Book Bridge to Abstract Math by Morash Exercises

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Chapter 1 Sets

Chapter 1.1 Basic Definitions and Notation

Methods of Describing Sets

Example 1. Let $B = \{$ Massachusetts, Michigan, California $\}$.

The set B consists of 3 elements, and Michigan $\in B$, but Ohio $\notin B$.

In this context, the universal set U is the set of all 50 states of the United States.

Example 2. Let $C = \{x|\ x$ is a natural number and $x \leq 100\}$.

Observe that $C = \{1, 2, 3, ..., 99, 100\}$, and $57 \in C$, but $126 \notin C$.

Observe that $C = \{x \in \mathbb{N} : x \leq 100\}$.

Example 3. Let $D = \{x|\ x$ is the name of a state in the United States beginning with the letter M.$\}$.

List the elements of set D.

Solution. Observe that $D = \{$Maine, Maryland, Massachusetts, Michigan, Minnesota, Mississippi, Missouri, Montana$\}$ $\square$

Example 4. Let $X = \{x|\ x$ is a male citizen of the United States.$\}$.

The set X is a large set, difficult to describe using the roster method to list all of its elements.

Example 5. Let $T = \{10, 20, 30, 40, ...\}$.

Observe that $50 \in T$ and $50^{10} \in T$, but $15 \notin T$.

Intervals

Example 6. Since $\mathbb{Z} \subseteq \mathbb{R}$, and $0 \in \mathbb{Z}$ and $1 \in \mathbb{Z}$ and $\frac{1}{2} \in \mathbb{R}$ and $0 < \frac{1}{2} < 1$, but $\frac{1}{2} \notin \mathbb{Z}$, then $\mathbb{Z}$ is not an interval.

Example 7. Since $\mathbb{Q} \subseteq \mathbb{R}$, and $1 \in \mathbb{Q}$ and $2 \in \mathbb{Q}$ and $\sqrt{2} \in \mathbb{R}$ and $1 < \sqrt{2} < 2$, but $\sqrt{2} \notin \mathbb{Q}$, then $\mathbb{Q}$ is not an interval.

Example 8. Assume the universal set is $\mathbb{R}$.
Find the solution set to the inequality $7x - 9 \leq 16$.

Solution. Let $x \in \mathbb{R}$.
Observe that $7x - 9 \leq 16$ iff $7x \leq 25$ iff $x \leq \frac{25}{7}$.
Therefore, the solution set is $\{x \in \mathbb{R} : x \leq \frac{25}{7}\} = (-\infty, \frac{25}{7}]$. $\square$

Example 9. Assume the universal set is $\mathbb{R}$.
Find the solution set to the inequality $|2x + 3| < 5$.

Solution. Let $x \in \mathbb{R}$.
Observe that

$$\begin{aligned} |2x+3| < 5 \quad &\Leftrightarrow \quad -5 < 2x + 3 < 5 \\ &\Leftrightarrow \quad -8 < 2x < 2 \\ &\Leftrightarrow \quad -4 < x < 1 \\ &\Leftrightarrow \quad x \in (-4, 1). \end{aligned}$$

Therefore, the set of all real numbers x such that $|2x + 3| < 5$ is the open interval $(-4, 1)$. $\square$

Example 10. Assume the universal set is $\mathbb{R}$.
Find the solution set to the inequality $2x^2 + x - 28 \leq 0$.

Solution. Let $S = \{x \in \mathbb{R} : 2x^2 + x - 28 \leq 0\}$.
Let $x \in S$.
Then $x \in \mathbb{R}$ and $2x^2 + x - 28 \leq 0$, so $2x^2 + x - 28 = (2x - 7)(x + 4) \leq 0$.
Thus, either $(2x - 7)(x + 4) < 0$ or $(2x - 7)(x + 4) = 0$, so either $2x - 7 > 0$ and $x + 4 < 0$, or $2x - 7 < 0$ and $x + 4 > 0$, or $2x - 7 = 0$, or $x + 4 = 0$.
We consider these cases separately.
Case 1: Suppose $x + 4 = 0$.
Then $x = -4$.
Case 2: Suppose $2x - 7 = 0$.
Then $x = \frac{7}{2}$.
Case 3: Suppose $2x - 7 > 0$ and $x + 4 < 0$.
Then $x > \frac{7}{2}$ and $x < -4$, an impossibility.
Therefore, this case cannot happen.
Case 4: Suppose $2x - 7 < 0$ and $x + 4 > 0$.
Then $x < \frac{7}{2}$ and $x > -4$, so $-4 < x < \frac{7}{2}$.

Hence, in all cases, either $x = -4$ or $x = \frac{7}{2}$, or $-4 < x < \frac{7}{2}$, so $-4 \leq x \leq \frac{7}{2}$.
Therefore, $x \in [-4, \frac{7}{2}]$, so $S \subseteq [-4, \frac{7}{2}]$.

Let $y \in [-4, \frac{7}{2}]$.

Then $y \in \mathbb{R}$ and $-4 \leq y \leq \frac{7}{2}$, so $-4 \leq y$ and $y \leq \frac{7}{2}$.
Hence, $0 \leq y + 4$ and $2y \leq 7$, so $y + 4 \geq 0$ and $2y - 7 \leq 0$.
Thus, $(y + 4)(2y - 7) \leq 0$, so $2y^2 + y - 28 \leq 0$.
Since $y \in \mathbb{R}$ and $2y^2 + y - 28 \leq 0$, then $y \in S$, so $[-4, \frac{7}{2}] \subseteq S$.

Since $S \subseteq [-4, \frac{7}{2}]$ and $[-4, \frac{7}{2}] \subseteq S$, then $S = [-4, \frac{7}{2}]$. $\square$

Example 11. Assume the universal set is $\mathbb{R}$.
Find the solution set to the inequality $5x^2 + 3x + 2 < 0$.

