binom{n-2}{k-2} + 2\binom{n-2}{k-1} + \binom{n-2}{k}\}$.
Basis:
Let $n = 4$.
Then $2 \leq k \leq 4 - 2 = 2$, so $2 \leq k \leq 2$.
Hence, $k = 2$.
Since $\binom{4-2}{2-2} + 2\binom{4-2}{2-1} + \binom{4-2}{2} = 6 = \binom{4}{2}$, then $4 \in S$.
Induction:
Suppose $m \in S$.
Then $m \in \mathbb{Z}^+$ and $m \geq 4$ and $\binom{m}{k} = \binom{m-2}{k-2} + 2\binom{m-2}{k-1} + \binom{m-2}{k}$.
Observe that

$$\begin{aligned}
\binom{m+1}{k} &= \binom{m}{k-1} + \binom{m}{k} \\
&= \binom{m}{k-1} + [\binom{m-2}{k-2} + 2\binom{m-2}{k-1} + \binom{m-2}{k}] \\
&= \binom{m}{k-1} + \binom{m-2}{k-2} + \binom{m-2}{k-1} + \binom{m-2}{k-1} + \binom{m-2}{k} \\
&= \binom{m}{k-1} + [\binom{m-2}{k-2} + \binom{m-2}{k-1}] + [\binom{m-2}{k-1} + \binom{m-2}{k}] \\
&= \binom{m}{k-1} + \binom{m-1}{k-1} + \binom{m-1}{k} \\
&= [\binom{m-1}{k-2} + \binom{m-1}{k-1}] + \binom{m-1}{k-1} + \binom{m-1}{k} \\
&= \binom{m-1}{k-2} + 2\binom{m-1}{k-1} + \binom{m-1}{k}.
\end{aligned}$$

Since $m \in \mathbb{Z}^+$, then $m + 1 \in \mathbb{Z}^+$.
Since $m + 1 > m \geq 4$, then $m + 1 > 4$.
Since $m + 1 \in \mathbb{Z}^+$ and $m + 1 > 4$ and $\binom{m+1}{k} = \binom{m-1}{k-2} + 2\binom{m-1}{k-1} + \binom{m-1}{k}$, then $m + 1 \in S$.
Hence, $m \in S$ implies $m + 1 \in S$.
Therefore, by PMI, $\binom{n}{k} = \binom{n-2}{k-2} + 2\binom{n-2}{k-1} + \binom{n-2}{k}$ for all $n \geq 4$. $\square$

Exercise 49. For every $n \in \mathbb{Z}^+$, $\sum_{k=0}^{n}(-1)^k\binom{n}{k} = 0$.

Proof. Let $n \in \mathbb{Z}^+$

Then either n is even or n is odd.

We consider each case separately.

Case 1: Suppose n is even.

Then $(-1)^n = 1$ and $n-1$ is odd, so $(-1)^{n-1} = -1$.

Observe that

$$\begin{aligned}
\sum_{k=0}^{n}(-1)^k\binom{n}{k} &= (-1)^0\binom{n}{0} + (-1)^1\binom{n}{1} + (-1)^2\binom{n}{2} + ... + (-1)^{n-1}\binom{n}{n-1} + (-1)^n\binom{n}{n} \\
&= 1 - \binom{n}{1} + \binom{n}{2} + ... - \binom{n}{n-1} + \binom{n}{n} \\
&= 1 - [\binom{n-1}{0} + \binom{n-1}{1}] + [\binom{n-1}{1} + \binom{n-1}{2}] + ... - [\binom{n-1}{n-2} + \binom{n-1}{n-1}] + [\binom{n-1}{n-1} \\
&= 1 - \binom{n-1}{0} + \binom{n-1}{n} \\
&= 1 - 1 + 0 \\
&= 0.
\end{aligned}$$

Case 2: Suppose n is odd.

Then $(-1)^n = -1$ and $n-1$ is even, so $(-1)^{n-1} = 1$.

Observe that

$$\begin{aligned}
\sum_{k=0}^{n}(-1)^k\binom{n}{k} &= (-1)^0\binom{n}{0} + (-1)^1\binom{n}{1} + (-1)^2\binom{n}{2} + ... + (-1)^{n-1}\binom{n}{n-1} + (-1)^n\binom{n}{n} \\
&= 1 - \binom{n}{1} + \binom{n}{2} + ... + \binom{n}{n-1} - \binom{n}{n} \\
&= 1 - [\binom{n-1}{0} + \binom{n-1}{1}] + [\binom{n-1}{1} + \binom{n-1}{2}] + ... + [\binom{n-1}{n-2} + \binom{n-1}{n-1}] - [\binom{n-1}{n-1} \\
&= 1 - \binom{n-1}{0} - \binom{n-1}{n} \\
&= 1 - 1 - 0 \\
&= 0.
\end{aligned}$$

□

Exercise 50. Show that $\sum_{k=1}^{n} k\binom{n}{k} = n2^{n-1}$ for all $n \in \mathbb{Z}^+$.

Proof. We prove the statement $\sum_{k=1}^{n} k\binom{n}{k} = n2^{n-1}$ for all $n \in \mathbb{Z}^+$ by induction on n.

Let $S = \{n \in \mathbb{Z}^+ : \sum_{k=1}^{n} k\binom{n}{k} = n2^{n-1}\}$.

Basis:

Since $1 \in \mathbb{Z}^+$ and $\sum_{k=1}^{1} k\binom{1}{k} = 1\binom{1}{1} = 1 = (1)2^{1-1}$, then $1 \in S$.

Induction:
Suppose $m \in S$.
Then $m \in \mathbb{Z}^+$ and $\sum_{k=1}^{m} k\binom{m}{k} = m2^{m-1}$.
We must prove $\sum_{k=1}^{m+1} k\binom{m+1}{k} = (m+1)2^m$.
TODO
We need to finish this proof!
Hence, $m \in S$ implies $m+1 \in S$.
Therefore, by PMI, $\sum_{k=1}^{n} k\binom{n}{k} = n2^{n-1}$ for all $n \in \mathbb{Z}^+$. □

Exercise 51. Show that $\sum_{k=0}^{n} 2^k\binom{n}{k} = 3^n$ for all $n \in \mathbb{Z}^+$.

Proof. Let $n \in \mathbb{Z}^+$.
Observe that

$$\begin{aligned}
\sum_{k=0}^{n} 2^k\binom{n}{k} &= 2^0\binom{n}{0} + 2^1\binom{n}{1} + ... + 2^{n-2}\binom{n}{n-2} + 2^{n-1}\binom{n}{n-1} + 2^n\binom{n}{n} \\
&= \binom{n}{0}2^0 + \binom{n}{1}2^1 + ... + \binom{n}{n-2}2^{n-2} + \binom{n}{n-1}2^{n-1} + \binom{n}{n}2^n \\
&= \binom{n}{n}2^0 + \binom{n}{n-1}2^1 + ... + \binom{n}{2}2^{n-2} + \binom{n}{1}2^{n-1} + \binom{n}{0}2^n \\
&= \binom{n}{0}2^n + \binom{n}{1}2^{n-1} + \binom{n}{2}2^{n-2} + ... + \binom{n}{n-1}2^1 + \binom{n}{n}2^0 \\
&= \binom{n}{0}2^{n-0} + \binom{n}{1}2^{n-1} + \binom{n}{2}2^{n-2} + ... + \binom{n}{n-1}2^{n-(n-1)} + \binom{n}{n}2^{n-n} \\
&= \sum_{k=0}^{n}\binom{n}{k}2^{n-k} \\
&= (2+1)^n \\
&= 3^n.
\end{aligned}$$

□

Proposition 52. *The sum of two even integers is even.*

Proof. Let a and b be any integers.
Suppose a is even and b is even.
Then $a = 2k$ for some $k \in \mathbb{Z}$ and $b = 2m$ for some $m \in \mathbb{Z}$.
Thus $a + b = 2k + 2m = 2(k + m)$.
Let $n = k + m$.
Then $a + b = 2n$.
Since $k \in \mathbb{Z}$ and $m \in \mathbb{Z}$, then $k + m \in \mathbb{Z}$, so $n \in \mathbb{Z}$.
Therefore, $a + b$ is even. □