Solution. Let S be the solution set to the inequality $5x^2 + 3x + 2 < 0$.
Then $S = \{x \in \mathbb{R} : 5x^2 + 3x + 2 < 0\}$.
Let $x \in \mathbb{R}$.
Observe that

$$\begin{aligned} 5x^2 + 3x + 2 &= 5(x^2 + \frac{3x}{5}) + 2 \\ &= 5(x + \frac{3}{10})^2 + 2 - 5 \cdot (\frac{3}{10})^2 \\ &= 5(x + \frac{3}{10})^2 + \frac{31}{20} \\ &> 0. \end{aligned}$$

Therefore, $5x^2 + 3x + 2 > 0$, so $5x^2 + 3x + 2 > 0$ for all $x \in \mathbb{R}$.
Hence, there is no $x \in \mathbb{R}$ such that $5x^2 + 3x + 2 < 0$, so $S = \emptyset$. $\square$

Relations between Sets

Example 12. Let $H = \{1, 2, 3\}$.
Let $N = \{2, 4, 6, 8, 10\}$.
Let $P = \{1, 2, 3, ..., 9, 10\}$.
Find the subset relationships among these sets.
Show that $P \not\subseteq H$ and $P \not\subseteq N$.
Show that neither H nor N is a subset of the other.

Solution. Observe that $H \subseteq H$ and $N \subseteq N$ and $P \subseteq P$ and $H \subset P$ and $N \subset P$.
Since $4 \in P$, but $4 \notin H$, then $P \not\subseteq H$.
Since $1 \in P$, but $1 \notin N$, then $P \not\subseteq N$.

We show that neither H nor N is a subset of the other.
Since $1 \in H$, but $1 \notin N$, then $H \not\subseteq N$.
Since $4 \in N$, but $4 \notin H$, then $N \not\subseteq H$.
Since $H \not\subseteq N$ and $N \not\subseteq H$, then neither H nor N is a subset of the other. $\square$

Example 13. Let $T = \{2, 4, 6, ...\}$.
Let $V = \{4, 8, 12, ...\}$.
Let $W = \{..., -8, -4, 0, 4, 8, ...\}$.
Find the subset relationships among these sets.

Solution. Observe that T is the set of all positive even integers, and V is the set of all positive multiples of 4, and W is the set of all multiples of 4.
Since every positive multiple of 4 is a positive even integer, then $V \subseteq T$.
Since every positive multiple of 4 is a multiple of 4, then $V \subseteq W$.
Since $2 \in T$, but $2 \notin V$, then $T \not\subseteq V$.
Since $-8 \in W$, but $-8 \notin V$, then $W \not\subseteq V$.

Since $6 \in T$, but $6 \notin W$, then $T \not\subseteq W$.
Since $-4 \in W$, but $-4 \notin T$, then $W \not\subseteq T$.
Since $T \not\subseteq W$ and $W \not\subseteq T$, then neither T nor W is a subset of the other. $\square$

Example 14. Find the subset relationships of the number sets: $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$.

Solution. We find the relationships below.
Observe that $\mathbb{N} \subset \mathbb{Z}$ and $\mathbb{Z} \subset \mathbb{Q}$ and $\mathbb{Q} \subset \mathbb{R}$ and $\mathbb{R} \subset \mathbb{C}$.
Since $\mathbb{Q} \subset \mathbb{R}$ and $\mathbb{R} \subset \mathbb{C}$, then $\mathbb{Q} \subset \mathbb{C}$.
Since $\mathbb{Z} \subset \mathbb{Q}$ and $\mathbb{Q} \subset \mathbb{R}$, then $\mathbb{Z} \subset \mathbb{R}$.
Since $\mathbb{Z} \subset \mathbb{R}$ and $\mathbb{R} \subset \mathbb{C}$, then $\mathbb{Z} \subset \mathbb{C}$.
Since $\mathbb{N} \subset \mathbb{Z}$ and $\mathbb{Z} \subset \mathbb{Q}$, then $\mathbb{N} \subset \mathbb{Q}$.
Since $\mathbb{N} \subset \mathbb{Z}$ and $\mathbb{Z} \subset \mathbb{R}$, then $\mathbb{N} \subset \mathbb{R}$.
Since $\mathbb{N} \subset \mathbb{Z}$ and $\mathbb{Z} \subset \mathbb{C}$, then $\mathbb{N} \subset \mathbb{C}$.
Therefore, $\mathbb{N} \subset \mathbb{Z}$ and $\mathbb{N} \subset \mathbb{Q}$ and $\mathbb{N} \subset \mathbb{R}$ and $\mathbb{N} \subset \mathbb{C}$, and $\mathbb{Z} \subset \mathbb{Q}$ and $\mathbb{Z} \subset \mathbb{R}$ and $\mathbb{Z} \subset \mathbb{C}$, and $\mathbb{Q} \subset \mathbb{R}$ and $\mathbb{Q} \subset \mathbb{C}$, and $\mathbb{R} \subset \mathbb{C}$. $\square$

Chapter 1.1 Exercises

Exercise 15. Let $A = \{x \in \mathbb{R} : x^3 + x^2 - 12x = 0\}$.
Express the set via the roster method; that is, list its elements.

Solution. Observe that A is the solution set to the equation $x^3 + x^2 - 12x = 0$.
Let $x \in A$.
Then $x \in \mathbb{R}$ and $x^3 + x^2 - 12x = 0$, so $0 = x(x^2 + x - 12) = x(x + 4)(x - 3)$.
Therefore, either $x = 0$ or $x = -4$ or $x = 3$, so $x \in \{-4, 0, 3\}$.
Hence, $A \subseteq \{-4, 0, 3\}$.

We verify the zeros of the polynomial $x^3 + x^2 - 12x$ are $-4, 0, 3$.
Since $-4 \in \mathbb{R}$ and $(-4)^3 + (-4)^2 - 12(-4) = 0$, then $-4 \in A$.
Since $-0 \in \mathbb{R}$ and $(0)^3 + (0)^2 - 12(0) = 0$, then $0 \in A$.
Since $3 \in \mathbb{R}$ and $(3)^3 + (3)^2 - 12(3) = 0$, then $3 \in A$.
Thus, $-4 \in A$ and $0 \in A$ and $3 \in A$, so $\{-4, 0, 3\} \subseteq A$.

Since $A \subseteq \{-4, 0, 3\}$ and $\{-4, 0, 3\} \subseteq A$, then $A = \{-4, 0, 3\}$. $\square$

Exercise 16. Let $B = \{x \in \mathbb{R} : \frac{3}{x+1} + \frac{3}{x^2+x} = -2\}$.
Express the set via the roster method; that is, list its elements.

Solution. Observe that B is the solution set to the equation $\frac{3}{x+1} + \frac{3}{x^2+x} = -2$.
The universal set is $\mathbb{R}$.
Since division by zero is not defined, then $x+1 \neq 0$ and $x^2+x = x(x+1) \neq 0$.
Since $x(x+1) \neq 0$, then $x \neq 0$.

Let $x \in B$.
Then $x \in \mathbb{R}$ and $\frac{3}{x+1} + \frac{3}{x^2+x} = -2$.
Observe that

$$\begin{aligned} -2 &= \frac{3}{x+1} + \frac{3}{x(x+1)} \\ &= \frac{3x+3}{x(x+1)} \\ &= \frac{3(x+1)}{x(x+1)} \\ &= \frac{3}{x}. \end{aligned}$$

Hence, $-2 = \frac{3}{x}$.
Since $x \neq 0$, then $-2x = 3$, so $x = -\frac{3}{2}$.
Thus, $x \in \{-\frac{3}{2}\}$, so $B \subseteq \{-\frac{3}{2}\}$.

We verify that $-\frac{3}{2}$ is a solution to the equation.
Since $-\frac{3}{2} \in \mathbb{R}$ and $\frac{3}{\frac{-3}{2}+1} + \frac{3}{(\frac{-3}{2})^2 + (\frac{-3}{2})} = -2$, then $-\frac{3}{2} \in B$, so $\{-\frac{3}{2}\} \subseteq B$.

Since $B \subseteq \{-\frac{3}{2}\}$ and $\{-\frac{3}{2}\} \subseteq B$, then $B = \{-\frac{3}{2}\}$. $\square$

Exercise 17. Let $C = \{x \in \mathbb{N} : -12 \leq x < 25\}$.
Express the set via the roster method; that is, list its elements.

Solution. Since C is the set of all natural numbers greater than or equal to -12 and less than 25, then $C = \{1, 2, 3, 4, ..., 23, 24\}$. $\square$

Exercise 18. Let $D = \{x \in \mathbb{N} : x \text{ is prime and divisible by } 2\}$.
Express the set via the roster method; that is, list its elements.

Solution. Let $x \in D$.

Then $x \in \mathbb{N}$ and x is prime and x is divisible by 2.

Since 2 is the only natural number that is prime and is divisible by 2, then $x = 2$, so $x \in \{2\}$.

Thus, $x \in D$ implies $x \in \{2\}$, so $D \subset \{2\}$.

Since $2 \in \mathbb{N}$ and 2 is prime and 2 is divisible by 2, then $2 \in D$, so $\{2\} \subseteq D$.

Since $D \subseteq \{2\}$ and $\{2\} \subseteq D$, then $D = \{2\}$. $\square$

Exercise 19. Let $E = \{x \in \mathbb{Z} : -5 < x < 4\}$.

Express the set via the roster method; that is, list its elements.

Solution. Since E is the set of all integers between -5 and 4, then $E = \{-4, -3, -2, -1, 0, 1, 2, 3\}$. $\square$

Exercise 20. Let $G = \{x :\ x \text{ is a planet in the Earth's solar system } \}$.

Express the set via the roster method; that is, list its elements.

Solution. Since G is the set of all planets in the Earth's solar system, then $G = \{$ Mercury, Venus, Earth, Mars, Jupiter, Saturn, Uranus, Neptune$\}$. $\square$

Exercise 21. Let $H = \{x :\ x \text{ is a month of the year } \}$.

Express the set via the roster method; that is, list its elements.

Solution. Since H is the set of all months of the year, then $H = \{$ January, February, March, April, May, June, July, August, September, October, November, December$\}$. $\square$

Exercise 22. Let $I = \{a \in \mathbb{R} : f(x) = \dfrac{x}{x^2 - 3x + 2} \text{ is discontinuous at } x = a \ \}$.

Express the set via the roster method; that is, list its elements.

Solution. The rational function f given by $f(x) = \dfrac{x}{x^2 - 3x + 2} = \dfrac{x}{(x-2)(x-1)}$ has domain all real numbers x except $x = 1$ and $x = 2$.

Therefore, the domain of f is the set of all real numbers excluding 1 and 2.

The function f is discontinuous at $x = 1$, since the line $x = 1$ is a vertical asymptote.

The function f is discontinuous at $x = 2$, since the line $x = 2$ is a vertical asymptote.

Therefore, $I = \{1, 2\}$. $\square$

Exercise 23. Let $J = \{a \in \mathbb{R} : f(x) = |x| \text{ fails to have a derivative at } x = a \ \}$.

Express the set via the roster method; that is, list its elements.

Solution. The absolute value function f is defined by $f(x) = |x|$ for all $x \in \mathbb{R}$.