Proposition 53. *The sum of two integers with opposite parity is odd.*

Proof. Let a and b be any integers.
Suppose a and b have opposite parity.
Then either a is even and b is odd, or a is odd and b is even.
We consider these cases separately.
Case 1: Suppose a is even and b is odd.
Then $a = 2k$ for some $k \in \mathbb{Z}$ and $b = 2m + 1$ for some $m \in \mathbb{Z}$.
Hence, $a + b = 2k + (2m + 1) = (2k + 2m) + 1 = 2(k + m) + 1$.
Let $n = k + m$.
Then $a + b = 2n + 1$.
Since $k \in \mathbb{Z}$ and $m \in \mathbb{Z}$, then $k + m \in \mathbb{Z}$ because $\mathbb{Z}$ is closed under addition.
Thus, $n \in \mathbb{Z}$.
Therefore, $a + b$ is odd.
Case 2: Suppose a is odd and b is even.
Then $a = 2k + 1$ for some $k \in \mathbb{Z}$ and $b = 2m$ for some $m \in \mathbb{Z}$.
Hence, $a+b = (2k+1)+2m = 2k+(1+2m) = 2k+(2m+1) = (2k+2m)+1 = 2(k + m) + 1$.
Let $n = k + m$.
Then $a + b = 2n + 1$.
Since $k \in \mathbb{Z}$ and $m \in \mathbb{Z}$, then $k + m \in \mathbb{Z}$ because $\mathbb{Z}$ is closed under addition.
Thus, $n \in \mathbb{Z}$.
Therefore, $a + b$ is odd. □

Proposition 54. *Let $n \in \mathbb{Z}^+$.*
Then n^2 is even iff n is even.

Proof. We prove if n is even, then n^2 is even.
Suppose n is even.
Then $n = 2a$ for some integer a.
Thus, $n^2 = (2a)^2 = 4a^2 = 2(2a^2)$.
Since $2a^2$ is an integer, then this implies n^2 is even, as desired.

Conversely, we prove if n^2 is even, then n is even.
We use proof by contrapositive.
Suppose n is not even.
Then n is odd, so there exists an integer b such that $n = 2b + 1$.
Thus, $n^2 = (2b + 1)^2 = 4b^2 + 4b + 1 = 2(2b^2 + 2b) + 1$.
Since $2b^2 + 2b$ is an integer, then this implies n^2 is odd.
Therefore, n^2 is not even, as desired. □

Let $n \in \mathbb{Z}^+$.
Then n^2 is even iff n is even.
Thus, n^2 is not even iff n is not even.
Therefore, n^2 is odd iff n is odd.

Proposition 55. *Let $m, n \in \mathbb{Z}$. Then $m + n$ is even if and only if m and n have the same parity.*

Proof. We first show that if $m + n$ is even then m and n have the same parity.

We use proof by contrapositive.

Suppose m and n do not have the same parity.

Then m and n have opposite parity, so we consider two cases.

Case 1: Suppose m is even and n is odd.

Then $m = 2a$ and $n = 2b + 1$ for some $a, b \in \mathbb{Z}$.

Thus $m + n = 2a + (2b + 1) = 2(a + b) + 1$.

Therefore $m + n$ is odd, implying that $m + n$ is not even.

Case 2: Suppose m is odd and n is even.

Then $m = 2a + 1$ and $n = 2b$ for some $a, b \in \mathbb{Z}$.

Thus $m + n = (2a + 1) + 2b = 2(a + b) + 1$.

Therefore $m + n$ is odd, so $m + n$ is not even.

Either way both cases show that $m + n$ is not even.

Conversely, we show that if m and n have the same parity then $m + n$ is even.

Suppose m and n have the same parity. Then we have two cases to consider.

Case 1: Suppose m and n are both even.

Then $m = 2a$ and $n = 2b$ for some $a, b \in \mathbb{Z}$.

Thus $m + n = 2a + 2b = 2(a + b)$.

Therefore $m + n$ is even.

Case 2: Suppose m and n are both odd.

Then $m = 2a + 1$ and $n = 2b + 1$ for some $a, b \in \mathbb{Z}$.

Thus $m + n = (2a + 1) + (2b + 1) = 2(a + b + 1)$.

Therefore $m + n$ is even.

Either way both cases show that $m + n$ is even. □

Proposition 56. *Let $a, b \in \mathbb{Z}$. If $a + b$ is odd, then $a^2 + b^2$ is odd.*

Proof. Suppose $a + b$ is odd.

Then a and b have opposite parity, for if they had the same parity then their sum would be even. So we consider two cases.

Case 1: Suppose a is even and b is odd.

Since a is even then a^2 is even.

Since b is odd then b^2 is odd.

Since a^2 and b^2 have opposite parity then their sum $a^2 + b^2$ is odd since we proved theorem **??**.

Case 2: Suppose a is odd and b is even.

Since a is odd then a^2 is odd.

Since b is even then b^2 is even.

Since a^2 and b^2 have opposite parity then their sum $a^2 + b^2$ is odd.

Either way both cases show that $a^2 + b^2$ is odd. □

Proposition 57. *Let $m, n \in \mathbb{Z}$.*

If m and n are odd, then mn is odd.

Proof. Suppose m and n are odd integers. Then $m = 2a+1$ and $n = 2b+1$ for $a, b \in \mathbb{Z}$.

Consequently the product $mn = (2a+1)(2b+1) = 4ab + 2a + 2b + 1 = 2(2ab + a + b) + 1$.

Therefore $mn = 2c+1$, where $c = 2ab+a+b \in \mathbb{Z}$, so mn is odd by definition of an odd integer. □

Proposition 58. *Let $m, n \in \mathbb{Z}$. If m is even, then mn is even.*

Proof. Suppose m and n are integers and m is even. Then $m = 2a$ for some $a \in \mathbb{Z}$.

Thus $mn = (2a)n = 2(an) = 2b$ where $b = an \in \mathbb{Z}$.
Therefore mn is even, by definition of an even number. □

Exercise 59. Let n be a positive integer.
Then n is even iff n is not odd.

Proof. We prove by contradiction : if n is even, then n is not odd.
Suppose n is even and n is odd.
Then n is both even and odd.
But, this contradicts the fact that n is not both even and odd.
Therefore, if n is even, then n is not odd.

Conversely, we prove: if n is not odd, then n is even.
Suppose n is not odd.
Since n is a positive integer, then either n is even or n is odd.
Since n is not odd, then this implies n is even.
Therefore, if n is not odd, then n is even. □

Exercise 60. No natural number is both even and odd.

Proof. Suppose there is a natural number that is both even and odd.
Then there is $n \in \mathbb{N}$ such that n is even and n is odd.
Since n is even, then $n = 2k$ for some integer k.
Since n is odd, then $n = 2m - 1$ for some integer m.
Thus, $2k = n = 2m - 1$, so $2k = 2m - 1$.
Hence, $1 = 2m - 2k = 2(m - k)$, so $\frac{1}{2} = m - k$.
Since m and k are integers, then $m - k$ is an integer.
Consequently, $\frac{1}{2}$ is an integer, a contradiction.
Therefore, there is no natural number that is both even and odd. □

Exercise 61. For every pair of odd integers m and n, the sum $m + n$ is even and the product mn is odd.

Proof. Let m and n be odd integers.

Since m is odd, then there exists an integer a such that $m = 2a + 1$.
Since n is odd, then there exists an integer b such that $n = 2b + 1$.
Observe that $m + n = (2a + 1) + (2b + 1) = 2a + 2b + 2 = 2(a + b + 1)$.
Since $a + b + 1$ is an integer, then this implies $m + n$ is even.
Observe that $mn = (2a+1)(2b+1) = 4ab+2a+2b+1 = 2(2ab+a+b)+1$.
Since $2ab + a + b$ is an integer, then this implies mn is odd. □

Exercise 62. The product of two even integers is a multiple of 4.