Since $0 \in \mathbb{R}$ and every real number is an accumulation point of $\mathbb{R}$, then 0 is an accumulation point of $\mathbb{R}$, so 0 is an accumulation point of $\mathbb{R} - \{0\} = \mathbb{R}^*$.

Let $q : \mathbb{R}^* \to \mathbb{R}$ be a function defined by $q(x) = \dfrac{f(x) - f(0)}{x - 0} = \dfrac{|x|}{x}$ for all $x \in \mathbb{R}^*$.

We consider the limit $\lim_{x\to 0} q(x) = \lim_{x\to 0} \frac{|x|}{x}$.
Let $x \in \mathbb{R}^*$.
Then $x \in \mathbb{R}$ and $x \neq 0$, so either $x > 0$ or $x < 0$.
If $x > 0$, then $\lim_{x\to 0} q(x) = \lim_{x\to 0} \frac{|x|}{x} = \lim_{x\to 0} \frac{x}{x} = \lim_{x\to 0} 1 = 1$.
If $x < 0$, then $\lim_{x\to 0} q(x) = \lim_{x\to 0} \frac{|x|}{x} = \lim_{x\to 0} \frac{-x}{x} = \lim_{x\to 0} -1 = -1$.
Therefore, $\lim_{x\to 0} q(x)$ does not exist, so f is not differentiable at 0.
Hence, f fails to have a derivative at $x = 0$.
Thus, $J = \{0\}$. □

Exercise 24. Let $K = \{a \in \mathbb{R} : f(x) = 3x^4+4x^3-12x^2$ has a relative maximum at $x = a$ $\}$.
Express the set via the roster method; that is, list its elements.

Solution. Let $f : \mathbb{R} \to \mathbb{R}$ be the polynomial function defined by $f(x) = 3x^4 + 4x^3 - 12x^2$ for all $x \in \mathbb{R}$.
We use the first derivative test to find the relative maxima of f, if any exist.
Since polynomial functions are continuous, then f is continuous, so f is continuous on $\mathbb{R}$.
Observe that $f'(x) = 12x^3 + 12x^2 - 24x$ for all $x \in \mathbb{R}$, so f is differentiable on $\mathbb{R}$.
Since $f'(x) = 0$ iff $0 = 12x^3+12x^2-24x = 12x(x^2+x-2) = 12x(x+2)(x-1)$, then $f'(x) = 0$ iff $12x(x+2)(x-1) = 0$.
Thus, $f'(x) = 0$ whenever $x = 0$ or $x = -2$ or $x = 1$.
The critical points are: $-2, 0, 1$.

We determine the sign of $f'(x)$ on the open intervals: $(-\infty, -2), (-2, 0), (0, 1), (1, \infty)$.
We determine the sign of $f'(x)$ on the interval $(-\infty, -2)$.
Let $x \in (-\infty, -2)$.
Then $x < -2$, so $x + 2 < 0$ and $x - 1 < -3$.
Since $x < -2 < 0$, then $x < 0$.
Since $x - 1 < -3 < 0$, then $x - 1 < 0$.
Since $12 > 0$ and $x < 0$ and $x+2 < 0$ and $x-1 < 0$, then $12x(x+2)(x-1) < 0$, so $f'(x) < 0$.
Therefore, $f'(x) < 0$ for all $x \in (-\infty, -2)$.

We determine the sign of $f'(x)$ on the interval $(-2, 0)$.
Let $x \in (-2, 0)$.
Then $-2 < x < 0$, so $-2 < x$ and $x < 0$.
Since $-2 < x$, then $0 < x + 2$, so $x + 2 > 0$.
Since $x < 0$, then $x - 1 < -1 < 0$, so $x - 1 < 0$.
Since $12 > 0$ and $x < 0$ and $x+2 > 0$ and $x-1 < 0$, then $12x(x+2)(x-1) < 0$, so $f'(x) > 0$.
Therefore, $f'(x) > 0$ for all $x \in (-2, 0)$.

We determine the sign of $f'(x)$ on the interval $(0, 1)$.

Let $x \in (0, 1)$.

Then $0 < x < 1$, so $0 < x$ and $x < 1$.

Since $0 < x$, then $x > 0$.

Since $x > 0$, then $x + 2 > 2 > 0$, so $x + 2 > 0$.

Since $x < 1$, then $x - 1 < 0$.

Since $12 > 0$ and $x > 0$ and $x+2 > 0$ and $x-1 < 0$, then $12x(x+2)(x-1) < 0$, so $f'(x) < 0$.

Therefore, $f'(x) < 0$ for all $x \in (0, 1)$.

We determine the sign of $f'(x)$ on the interval $(1, \infty)$.

Let $x \in (1, \infty)$.

Then $x > 1$, so $x - 1 > 0$.

Since $x > 1 > 0$, then $x > 0$.

Since $x > 1$, then $x + 2 > 3 > 0$, so $x + 2 > 0$.

Since $12 > 0$ and $x > 0$ and $x+2 > 0$ and $x-1 > 0$, then $12x(x+2)(x-1) < 0$, so $f'(x) > 0$.

Therefore, $f'(x) > 0$ for all $x \in (1, \infty)$.

Consequently, the sign of $f'(x)$ for each interval is as follows.

$f'(x) < 0$ for all $x \in (-\infty, -2)$

$f'(x) > 0$ for all $x \in (-2, 0)$

$f'(x) < 0$ for all $x \in (0, 1)$

$f'(x) > 0$ for all $x \in (1, \infty)$

We find the relative extrema of f at each critical point.

We find the relative extremum at critical point $c = -2$.

Let $\delta = 2$.

The δ neighborhood $N(-2; 2) = (-4, 0)$ is a subset of $\mathbb{R}$, the domain of f.

Since f is continuous on $\mathbb{R}$, then f is continuous on the open interval $(-4, 0)$.

Observe that f is differentiable on the open intervals $(-4, -2)$ and $(-2, 0)$.

Since $N(-2; 2) = (-4, 0)$ is a subset of the domain of f, and f is continuous on the open interval $(-4, 0)$, and f is differentiable on the open intervals $(-4, -2)$ and $(-2, 0)$, and $f'(x) < 0$ for all $x \in (-4, -2)$ and $f'(x) > 0$ for all $x \in (-2, 0)$, then by the first derivative test, $f(-2)$ is a relative minimum.

We find the relative extremum at critical point $c = 0$.

Let $\delta = 1$.

The δ neighborhood $N(0; 1) = (-1, 1)$ is a subset of $\mathbb{R}$, the domain of f.

Since f is continuous on $\mathbb{R}$, then f is continuous on the open interval $(-1, 1)$.

Observe that f is differentiable on the open intervals $(-1, 0)$ and $(0, 1)$.

Since $N(0; 1) = (-1, 1)$ is a subset of the domain of f, and f is continuous on the open interval $(-1, 1)$, and f is differentiable on the open intervals $(-1, 0)$ and $(0, 1)$, and $f'(x) > 0$ for all $x \in (-1, 0)$ and $f'(x) < 0$ for all $x \in (0, 1)$, then by the first derivative test, $f(0)$ is a relative maximum.

We find the relative extremum at critical point $c = 1$.

Let $\delta = 1$.

The δ neighborhood $N(1;1) = (0,2)$ is a subset of $\mathbb{R}$, the domain of f.

Since f is continuous on $\mathbb{R}$, then f is continuous on the open interval $(0,2)$.

Observe that f is differentiable on the open intervals $(0,1)$ and $(1,2)$.

Since $N(1;1) = (0,2)$ is a subset of the domain of f, and f is continuous on the open interval $(0,2)$, and f is differentiable on the open intervals $(0,1)$ and $(1,2)$, and $f'(x) < 0$ for all $x \in (0,1)$ and $f'(x) > 0$ for all $x \in (1,2)$, then by the first derivative test, $f(1)$ is a relative minimum.

Therefore, $f(-2)$ and $f(1)$ are relative minima of f, and $f(0)$ is the relative maximum of f.

Since $f(0)$ is the only relative maximum of f, then $K = \{0\}$. $\square$

Exercise 25. Let $L = \{x \in \mathbb{R} : \sqrt{x+2} = \sqrt{7-x} - 3\}$.

Express the set via the roster method; that is, list its elements.

Solution. Let $x \in L$.

Then $x \in \mathbb{R}$ and $\sqrt{x+2} = \sqrt{7-x} - 3$.

The universal set is $\mathbb{R}$.

Since the square root of a real number is always non-negative, then $x+2 \geq 0$ and $7 - x \geq 0$, so $x \geq -2$ and $7 \geq x$.

Thus, $-2 \leq x$ and $x \leq 7$, so $-2 \leq x \leq 7$.

Since $\sqrt{x+2} = \sqrt{7-x} - 3$, then $3 = \sqrt{7-x} - \sqrt{x+2}$.

We square both sides to obtain $9 = (7-x) - 2\sqrt{(7-x)(x+2)} + (x+2)$.

Hence, $9 = 9 - 2\sqrt{(7-x)(x+2)}$, so $2\sqrt{(7-x)(x+2)} = 0$.

Thus, $\sqrt{(7-x)(x+2)} = 0$, so $(7-x)(x+2) = 0$.

Therefore, either $x = 7$ or $x = -2$.

If $x = -2$, then $\sqrt{-2+2} = 0$ and $\sqrt{7-(-2)} - 3 = 0$, so $-2 \in L$.

If $x = 7$, then $\sqrt{7+2} = 3$ and $\sqrt{7-7} - 3 = -3$, so $7 \notin L$.

Consequently, the only solution is -2.

Therefore, $x \in \{-2\}$, so $L \subseteq \{-2\}$.

Since $-2 \in L$, then $\{-2\} \subseteq L$.

Since $L \subseteq \{-2\}$ and $\{-2\} \subseteq L$, then $L = \{-2\}$. $\square$

Exercise 26. Let $M = \{x \in \mathbb{R} : |2x^2 + 2x - 1| = |x^2 - 4x - 6|\}$.

Express the set via the roster method; that is, list its elements.

Solution. Let $x \in M$.

Then $x \in \mathbb{R}$ and $|2x^2 + 2x - 1| = |x^2 - 4x - 6|$.

Let $a = x^2 - 4x - 6$.

Then $|2x^2 + 2x - 1| = |a|$, so either $2x^2 + 2x - 1 = |a|$ or $2x^2 + 2x - 1 = -|a|$.

Hence, either $|a| = 2x^2 + 2x - 1$, or $|a| = -(2x^2 + 2x - 1)$, so either $a = 2x^2 + 2x - 1$ or $a = -(2x^2 + 2x - 1)$, or $a = -(2x^2 + 2x - 1)$ or $a = 2x^2 + 2x - 1$.

Therefore, either $a = 2x^2 + 2x - 1$ or $a = -(2x^2 + 2x - 1)$.

We consider these cases separately.

Case 1: Suppose $a = 2x^2 + 2x - 1$.
Then $x^2 - 4x - 6 = 2x^2 + 2x - 1$, so $x^2 + 6x + 5 = 0$.
Hence, $(x+5)(x+1) = 0$, so either $x = -5$ or $x = -1$.
Case 2: Suppose $a = -(2x^2 + 2x - 1)$.
Then $x^2 - 4x - 6 = -2x^2 - 2x + 1$, so $3x^2 - 2x - 7 = 0$.
Hence, either $x = \frac{1+\sqrt{22}}{3}$ or $x = \frac{1-\sqrt{22}}{3}$.