Solution. We must first translate the English statement into logical symbols.
The English statement means:
If a and b are even integers, then ab is a multiple of 4.
Thus,
a is even and b is even $\Rightarrow (4|ab)$.
Hence the statement to prove is:
$(\forall a, b \in \mathbb{Z})[a \text{ is even} \land b \text{ is even} \to (4|ab)]$.
We let $a, b \in \mathbb{Z}$ be arbitrary such that a is even and b is even.
To prove $4|ab$, we must find some $n \in \mathbb{Z}$ such that $ab = 4n$.
Since a is even, then $a = 2n_1$ for some $n_1 \in \mathbb{Z}$.
Since b is even, then $b = 2n_2$ for some $n_2 \in \mathbb{Z}$.
We want $ab = 4n$, so we need to find n in terms of n_1 and n_2.
Thus, $(2n_1)(2n_2) = 4n$.
We solve for n to get $n = n_1 n_2$.
Hence, we let $n = n_1 n_2$. □

Proof. Suppose a and b are arbitrary even integers.

To prove ab is a multiple of 4, we must prove $4|ab$; that is, we must find an integer n such that $ab = 4n$.

Since a and b are even, then $a = 2n_1$ and $b = 2n_2$ for some integers n_1 and n_2.

Let $n = n_1 n_2$.
Since $\mathbb{Z}$ is closed under multiplication, then clearly n is an integer.
Observe that

$$\begin{aligned} ab &= (2n_1)(2n_2) \\ &= 4n_1 n_2 \\ &= 4(n_1 n_2) \\ &= 4n, \text{ as desired.} \end{aligned}$$

□

Exercise 63. Prove or disprove the conjecture: Every odd integer is the sum of three odd integers.

Solution. We must either show for any odd integer n there exist three odd integers that when added yield n, or show that no such integers exist.

For any n we can add n and 1 and -1, or n and 3 and -3, or n and 5 and -5, etc. We only need choose one example. □

Proof. The conjecture is true.

Suppose n is an odd integer.

Then $n = n + 1 + (-1)$.

Since 1 and -1 are odd integers, then n is the sum of three odd integers. □

Exercise 64. Every even integer that is the square of an integer is a multiple of 4.

Solution. We must first translate the English statement into logical symbols.

The statement means:

if a is an even integer that is the square of some integer b, then a is a multiple of 4.

Thus,

a is even and $a = b^2 \Rightarrow 4|a$.

Hence, the statement to prove is:

$(\forall a, b \in \mathbb{Z})[a \text{ is even} \wedge (a = b^2) \rightarrow (4|a)]$.

Thus, we let $a, b \in \mathbb{Z}$ be arbitrary such that a is even and $a = b^2$.

To prove $4|a$, we must find some $n \in \mathbb{Z}$ such that $a = 4n$.

Since a is even, then $a = 2k_1$ for some $k_1 \in \mathbb{Z}$.

Work backwards.

Suppose $a = 4n$ for some $n \in \mathbb{Z}$.

Then $2k_1 = 4n$, so $n = k_1/2$.

Since n is an integer, then this implies k_1 must be even (otherwise, k_1 is odd which forces n to be a non integer).

Thus, we must find some $k_2 \in \mathbb{Z}$ such that $k_1 = 2k_2$.

Since $a = b^2$ and $a = 2k_1$, then $2k_1 = b^2$, so $2(2k_2) = b^2$.

Hence, $4k_2 = b^2$, so $(2k_3)^2 = b^2$ where $k_2 = k_3^2$.

Thus, $b = 2k_3$, so b is even.

Since $a = b^2$ and a is even, then b^2 must be even.

Somehow we must show that b^2 is even implies b is even.

Is there an associated theorem?

That is, does b^2 is even imply b is even?

Yes, we know that for every $x \in \mathbb{Z}$, x^2 is even if and only if x is even.

Thus, we use this theorem to devise our proof.

Since $a = b^2$ and a is even, then b^2 is even.

We know that for every $x \in \mathbb{Z}$, x^2 is even if and only if x is even.

Hence, in particular, b^2 is even if and only if b is even.

Since b^2 is even, then by modus ponens, we conclude b is even.

Thus, $b = 2n_1$ for some $n_1 \in \mathbb{Z}$.

Hence, $a = (2n_1)^2 = 4n_1^2$.

Thus, let $n = n_1^2$. □

Proof. Suppose a and b are arbitrary integers such that a is even and $a = b^2$.

To prove a is a multiple of 4, we must prove $4|a$; that is, we must find an integer n such that $a = 4n$.

Since $a = b^2$ and a is even, then b^2 is even.

For every $n \in \mathbb{Z}$, n^2 is even if and only if n is even.

Hence, in particular, b^2 is even if and only if b is even.

Since b^2 is even, then we conclude b is even.

Thus, $b = 2m$ for some integer m.

Let $n = m^2$.

Clearly, n is an integer, since $\mathbb{Z}$ is closed under multiplication.

Observe that

$$\begin{aligned} a &= b^2 \\ &= (2m)^2 \\ &= 4m^2 \\ &= 4n, \text{ as desired.} \end{aligned}$$

□

Proposition 65. *Let $n \in \mathbb{Z}$.*

If n is odd, then n^3 is odd.

Proof. Suppose n is odd.

Then there exists an integer k such that $n = 2k + 1$.

Hence, $n^3 = (2k + 1)^3 = 8k^3 + 12k^2 + 6k + 1 = 2(4k^3 + 6k^2 + 3k) + 1$.

Since $4k^3 + 6k^2 + 3k$ is an integer, then this implies n^3 is odd, as desired. □

Exercise 66. Prove or disprove: Some even numbers are odd.

Solution. Clearly, the statement is false.

Thus, we need to disprove the statement: Some even numbers are odd.

Let P : some even numbers are odd.

Define predicate $p(n) : n$ is even and $q(n) : n$ is odd.

Our domain of discourse is $\mathbb{Z}$.

P has the form some p's are q's, so $P : (\exists n)(p(n) \wedge q(n))$.

To disprove P, we must prove $\neg P$ where $\neg P : \neg(\exists n)(p(n) \wedge q(n))$.

We use proof by contradiction.

Hence, we assume $(\exists n)(p(n) \wedge q(n))$.

That is, we assume there exists an integer n such that n is even and odd.

We must derive a contradiction. □

Proof. Suppose there exists an integer that is even and odd.

Then there exists an integer n such that n is even and n is odd.

Thus, $n = 2a$ and $n = 2b + 1$ for some integers a and b.

Hence, $2a = 2b + 1$, so $2a - 2b = 1$.

Thus, $2(a - b) = 1$.

Let $c = a - b$.

Then c is an integer and $2c = 1$.
Hence, 1 is even.
But, 1 is not even.
Therefore, there is no integer that is even and odd.
Therefore, it is false that some even numbers are odd.
This implies no even numbers are odd. In other words, all even numbers are not odd. □

Exercise 67. Suppose $n \in \mathbb{Z}$. If $7n + 9$ is even, then n is odd.

Proof. We use proof by contrapositive.
Suppose n is not odd.
Then n is even, so $n = 2a$ for some $a \in \mathbb{Z}$.
Thus $7n + 9 = 7(2a) + 9 = 14a + 8 + 1 = 2(7a + 4) + 1$.
Therefore $7n + 9 = 2b + 1$, where $b = 7a + 4 \in \mathbb{Z}$, so $7n + 9$ is odd.
Therefore $7n + 9$ is not even. □

Exercise 68. If n is an odd integer, then $n^2 + 3n + 5$ is odd.

Proof. Suppose n is odd. Then $n = 2a + 1$ for some $a \in \mathbb{Z}$.

Consequently $n^2+3n+5 = (2a+1)^2+3(2a+1)+5 = 4a^2+4a+1+6a+8 = 4a^2 + 10a + 9 = 2(2a^2 + 5a + 4) + 1 = 2b + 1$ where $b = 2a^2 + 5a + 4 \in \mathbb{Z}$.