Thus, in all cases, either $x = -5$ or $x = -1$ or $x = \frac{1+\sqrt{22}}{3}$ or $x = \frac{1-\sqrt{22}}{3}$, so $x \in \{-5, -1, \frac{1+\sqrt{22}}{3}, \frac{1+\sqrt{22}}{3}\}$.

Therefore, $M \subseteq \{-5, -1, \frac{1+\sqrt{22}}{3}, \frac{1-\sqrt{22}}{3}\}$.

We verify each of the four solutions satisfies the equation $|2x^2 + 2x - 1| = |x^2 - 4x - 6|$.

Hence, $\{-5, -1, \frac{1+\sqrt{22}}{3}, \frac{1-\sqrt{22}}{3}\} \subseteq M$.

Since $M \subseteq \{-5, -1, \frac{1+\sqrt{22}}{3}, \frac{1-\sqrt{22}}{3}\}$ and $\{-5, -1, \frac{1+\sqrt{22}}{3}, \frac{1-\sqrt{22}}{3}\} \subseteq M$, then $M = \{-5, -1, \frac{1+\sqrt{22}}{3}, \frac{1-\sqrt{22}}{3}\}$. $\square$

Exercise 27. Let $N = \{z \in \mathbb{C} : z^2 = -1\}$.
Express the set via the roster method; that is, list its elements.

Solution. Let $z \in N$.
Then $z \in \mathbb{C}$ and $z^2 = -1$, so $z^2 + 1 = 0$.
Hence, $(z - i)(z + i) = 0$, so either $z = i$ or $z = -i$.
Therefore, $z \in \{i, -i\}$, so $N \subseteq \{i, -i\}$.

Since $i \in \mathbb{C}$ and $i^2 = -1$, then $i \in N$.
Since $-i \in \mathbb{C}$ and $(-i)^2 = -1$, then $-i \in N$.
Since $i \in N$ and $-i \in N$, then $\{i, -i\} \subseteq N$.

Since $N \subseteq \{i, -i\}$ and $\{i, -i\} \subseteq N$, then $N = \{i, -i\}$. $\square$

Exercise 28. Let $O = \{z \in \mathbb{C} : z^4 = 1\}$.
Express the set via the roster method; that is, list its elements.

Solution. Let $z \in O$.
Then $z \in \mathbb{C}$ and $z^4 = 1$, so $0 = z^4 - 1 = (z^2 - 1)(z^2 + 1) = (z - 1)(z + 1)(z - i)(z + i)$.
Hence, either $z = 1$ or $z = -1$ or $z = i$ or $z = -i$.
Therefore, $z \in \{1, -1, i, -i\}$, so $O \subseteq \{1, -1, i, -i\}$.

Since $1 \in \mathbb{C}$ and $1^4 = 1$, then $1 \in O$.
Since $-1 \in \mathbb{C}$ and $(-1)^4 = 1$, then $-1 \in O$.
Since $i \in \mathbb{C}$ and $i^4 = 1$, then $i \in O$.
Since $-i \in \mathbb{C}$ and $(-i)^4 = 1$, then $-i \in O$.
Since $1 \in O$ and $-1 \in O$ and $i \in O$ and $-i \in O$, then $\{1, -1, i, -i\} \subseteq O$.

Since $O \subseteq \{1, -1, i, -i\}$ and $\{1, -1, i, -i\} \subseteq O$, then $O = \{1, -1, i, -i\}$. $\square$

Exercise 29. Let $A = \{x \in \mathbb{R} : \sin(2x) = 2\sin x \cos x\}$.
Express the set by either interval notation(including $\emptyset$), or by $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$, or $\mathbb{C}$.

Solution. Let $x \in \mathbb{R}$.
Observe that

$$\begin{aligned} \sin 2x &= \sin(x + x) \\ &= \sin x \cos x + \cos x \sin x \\ &= \sin x \cos x + \sin x \cos x \\ &= 2\sin x \cos x. \end{aligned}$$

Therefore, $\sin 2x = 2\sin x \cos x$, so $\sin 2x = 2\sin x \cos x$ for all $x \in \mathbb{R}$.
Since A is the set of all real numbers x such that $\sin 2x = 2\sin x \cos x$, then $A = \mathbb{R}$. $\square$

Exercise 30. Let $B = \{x \in \mathbb{R} : \sin(\pi x) = 0\}$.
Express the set by either interval notation(including $\emptyset$), or by $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$, or $\mathbb{C}$.

Solution. Let $x \in \mathbb{R}$.
Since $\sin x = 0$ iff $x = n\pi$ for any integer n, then $\sin(\pi x) = 0$ iff $\pi x = n\pi$ for any integer n iff $x = n$ for any integer n iff x is any integer n iff $x \in \mathbb{Z}$.
Therefore, $\sin(\pi x) = 0$ iff $x \in \mathbb{Z}$, so $x \in B$ iff $x \in \mathbb{R}$ and $x \in \mathbb{Z}$.
Hence, $x \in B$ iff $x \in \mathbb{R} \cap \mathbb{Z}$.
Since $\mathbb{Z} \subseteq \mathbb{R}$, then $\mathbb{Z} \cap \mathbb{R} = \mathbb{Z}$.
Thus, $B = \mathbb{Z}$. $\square$

Exercise 31. Let $C = \{x \in \mathbb{R} : x^2 = -1\}$.
Express the set by either interval notation(including $\emptyset$), or by $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$, or $\mathbb{C}$.

Solution. Since $x^2 \geq 0$ for all $x \in \mathbb{R}$, then there is no real number x such that $x^2 = -1$.
Therefore, $C = \emptyset$. $\square$

Exercise 32. Let $D = \{x \in \mathbb{R} : x^2 - 5x + 7 < 0\}$.
Express the set by either interval notation(including $\emptyset$), or by $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$, or $\mathbb{C}$.

Solution. Let $x \in \mathbb{R}$.

Observe that

$$\begin{aligned} x^2 - 5x + 7 &= (x - \frac{5}{2})^2 + 7 - (\frac{5}{2})^2 \\ &= (x - \frac{5}{2})^2 + \frac{3}{4} \\ &> 0. \end{aligned}$$

Hence, $x^2 - 5x + 7 > 0$ for all $x \in \mathbb{R}$, so there is no $x \in \mathbb{R}$ such that $x^2 - 5x + 7 < 0$.

Therefore, $D = \emptyset$. $\square$

Exercise 33. Let $E = \{x \in \mathbb{R} : 10x^2 - 7x - 12 \leq 0\}$.

Express the set by either interval notation(including $\emptyset$), or by $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$, or $\mathbb{C}$.

Solution. Let $x \in E$.

Then $x \in \mathbb{R}$ and $10x^2 - 7x - 12 \leq 0$, so $(2x - 3)(5x + 4) \leq 0$.

Hence, either $(2x-3)(5x+4) < 0$ or $(2x-3)(5x+4) = 0$, so either $2x-3 > 0$ and $5x + 4 < 0$, or $2x - 3 < 0$ and $5x + 4 > 0$, or $(2x - 3)(5x + 4) = 0$.

We consider these cases separately.

Case 1: Suppose $(2x - 3)(5x + 4) = 0$.

Then either $2x - 3 = 0$ or $5x + 4 = 0$, so either $x = \frac{3}{2}$ or $x = -\frac{4}{5}$.

Case 2: Suppose $2x - 3 > 0$ and $5x + 4 < 0$.

Then $x > \frac{3}{2}$ and $x < -\frac{4}{5}$.

But, there is no real number that is both greater than $\frac{3}{2}$ and less than $-\frac{4}{5}$.

Therefore, this case is not possible.

Case 3: Suppose $2x - 3 < 0$ and $5x + 4 > 0$.

Then $x < \frac{3}{2}$ and $x > -\frac{4}{5}$, so $-\frac{4}{5} < x < \frac{3}{2}$.

Thus, in all cases, either $x = \frac{3}{2}$ or $x = -\frac{4}{5}$, or $-\frac{4}{5} < x < \frac{3}{2}$, so $-\frac{4}{5} \leq x \leq \frac{3}{2}$.

Hence, $x \in [-\frac{4}{5}, \frac{3}{2}]$.

Therefore, $x \in E$ implies $x \in [-\frac{4}{5}, \frac{3}{2}]$, so $E \subseteq [-\frac{4}{5}, \frac{3}{2}]$.

Let $y \in [-\frac{4}{5}, \frac{3}{2}]$.

Then $y \in \mathbb{R}$ and $-\frac{4}{5} \leq y \leq \frac{3}{2}$, so $-\frac{4}{5} \leq y$ and $y \leq \frac{3}{2}$.

Hence, $-4 \leq 5y$ and $2y \leq 3$, so $0 \leq 5y + 4$ and $2y - 3 \leq 0$.

Since $5y+4 \geq 0$ and $2y-3 \leq 0$, then $(5y+4)(2y-3) \leq 0$, so $10y^2-7y-12 \leq 0$.

Since $y \in \mathbb{R}$ and $10y^2 - 7y - 12 \leq 0$, then $y \in E$.

Thus, $y \in [-\frac{4}{5}, \frac{3}{2}]$ implies $y \in E$, so $[-\frac{4}{5}, \frac{3}{2}] \subseteq E$.

Since $E \subseteq [-\frac{4}{5}, \frac{3}{2}]$ and $[-\frac{4}{5}, \frac{3}{2}] \subseteq E$, then $E = [-\frac{4}{5}, \frac{3}{2}]$. $\square$

Exercise 34. Let $F = \{x \in \mathbb{R} : |6x - 8| \leq 4\}$.

Express the set by either interval notation(including $\emptyset$), or by $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$, or $\mathbb{C}$.

Solution. Let $x \in F$.

Then $x \in \mathbb{R}$ and $|6x - 8| \leq 4$.

Observe that

$$\begin{aligned} |6x - 8| \leq 4 &\Leftrightarrow -4 \leq 6x - 8 \leq 4 \\ &\Leftrightarrow \frac{2}{3} \leq x \leq 12 \\ &\Leftrightarrow x \in [\frac{2}{3}, 2]. \end{aligned}$$

Hence, $x \in [\frac{2}{3}, 2]$.

Thus, $x \in F$ implies $x \in [\frac{2}{3}, 2]$, so $F \subseteq [\frac{2}{3}, 2]$.

Let $y \in [\frac{2}{3}, 2]$.

Then $y \in \mathbb{R}$ and $\frac{2}{3} \leq y \leq 2$, so $4 \leq 6y \leq 12$.

Hence, $-4 \leq 6y - 8 \leq 4$, so $|6y - 8| \leq 4$.

Since $y \in \mathbb{R}$ and $|6y - 8| \leq 4$, then $y \in F$.

Thus, $y \in [\frac{2}{3}, 2]$ implies $y \in F$, so $[\frac{2}{3}, 2] \subseteq F$.

Since $F \subseteq [\frac{2}{3}, 2]$ and $[\frac{2}{3}, 2] \subseteq F$, then $F = [\frac{2}{3}, 2]$. $\square$

Exercise 35. Let $G = \{x \in \mathbb{R} : |7x - 12| < 0\}$.

Express the set by either interval notation(including $\emptyset$), or by $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$, or $\mathbb{C}$.