Therefore $n^2 + 3n + 5$ is odd, by definition of an odd number. □

Exercise 69. Let m and n be integers.
If mn is odd, then m is odd and n is odd.

Proof. We prove by contrapositive.
Suppose either m is not odd or n is not odd.
We consider these cases separately.
Case 1: Suppose m is not odd.
Then m is even, so $m = 2k$ for some integer k.
Thus, $mn = (2k)n = 2(kn)$.
Since kn is an integer, then this implies mn is even, so mn is not odd.
Case 2: Suppose n is not odd.
Then n is even, so $n = 2k$ for some integer k.
Thus, $mn = m(2k) = 2(km)$.
Since km is an integer, then this implies mn is even, so mn is not odd.
Therefore, in all cases, mn is not odd, as desired. □

Exercise 70. Let $n \in \mathbb{Z}$.
If $3n$ is even, then n is even.

Proof. Suppose $3n$ is even.
Then $3n = 2k$ for some integer k.
Since $n = 3n - 2n = 2k - 2n = 2(k - n)$ and $k - n$ is an integer, then n is even. □

Exercise 71. If n is an even integer, then $n^2 - 6n + 5$ is odd.

Solution. We use direct proof and the definition of odd number. This is an easy straightforward proof. □

Proof. Suppose n is an even integer.
Then $n = 2a$ for some $a \in \mathbb{Z}$, by definition of an even integer.
So $n^2 - 6n + 5 = (2a)^2 - 6(2a) + 5 = 2(2a^2 - 6a + 2) + 1$.
Therefore, $n^2 - 6n + 5 = 2b + 1$, where $b = 2a^2 - 6a + 2 \in \mathbb{Z}$.
Consequently, $n^2 - 6n + 5$ is odd, by definition of an odd integer. □

Exercise 72. Let n be an integer.
If $n^2 - 6n + 5$ is even, then n is odd.

Proof. We prove by contrapositive.
Suppose n is not odd.
Then n is even, so $n = 2a$ for some integer a.
To prove $n^2 - 6n + 5$ is not even, we prove $n^2 - 6n + 5$ is odd.
Thus, we must find an integer b such that $n^2 - 6n + 5 = 2b + 1$.
Let $b = 2a^2 - 6a + 2$.
Then clearly b is an integer.
Observe that

$$\begin{aligned} n^2 - 6n + 5 &= (2a)^2 - 6(2a) + 5 \\ &= 4a^2 - 12a + 5 \\ &= 4a^2 - 12a + 4 + 1 \\ &= 2(2a^2 - 6a + 2) + 1 \\ &= 2b + 1, \text{ as desired.} \end{aligned}$$

□

Exercise 73. There exists an integer that can be expressed as the sum of two perfect cubes in two different ways.

Proof. Consider the number 1729.
Note that $1^3 + 12^3 = 1729$ and $9^3 + 10^3 = 1729$.
Thus the number 1729 can be expressed as the sum of two perfect cubes in two different ways. □

Exercise 74. Suppose $n \in \mathbb{Z}$. Then n is even if and only if $3n + 5$ is odd.

Proof. We first use direct proof to prove that if n is even then $3n + 5$ is odd.
Suppose n is even.
Then there is some integer a for which $n = 2a$.
Thus $3n + 5 = 3(2a) + 5 = 6a + 5 = 2(3a + 2) + 1$, so $3n + 5$ is odd because it has form $2b + 1$, where $b = 3a + 2 \in \mathbb{Z}$.

Conversely, we show that if $3n + 5$ is odd then n is even.
We will prove this using contrapositive proof.
Suppose n is not even.
Then n is odd, so there is an integer a for which $n = 2a + 1$.
Thus $3n + 5 = 3(2a + 1) + 5 = 6a + 8 = 2(3a + 4)$, so $3n + 5$ is even.
Therefore $3n + 5$ is not odd. □

Exercise 75. Given an integer n, then $n^3 + n^2 + n$ is even if and only if n is even.

Proof. We first use contrapositive proof to show that if $n^3 + n^2 + n$ is even then n is even.
Suppose n is not even.
Then n is odd, so $n = 2k + 1$ for some $k \in \mathbb{Z}$.
Thus $n^3 + n^2 + n = n(n^2 + n + 1) = (2k + 1)((2k + 1)^2 + (2k + 1) + 1) = (2k+1)(4k^2+4k+1+2k+2) = (2k+1)(4k^2+6k+3) = (2k+1)(2(2k^2+3k+1)+1)$.
The first factor $2k + 1$ is an odd integer and the second factor $(2(2k^2 + 3k + 1) + 1)$ is odd, thus the product $n^3 + n^2 + n$ is odd.
Therefore $n^3 + n^2 + n$ is not even.

Conversely, we show that if n is even then $n^3 + n^2 + n$ is even.
Suppose n is even.
Then $n = 2k$ for some $k \in \mathbb{Z}$.
Thus $n^3+n^2+n = n(n^2+n+1) = (2k)((2k)^2+2k+1) = (2k)(4k^2+2k+1)$.
The first factor $2k$ is even. Therefore the product $n^3 + n^2 + n$ is even. □

Exercise 76. Suppose $a, b \in \mathbb{Z}$. Then $(a - 3)b^2$ is even if and only if a is odd or b is even.

Proof. We first prove that if $(a - 3)b^2$ is even then a is odd or b is even.
We use proof by contrapositive.
Suppose it is not the case that a is odd or b is even.
Then a is even and b is odd.
Thus $a = 2c$ and $b = 2d + 1$ for some $c, d \in \mathbb{Z}$.
Thus $(a - 3)b^2 = (2c - 3)(2d + 1)^2 = (2(c - 2) + 1)(2d + 1)^2$.
The first factor $(2(c - 2) + 1)$ is odd and the second factor $(2d + 1)^2$ is odd since the square of an odd number is always odd.
Therefore, the product $(a - 3)b^2$ is odd.
Consequently, $(a - 3)b^2$ is not even.

Conversely, we show that if a is odd or b is even then $(a - 3)b^2$ is even.
Suppose a is odd or b is even. There are two cases to consider.
Case 1: Suppose a is odd.
Then $a = 2c + 1$ for some $c \in \mathbb{Z}$.
Thus $(a - 3)b^2 = (2c + 1 - 3)b^2 = 2(c - 1)b^2$.
The first factor $2(c - 1)$ is even.
Therefore the product $(a - 3)b^2$ is even.

Case 2: Suppose b is even.
Then $b = 2d$ for some $d \in \mathbb{Z}$.
Thus $(a-3)b^2 = (a-3)(2d)^2 = (a-3)(2(2d^2))$.
The second factor $2(2d^2)$ is even, thus the product $(a-3)b^2$ is even.
Either way both cases show that $(a-3)b^2$ is even. □

Lemma 77. *Let $m, n \in \mathbb{Z}$. If mn is even and m is odd, then n is even.*

Proof. Let $m, n \in \mathbb{Z}$.
Suppose mn is even and m is odd.
Then $mn = 2a$ and $m = 2b + 1$ for some $a, b \in \mathbb{Z}$.
Substitution yields $(2b + 1)n = 2a$.
It follows that:

$$\begin{aligned} 2bn + n &= 2a \\ n &= 2a - 2bn \\ n &= 2(a - nb). \end{aligned} \tag{6}$$

Hence $n = 2(a - nb)$ where $a - nb \in \mathbb{Z}$.
Therefore n is even, by definition of an even integer. □

Exercise 78. Let x and y be integers.
If $x(x + y)$ is odd, then y is even.

Proof. Suppose $x(x + y)$ is odd.
Then x is odd and $x + y$ is odd.
Therefore, y is even. □

Exercise 79. Let x be an integer.
If xy is even for any integer y, then x is even.

Proof. Suppose xy is even for any integer y.
Let $y = 1$.
Then $xy = x * 1 = x$ is even. □

Exercise 80. Let y be an integer.
If $x(x + y)$ is even for any integer x, then y is odd.

Proof. Suppose $x(x + y)$ is even for any integer x.
Let $x = 1$.
Then $x(x + y) = 1(1 + y) = 1 + y$ is even.
Therefore, y is odd. □

Exercise 81. The product $n(n + 1)$ is even for every integer n.

Proof. We prove by contradiction.

Suppose the product $n(n+1)$ is not even for every integer n.

Then there exists an integer x such that $x(x+1)$ is not even.

Hence, $x(x+1)$ is odd.

Thus, x is odd and $x+1$ is odd.

Therefore, $x = 2k+1$ and $x+1 = 2m+1$ for integers k and m.

Since $x+1 = 2m+1$, then $x = 2m$, so $2k+1 = 2m$.

Hence, $1 = 2m - 2k = 2(m-k)$.

Since $m-k$ is an integer, then this implies 1 is even, a contradiction.

Therefore, $n(n+1)$ is even for every integer n. ☐

Exercise 82. Let $a, b \in \mathbb{Z}$. If $a^2(b^2 - 2b)$ is odd, then a and b are odd.

Solution. We use proof by contrapositive since direct proof doesn't help much. ☐

Proof. Suppose neither a nor b is odd.

Then a is not odd or b is not odd, so a is even or b is even.

We consider these two cases separately.

Case 1: Suppose a is even.

Then $a = 2c$ for some integer c.

Thus $a^2(b^2 - 2b) = (2c)^2(b^2 - 2b) = 2(2c^2(b^2 - 2b))$, which is even.

Therefore $a^2(b^2 - 2b)$ is not odd.

Case 2: Suppose b is even.

Then $b = 2c$ for some integer c.

Thus $a^2(b^2 - 2b) = a^2((2c)^2 - 2(2c)) = a^2(4c^2 - 4c) = 2(a^2(2c^2 - 2c))$, which is even.

Therefore $a2(b^2 - 2b)$ is not odd.

Thus in either case $a^2(b^2 - 2b)$ is not odd. ☐

Exercise 83. Let $a, b \in \mathbb{Z}$.

If both ab and $a+b$ are even, then both a and b are even.

Solution. Direct proof doesn't help too much. Let's try proof by contrapositive. ☐

Proof. Suppose it is not the case that both a and b are even.

Then neither a nor b is even, so either a is not even or b is not even.

Hence, either a is odd or b is odd or both are odd.

There are three cases to consider.

Case 1: Suppose a is odd and b is even.

Then there are integers c and d for which $a = 2c+1$ and $b = 2d$.

Therefore $ab = (2c+1)(2d) = 2(d(2c+1))$, which is even; and $a+b = (2c+1) + 2d = 2(c+d) + 1$, which is odd.

Thus it is not the case that both ab and $a+b$ are even.

Case 2: Suppose a is even and b is odd.

Then there are integers c and d for which $a = 2c$ and $b = 2d+1$.

Therefore $ab = 2c(2d+1)$, which is even; and $a+b = 2c+(2d+1) = 2(c+d)+1$, which is odd.

Thus it is not the case that both ab and $a + b$ are even.

Case 3: Suppose a is odd and b is odd.

Then there are integers c and d for which $a = 2c + 1$ and $b = 2d + 1$.

Therefore $ab = (2c + 1)(2d + 1) = 4cd + 2c + 2d + 1 = 2(2cd + c + d) + 1$, which is odd; and $a + b = (2c + 1) + (2d + 1) = 2(c + d + 1)$, which is even. Thus it is not the case that both ab and $a + b$ are even.

These cases show that it is not the case that both ab and $a + b$ are even. □

Exercise 84. Let $x, y \in \mathbb{Z}$.

If $x^2(y + 3)$ is even, then x is even or y is odd.

Solution. Proof by contrapositive seems to be a better approach than direct proof. □

Proof. Suppose it is not the case that x is even or y is odd.

Then x is not even and y is not odd, which implies that x is odd and y is even.

Thus $x = 2a + 1$ and $y = 2b$ for some $a, b \in \mathbb{Z}$.

Consequently $x^2(y + 3) = (2a + 1)^2(2b + 3) = (4a^2 + 4a + 1)(2b + 3) = 8a^2b + 12a^2 + 8ab + 12a + 2b + 3 = 2(4a^2b + 6a^2 + 4ab + 6a + b + 1) + 1$.

This shows $x^2(y + 3) = 2c + 1$ where $c = 4a^2b + 6a^2 + 4ab + 6a + b + 1 \in \mathbb{Z}$, so $x^2(y + 3)$ is odd.

Therefore $x^2(y + 3)$ is not even. □

Exercise 85. Let $n \in \mathbb{Z}$.

If $n^3 - 1$ is even, then n is odd.

Proof. Suppose n is not odd.

Then n is even, so $n = 2a$ for some integer a.

Thus $n^3 - 1 = (2a)^3 - 1 = 8a^3 - 1 = 2(4a^3 - 1) + 1$.

Therefore $n^3 - 1 = 2b + 1$ where $b = 4a^3 - 1 \in \mathbb{Z}$, so $n^3 - 1$ is odd.

Thus $n^3 - 1$ is not even. □

Exercise 86. If $n \in \mathbb{Z}$, then $n^2 + 3n + 4$ is even.

Proof. Let $n \in \mathbb{Z}$.

Then n is even or odd. There are two cases to consider.

Case 1. Suppose n is odd.

Then $n = 2a + 1$ for some $a \in \mathbb{Z}$.

Thus $n^2 + 3n + 4 = (2a + 1)^2 + 3(2a + 1) + 4 = 4a^2 + 4a + 1 + 6a + 7 = 4a^2 + 10a + 8 = 2(2a^2 + 5a + 4) = 2b$ where $b = 2a^2 + 5a + 4 \in \mathbb{Z}$.

Hence $n^2 + 3n + 4$ is even, by definition of an even number.

Case 2. Suppose n is even.

Then $n = 2a$ for some $a \in \mathbb{Z}$.

Thus $n^2 + 3n + 4 = (2a)^2 + 3(2a) + 4 = 4a^2 + 6a + 4 = 2(2a^2 + 3a + 2) = 2b$ where $b = 2a^2 + 3a + 2 \in \mathbb{Z}$.

Hence $n^2 + 3n + 4$ is even, by definition of an even number.

In either case $n^2 + 3n + 4$ is even. □

Proposition 87. *If two integers have opposite parity, then their product is even.*

Proof. Suppose m and n are integers with opposite parity. Then one of them is odd while the other is even.

Without loss of generality, suppose m is even and n is odd.

Then $m = 2a$ and $n = 2b + 1$ for some $a, b \in \mathbb{Z}$.

It follows that $mn = (2a)(2b + 1) = 2(2ab + a) = 2c$ where $c = 2ab + a \in \mathbb{Z}$

Therefore mn is even, by definition of even. □

Exercise 88. Prove or disprove the conjecture: If m and n are odd integers, then $m^2 - n^2$ is divisible by 8.

Solution. Let's try some example values of m,n to get a feel for if this conjecture looks true or not.

$m = 3, n = 5 \Rightarrow 3^2 - 5^2 = -16 = 8 * (-2)$

$m = 5, n = 1 \Rightarrow 5^2 - 1^2 = 24 = 8 * 3$

$m = 9, n = 7 \Rightarrow 9^2 - 7^2 = 32 = 8 * 4$

$m = 17, n = 11 \Rightarrow 17^2 - 11^2 = 168 = 8 * 21$

So, this appears it might be true. So, let's try to prove this conjecture.

We can translate this conjecture into logical symbols as:

m and n are odd integers $\Rightarrow 8|(m^2 - n^2)$.

We can use direct proof strategy and elementary number theory to devise a proof. □

Proof. Suppose m and n are odd integers.

Then the sum $m + n$ is even and the difference $m - n$ is even because the sum of two odd integers is even and the difference of two odd integers is even.

Thus for some $k_1, k_2 \in \mathbb{Z}$, we have

$$\begin{aligned} m - n &= 2k_1 \\ m + n &= 2k_2 \end{aligned}$$

We can add both equations to get $2m = 2k_1 + 2k_2$ which implies $m = k_1 + k_2$.

Since m is odd, then k_1 and k_2 must have opposite parity, for if both k_1 and k_2 were even, or if both k_1 and k_2 were odd, their sum $k_1 + k_2$ would be even.

Thus either k_1 is even and k_2 is odd, or k_1 is odd and k_2 is even.

We consider these cases separately.

Case 1: Suppose k_1 is even and k_2 is odd.

Then $k_1 = 2k_3$ and $k_2 = 2k_4 + 1$ for some $k_3, k_4 \in \mathbb{Z}$.

Hence $m^2 - n^2 = (m - n)(m + n) = (2k_1)(2k_2) = 4k_1k_2 = 4(2k_3)(2k_4 + 1) = 8k_3(2k_4 + 1)$.

Consequently $8|m^2 - n^2$.

Case 2: Suppose k_1 is odd and k_2 is even.

Then $k_1 = 2k_3 + 1$ and $k_2 = 2k_4$ for some $k_3, k_4 \in \mathbb{Z}$.

Hence $m^2 - n^2 = (m-n)(m+n) = (2k_1)(2k_2) = 4k_1k_2 = 4(2k_3+1)(2k_4) = 8k_4(2k_3+1)$.

Consequently $8|m^2 - n^2$.

Thus both cases show $8|m^2 - n^2$. Therefore this conjecture is a true proposition. $\square$

Exercise 89. Let a, b, r, s be coprime and $a^2 + b^2 = r^2$ and $a^2 - b^2 = s^2$.

Then a, r, s are odd and b is even.

Proof. We add both equations to obtain $2a^2 = r^2 + s^2$.

Since a^2 is an integer, then this implies $r^2 + s^2$ is even.

For any integers x and y, the sum $x + y$ is even iff either x, y are both even or x, y are both odd.

Thus, the sum $r^2 + s^2$ is even iff either r^2, s^2 are both even or r^2, s^2 are both odd.

Thus, either r^2, s^2 are both even or r^2, s^2 are both odd.

For any integer x, x^2 is even iff x is even and x^2 is odd iff x is odd.

Thus, either r, s are both even or r, s are both odd.

Suppose r, s are both even.

Then $2|r$ and $2|s$, so 2 is a common divisor of r and s.

Every common divisor of r and s divides $\gcd(r, s)$.

Thus, 2 divides $\gcd(r, s)$.

Since r, s are coprime, then $\gcd(r, s) = 1$.

Hence, $2|1$, a contradiction.

Therefore, r and s cannot be both even.

Since either r, s are both even or both odd and r, s are not both even, then this implies r and s are both odd.

Thus, $r = 2k + 1$ and $s = 2m + 1$ for some integers k and m.

Observe that

$$\begin{aligned} 2a^2 &= r^2 + s^2 \\ &= (2k+1)^2 + (2m+1)^2 \\ &= 4k^2 + 4k + 1 + 4m^2 + 4m + 1 \\ &= 4k^2 + 4m^2 + 4k + 4m + 2 \\ &= 2(2k^2 + 2m^2 + 2k + 2m + 1). \end{aligned}$$

Thus, $a^2 = 2k^2 + 2m^2 + 2k + 2m + 1 = 2(k^2 + m^2 + k + m) + 1$.

Since $k^2 + m^2 + k + m$ is an integer, then this implies a^2 is odd, so a is odd.

Since $a^2 + b^2 = r^2$, then $b^2 = r^2 - a^2$.
Since r is odd, then r^2 is odd.
Since a is odd, then a^2 is odd.
Thus, the difference $r^2 - a^2$ is even, so b^2 is even.
Therefore, b is even.

Consequently, a, r, s are odd and b is even, as desired. □

Exercise 90. Let $a, b, c \in \mathbb{Z}$.
If $a^2 + b^2 = c^2$, then a or b is even.

Proof. We prove by contradiction.
Suppose $a^2 + b^2 = c^2$ and neither a nor b is even.
Then a and b are both not even, so a is not even and b is not even.
Thus, a is odd and b is odd, so there are integers k and m such that $a = 2k+1$ and $b = 2m + 1$.
Thus,

$$\begin{aligned} c^2 &= a^2 + b^2 \\ &= (2k+1)^2 + (2m+1)^2 \\ &= 4k^2 + 4k + 1 + 4m^2 + 4m + 1 \\ &= 4k^2 + 4m^2 + 4k + 4m + 2 \\ &= 2(2k^2 + 2m^2 + 2k + 2m + 1) \end{aligned}$$

Since $2k^2 + 2m^2 + 2k + 2m + 1$ is an integer, then c^2 is even, so c is even.
Thus, there is an integer n such that $c = 2n$, so $c^2 = 4n^2$.
Hence, $\frac{c^2}{2} = 2n^2$, so $2n^2 = 2k^2 + 2m^2 + 2k + 2m + 1$.
Therefore, $2(n^2 - k^2 - m^2 - k - m) = 2n^2 - 2k^2 - 2m^2 - 2k - 2m = 1$, so $2(n^2 - k^2 - m^2 - k - m) = 1$.
Since $n^2 - k^2 - m^2 - k - m$ is an integer, then this implies 1 is even, a contradiction.
Therefore, if $a^2 + b^2 = c^2$, then either a is even or b is even, as desired. □

Exercise 91. If $a, b \in \mathbb{Z}$, then $a^2 - 4b - 3 \neq 0$.

Proof. Suppose for the sake of contradiction that $a, b \in \mathbb{Z}$, but $a^2 - 4b - 3 = 0$.
Then $a^2 = 4b+3 = 2(2b+1)+1$, which means a^2 is odd. Therefore a is odd also (since we previously proved that if a is even, then a^2 is even), so $a = 2c+1$ for some integer c.
Substitution gives

$$\begin{aligned} (2c+1)^2 - 4b - 3 &= 0 \\ 4c^2 + 4c + 1 - 4b - 3 &= 0 \\ 4c^2 + 4c - 4b &= 2 \\ 2c^2 + 2c - 2b &= 1 \\ 2(c^2 + c - b) &= 1. \end{aligned}$$

From this last equation we conclude that 1 is an even number, a contradiction. □

Exercise 92. There exist no integers a and b for which $18a + 6b = 1$.

Proof. Suppose for the sake of contradiction that there exist integers a and b for which $18a + 6b = 1$.

Then $2(9a + 3b) = 1$, which means 1 is even, a contradiction. □

Exercise 93. If $a, b \in \mathbb{Z}$, then $a^2 - 4b - 2 \neq 0$.

Proof. Suppose for the sake of contradiction that $a, b \in \mathbb{Z}$, but $a^2 - 4b - 2 = 0$.

Then $a^2 = 4b + 2 = 2(2b + 1)$, so a^2 is even. Consequently a is even.

Thus there is an integer c for which $a = 2c$.

Substitution gives

$$\begin{aligned} (2c)^2 &= 4b + 2 \\ 4c^2 &= 4b + 2 \\ 2c^2 &= 2b + 1 \\ 2c^2 - 2b &= 1 \\ 2(c^2 - b) &= 1 \end{aligned}$$

Since $c^2 - b \in \mathbb{Z}$ the last equation means that 1 is even, a contradiction. □

Exercise 94. Prove or disprove the conjecture: There exist three integers a, b, c, all greater than 1 and no two equal, for which $a^b = b^c$.

Solution. We deduce that if b is odd, then a is odd based on the equation $a^b = b^c$ by considering what must be true if b is odd.

Also, we can deduce that if b is even, then a is even and c is even.

So we substitute various values of a, b, and c into the equation that fit these criteria.

There are many examples that demonstrate such integers exist that satisfy the equation.

For example: $9^3 = 3^6, 8^2 = 2^6, 16^2 = 2^8, 32^2 = 2^{10}, 8^4 = 4^6, 16^4 = 4^8, 32^4 = 4^{10}, 64^4 = 4^{12}, 36^6 = 6^{12}, 216^6 = 6^{18}$, etc.

Of course one example suffices to prove the conjecture since this is an existence assertion, so the proof below works. □

Proof. The conjecture is true.

Note that if $a = 9, b = 3$, and $c = 6$, then $a^b = 9^3 = (3^2)^3 = 3^6 = b^c$. □

Exercise 95. Prove or disprove the conjecture: If $n \in \mathbb{Z}$ and $n^5 - n$ is even, then n is even.

Solution. We can factor $n^5 - n = n(n^4 - 1) = n(n^2 - 1)(n^2 + 1)$.

The integer n is either odd or even.

If n is odd, then we know n^2 is odd.

Thus $n^2 - 1$ is even and $n^2 + 1$ is even.

Therefore the product $n(n^2 - 1)(n^2 + 1)$ is even since at least one of the factors is even.

Hence, $n^5 - n$ is even.

If n is even, then $n(n^2 - 1)(n^2 + 1)$ is even, since at least one of the factors is even.

Hence, $n^5 - n$ is even.

So, $n^5 - n$ is always even, but n can be either even or odd.

Consequently, the conjecture is false.

One counterexample suffices to disprove the conjecture since this is a universal quantification $\forall(n \in \mathbb{Z})(2|n^5 - n) \to 2|n$, so the proof below works. □

Proof. The conjecture is false.

Here is a counterexample: Let $n = 1$. Then $n^5 - n = 1^5 - 1 = 0$ which is even, but $n = 1$ is not even. □

Exercise 96. For every integer $n \geq 0$, $\sum_{i=0}^{n} i \cdot i! = (n+1)! - 1$.

Solution. We prove by induction(weak).

The statement S_n is $\sum_{i=0}^{n} i \cdot i! = (n+1)! - 1$.

The statement S_k is $\sum_{i=0}^{k} i \cdot i! = (k+1)! - 1$.

The statement S_{k+1} is $\sum_{i=0}^{k+1} i \cdot i! = [(k+1)+1]! - 1$. □

Proof. We prove by induction.

Basis:

If $n = 0$, then the statement is $\sum_{i=0}^{0} i \cdot i! = (0+1)! - 1$.

The left-hand side is $0 \cdot 0! = 0$ and the right-hand side is $1! - 1 = 0$, so the statement is obviously true for $n = 0$.

Induction:

Suppose $k \geq 0$.

We must prove if $\sum_{i=0}^{k} i \cdot i! = (k+1)! - 1$, then $\sum_{i=0}^{k+1} i \cdot i! = [(k+1)+1]! - 1$.

We use direct proof.

Suppose $\sum_{i=0}^{k} i \cdot i! = (k+1)! - 1$.

Observe that

$$\begin{aligned}
\sum_{i=0}^{k+1} i \cdot i! &= \sum_{i=0}^{k} i \cdot i! + (k+1)(k+1)! \\
&= [(k+1)! - 1] + (k+1)(k+1)! \\
&= (k+1)!(1 + k + 1) - 1 \\
&= (k+1)!(k+2) - 1 \\
&= (k+2)! - 1 \\
&= [(k+1)+1]! - 1
\end{aligned}$$

It follows by induction that $\sum_{i=0}^{n} i \cdot i! = (n+1)! - 1$ for every integer $n \geq 0$. $\square$

Exercise 97. Define the sequence of **Fibonacci numbers** by $f_1 = 1$ and $f_2 = 1$ and $f_{n+2} = f_{n+1} + f_n$ for all $n \in \mathbb{Z}^+$.

Then $f_n < 2^n$ for all $n \in \mathbb{Z}^+$.

Proof. Let $p(n)$ be the predicate defined by $p(n) : f_n < 2^n$ for $n \in \mathbb{Z}^+$.

To prove $p(n)$ is true for all $n \in \mathbb{Z}^+$, we prove by strong induction on n.

Basis:

Let $n = 1$. Then $f(1) = 1 < 2 = 2^1$, so $p(1)$ is true.

Let $n = 2$. Then $f(2) = 1 < 4 = 2^2$, so $p(2)$ is true.

Induction:

Let $k \in \mathbb{Z}^+$ such that $p(1)$ and $p(2)$ and ... and $p(k)$ are all true for $k \geq 2$.

Then $p(k-1)$ and $p(k)$ are true, so $f_{k-1} < 2^{k-1}$ and $f_k < 2^k$.

Since $k \in \mathbb{Z}^+$, then $k > 0$, so $2^k > 0$.

Observe that $f_{k+1} = f_k + f_{k-1} < 2^k + 2^{k-1} = 2^k(1 + 2^{-1}) = 2^k(\frac{3}{2}) < 2^k(2) = 2^{k+1}$.

Hence, $f_{k+1} < 2^{k+1}$, so $p(k+1)$ is true.

Therefore, by strong induction, $f_n < 2^n$ for all $n \in \mathbb{Z}^+$. $\square$

Exercise 98. Define the sequence of **Fibonacci numbers** by $f_1 = 1$ and $f_2 = 1$ and $f_{n+2} = f_{n+1} + f_n$ for all $n \in \mathbb{Z}^+$.

Then $f_n < (\frac{7}{4})^n$ for all $n \in \mathbb{Z}^+$.

Proof. Let $p(n)$ be the predicate defined by $p(n) : f_n < (\frac{7}{4})^n$ for $n \in \mathbb{Z}^+$.

To prove $p(n)$ is true for all $n \in \mathbb{Z}^+$, we prove by strong induction on n.

Basis:

Let $n = 1$. Then $f(1) = 1 < \frac{7}{4} = (\frac{7}{4})^1$, so $p(1)$ is true.

Let $n = 2$. Then $f(2) = 1 < 3 < \frac{49}{16} = (\frac{7}{4})^2$, so $p(2)$ is true.

Induction:

Let $k \in \mathbb{Z}^+$ such that $p(1)$ and $p(2)$ and ... and $p(k)$ are all true for $k \geq 2$.

Then $p(k-1)$ and $p(k)$ are true, so $f_{k-1} < (\frac{7}{4})^{k-1}$ and $f_k < (\frac{7}{4})^k$.

Since $k \in \mathbb{Z}^+$, then $k > 0$, so $(\frac{7}{4})^k > 0$.

Observe that

$$\begin{aligned}
f_{k+1} &= f_k + f_{k-1} \\
&< (\frac{7}{4})^k + (\frac{7}{4})^{k-1} \\
&= (\frac{7}{4})^{k-1}(\frac{7}{4} + 1) \\
&= (\frac{7}{4})^{k-1}(\frac{11}{4}) \\
&< (\frac{7}{4})^{k-1}(3) \\
&< (\frac{7}{4})^{k-1}(\frac{49}{16}) \\
&= (\frac{7}{4})^{k-1}(\frac{7}{4})^2 \\
&= (\frac{7}{4})^{k+1}.
\end{aligned}$$

Hence, $f_{k+1} < (\frac{7}{4})^{k+1}$, so $p(k+1)$ is true.

Therefore, by strong induction, $f_n < (\frac{7}{4})^n$ for all $n \in \mathbb{Z}^+$. □

Exercise 99. Define the sequence of **Fibonacci numbers** by $f_1 = 1$ and $f_2 = 1$ and $f_{n+2} = f_{n+1} + f_n$ for all $n \in \mathbb{Z}^+$.

For all positive integers $n \geq 2$, $f_{n+1}f_{n-1} = f_n^2 + (-1)^n$.

Proof. Let $p(n)$ be the predicate defined by $p(n) : f_{n+1}f_{n-1} = f_n^2 + (-1)^n$ for $n \in \mathbb{Z}^+$.

To prove $p(n)$ is true for $n \geq 2$, we prove by induction on n.

Basis:

Let $n = 2$. Then $f_3 f_1 = 2 \cdot 1 = 2$ and $(f_2)^2 + (-1)^2 = 1^2 + 1 = 2$, so $p(2)$ is true.

Induction:

Let $k \in \mathbb{Z}^+$ such that $p(k)$ is true for $k \geq 2$.

Then $f_{k+1}f_{k-1} = f_k^2 + (-1)^k$.

Observe that

$$\begin{aligned}
f_{k+2}f_k &= (f_{k+1} + f_k)f_k \\
&= f_{k+1}f_k + f_k^2 \\
&= f_{k+1}f_k + (f_{k+1}f_{k-1} - (-1)^k) \\
&= f_{k+1}(f_k + f_{k-1}) - (-1)^k \\
&= f_{k+1}f_{k+1} - (-1)^k \\
&= (f_{k+1})^2 - (-1)^k \\
&= (f_{k+1})^2 + (-1)(-1)^k \\
&= (f_{k+1})^2 + (-1)^{k+1}.
\end{aligned}$$

Hence, $f_{k+2}f_k = (f_{k+1})^2 + (-1)^{k+1}$, so $p(k+1)$ is true.

Therefore, by induction, $p(n)$ is true for all integers $n \geq 2$, so $f_{n+1}f_{n-1} = f_n^2 + (-1)^n$ for all integers $n \geq 2$. □

Exercise 100. Define the sequence of **Fibonacci numbers** by $f_1 = 1$ and $f_2 = 1$ and $f_{n+2} = f_{n+1} + f_n$ for all $n \in \mathbb{Z}^+$.

Let f_n be the n^{th} Fibonacci number.

Then f_n and f_{n+1} have no common factor greater than 1 for all $n \in \mathbb{Z}^+$.

(Therefore, $\gcd(f_n, f_{n+1}) = 1$, so that f_n and f_{n+1} are relatively prime.)

Proof. Let $p(n)$ be the statement: f_n and f_{n+1} have no common factor greater than 1.

To prove $p(n)$ is true for all $n \in \mathbb{Z}^+$, we prove by induction on n.

Basis:

Let $n = 1$.

Since $f_1 = 1$ and $f_2 = 1$, then f_1 and f_2 have 1 as the greatest common factor, so f_1 and f_2 have no common factor greater than 1.

Therefore, $p(1)$ is true.

Induction:

Let $k \in \mathbb{Z}^+$ such that $p(k)$ is true.

Then f_k and f_{k+1} have no common factor greater than 1.

Suppose f_{k+1} and f_{k+2} have a common factor greater than 1, say $d > 1$.

Then d divides both f_{k+1} and f_{k+2}, so d divides any linear combination of f_{k+1} and f_{k+2}.

Since $f_{k+2} = f_{k+1} + f_k$, then $f_k = f_{k+2} - f_{k+1}$ is a linear combination of f_{k+1} and f_{k+2}, so d divides f_k.

Since d divides f_k and d divides f_{k+1}, then f_k and f_{k+1} have a common divisor greater than 1.

But, this contradicts the induction hypothesis.

Thus, f_{k+1} and f_{k+2} have no common factor greater than 1, so $p(k+1)$ is true.

Hence, by the principle of induction, $p(n)$ is true for all $n \in \mathbb{Z}^+$.

Therefore, f_n and f_{n+1} have no common factor greater than 1 for all $n \in \mathbb{Z}^+$. □

Proposition 101. *Let f_n be the n^{th} Fibonacci number.*

Then

$$f_n = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n}{\sqrt{5}}$$

Solution. Let $A = \frac{1+\sqrt{5}}{2}$. We note that $\frac{1+\sqrt{5}}{2}$ is the **Golden ratio**.

Let $B = \frac{1-\sqrt{5}}{2}$.

We must prove: $\forall n \in \mathbb{N}, S_n$ where S_n is the statement:

$$f_n = \frac{A^n - B^n}{\sqrt{5}}$$

The trick to this proof is to work backwards to find the appropriate relationships among the terms. □

Proof. Let $A = \frac{1+\sqrt{5}}{2}$.

Let $B = \frac{1-\sqrt{5}}{2}$.

Observe that $A + B = 1$ and $A - B = \sqrt{5}$ and $A^2 = A + 1$ and $B^2 = B + 1$.

Let $S = \{n \in \mathbb{Z}^+ : f_n = \frac{A^n - B^n}{\sqrt{5}}\}$.

We must prove $S = \mathbb{Z}^+$.

We prove by strong induction on n.

Basis:

If $n = 1$, the statement is $f_1 = \frac{A^1 - B^1}{\sqrt{5}}$.

The left hand side is $f_1 = 1$ and the right hand side is $\frac{A-B}{\sqrt{5}} = \frac{\sqrt{5}}{\sqrt{5}} = 1$.

Therefore, this statement is true, so $1 \in S$.

If $n = 2$, the statement is $f_2 = \frac{A^2 - B^2}{\sqrt{5}}$.

The left hand side is $f_2 = 1$ and the right hand side is $\frac{A^2-B^2}{\sqrt{5}} = \frac{(A-B)(A+B)}{\sqrt{5}} = \frac{\sqrt{5}\cdot 1}{\sqrt{5}} = 1$.

Therefore, this statement is true, so $2 \in S$.

Induction:

Suppose $k - 1 \in S$ and $k \in S$ for $k \geq 2$.

Then $f_{k-1} = \frac{A^{k-1} - B^{k-1}}{\sqrt{5}}$ and $f_k = \frac{A^k - B^k}{\sqrt{5}}$.

To prove $k + 1 \in S$, we must prove $f_{k+1} = \frac{A^{k+1} - B^{k+1}}{\sqrt{5}}$.

Since $f_{k+1} = f_k + f_{k-1}$ then

$$\begin{aligned} f_{k+1} &= \frac{A^k - B^k}{\sqrt{5}} + \frac{A^{k-1} - B^{k-1}}{\sqrt{5}} \\ &= \frac{(A^k + A^{k-1}) - (B^k + B^{k-1})}{\sqrt{5}}. \end{aligned}$$

Since $A^2 = A + 1$, we multiply both sides of the equation by A^{k-1} to get $A^{k+1} = A^k + A^{k-1}$.

Since $B^2 = B + 1$, we multiply both sides of the equation by B^{k-1} to get $B^{k+1} = B^k + B^{k-1}$.

Substituting these into the previous equation we get $f_{k+1} = \frac{A^{k+1} - B^{k+1}}{\sqrt{5}}$.

It follows by induction that $f_n = \frac{A^n - B^n}{\sqrt{5}}$ for all $n \in \mathbb{Z}^+$. □

Exercise 102. Let $x \in \mathbb{R}$.

For all $n \in \mathbb{N}$, $x + 4x + 7x + ... + (3n - 2)x = \frac{n(3n-1)x}{2}$.

Proof. Let $p(n)$ be the predicate $x + 4x + 7x + ... + (3n - 2)x = \frac{n(3n-1)x}{2}$ defined over $\mathbb{Z}^+$.

We prove $p(n)$ is true for all positive integers n by induction on n.

Basis:

Since $x = \frac{1(3*1-1)x}{2} = x$, then $p(1)$ is true.

Induction:

Suppose $p(k)$ is true for $k \geq 1$.

Then $x + 4x + 7x + ... + (3k-2)x = \frac{k(3k-1)x}{2}$.
Observe that

$$\begin{aligned}
x + 4x + 7x + ... + (3k-2)x + [3(k+1)-2]x &= \frac{k(3k-1)x}{2} + [3(k+1)-2]x \\
&= \frac{(3k^2-k)x}{2} + (3k+1)x \\
&= \frac{3k^2x - kx + 2(3k+1)x}{2} \\
&= \frac{3k^2x - kx + 6kx + 2x}{2} \\
&= \frac{3k^2x + 5kx + 2x}{2} \\
&= \frac{(3k^2+5k+2)x}{2} \\
&= \frac{(k+1)(3k+2)x}{2} \\
&= \frac{(k+1)[3(k+1)-1]x}{2}.
\end{aligned}$$

Thus, $p(k+1)$ is true.

Therefore, by PMI, $p(n)$ is true for any positive integer n, so $x + 4x + 7x + ... + (3n-2)x = \frac{n(3n-1)x}{2}$ for all $n \in \mathbb{Z}^+$. □