Solution. Let $x \in \mathbb{R}$.

Then $7x - 12 \in \mathbb{R}$.

Since $|x| \geq 0$ for all $x \in \mathbb{R}$, then in particular, $|7x - 12| \geq 0$.

Hence, $|7x - 12| \geq 0$ for all $x \in \mathbb{R}$, so there is no real number x such that $|7x - 12 < 0$.

Therefore, $G = \emptyset$. $\square$

Exercise 36. Let $H = \{x \in \mathbb{R} : |9x + 13| \leq 0\}$.

Express the set by either interval notation(including $\emptyset$), or by $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$, or $\mathbb{C}$.

Solution. Let $x \in H$.

Then $x \in \mathbb{R}$ and $|9x + 13| \leq 0$.

Since $x \in \mathbb{R}$, then $9x + 13 \in \mathbb{R}$.

Since $|x| \geq 0$ for all $x \in \mathbb{R}$, then $|9x + 13| \geq 0$.

Since $|9x + 13| \leq 0$ and $|9x + 13| \geq 0$, then $|9x + 13| = 0$, so $9x + 13 = 0$.

Hence, $x = -\frac{13}{9}$, so $x \in \{-\frac{13}{9}\}$.

Thus, $x \in H$ implies $x \in \{-\frac{13}{9}\}$, so $H \subseteq \{-\frac{13}{9}\}$.

Since $-\frac{13}{9} \in \mathbb{R}$ and $|9(-\frac{13}{9}) + 13| = 0$, then $-\frac{13}{9} \in H$.

Hence, $\{-\frac{13}{9}\} \subseteq H$.

Since $H \subseteq \{-\frac{13}{9}\}$ and $\{-\frac{13}{9}\} \subseteq H$, then $H = \{-\frac{13}{9}\} = [-\frac{13}{9}, -\frac{13}{9}]$. $\square$

Exercise 37. Let $I = \{x \in \mathbb{R} : x > 0 \text{ and } \cos(\pi x) = \cot(\pi x)\}$.

Express the set by either interval notation(including $\emptyset$), or by $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$, or $\mathbb{C}$.

Solution. Let $x \in I$.

Then $x \in \mathbb{R}$ and $x > 0$ and $\cos(\pi x) = \cot(\pi x)$, so $\cos(\pi x) - \cot(\pi x) = 0$.

Thus, $\cos(\pi x) - \frac{\cos(\pi x)}{\sin(\pi x)} = 0$, so $\cos(\pi x)[1 - \frac{1}{\sin(\pi x)}] = 0$ and $\sin(\pi x) \neq 0$.

Since $\cos(\pi x)[1 - \frac{1}{\sin(\pi x)}] = 0$, then either $\cos(\pi x) = 0$ or $1 - \frac{1}{\sin(\pi x)} = 0$.

We consider these cases separately.

Case 1: Suppose $\cos(\pi x) = 0$.

Observe that $\cos(x) = 0$ iff x is an odd multiple of $\frac{\pi}{2}$ for all $x \in \mathbb{R}$.

Since $x > 0$, then $\cos(x) = 0$ iff x is an odd positive multiple of $\frac{\pi}{2}$.

Hence, $\cos(\pi x) = 0$ iff πx is an odd positive multiple of $\frac{\pi}{2}$.

Therefore, $\cos(\pi x) = 0$ iff $\pi x = \frac{(2k-1)\pi}{2}$ for all $k \in \mathbb{Z}^+$.

Since $\cos(\pi x) = 0$, then $\pi x = \frac{(2k-1)\pi}{2}$ for all $k \in \mathbb{Z}^+$, so $x = \frac{(2k-1)}{2}$ for all $k \in \mathbb{Z}^+$.

Case 2: Suppose $1 - \frac{1}{\sin(\pi x)} = 0$.

Then $\frac{1}{\sin(\pi x)} = 1$, so $\sin(\pi x) = 1$ provided that $\sin(\pi x) \neq 0$.

Since $x > 0$, then $\sin(x) = 1$ iff $x = \frac{\pi}{2}, \frac{5\pi}{2}, \frac{9\pi}{2} \frac{13\pi}{2}, ...$

Observe that $x = \frac{\pi}{2}, \frac{5\pi}{2}, \frac{9\pi}{2} \frac{13\pi}{2}, ...$ iff $x = \frac{(4k-3)\pi}{2}$ for all $k \in \mathbb{Z}^+$.

Hence, $\sin(x) = 1$ iff $x = \frac{(4k-3)\pi}{2}$ for all $k \in \mathbb{Z}^+$.
Since $x > 0$, then $\pi x > 0$, so $\sin(\pi x) = 1$ iff $\pi x = \frac{(4k-3)\pi}{2}$ for all $k \in \mathbb{Z}^+$.
Thus, $\sin(\pi x) = 1$ iff $x = \frac{4k-3}{2}$ for all $k \in \mathbb{Z}^+$.
Since $\sin(\pi x) = 1$, then $x = \frac{4k-3}{2}$ for all $k \in \mathbb{Z}^+$.

In all cases, either $x = \frac{(2k-1)}{2}$ for all $k \in \mathbb{Z}^+$ or $x = \frac{4k-3}{2}$ for all $k \in \mathbb{Z}^+$.
Hence, either $x = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \frac{7}{2}, \frac{9}{2}, \frac{11}{2}, \frac{13}{2}, ...$ or $x = \frac{1}{2}, \frac{5}{2}, \frac{9}{2}, \frac{13}{2}, ...$
We conclude $x = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \frac{7}{2}, \frac{9}{2}, \frac{11}{2}, \frac{13}{2}, ...$, so $x \in \{\frac{2k-1}{2} : k \in \mathbb{Z}^+\}$.
Therefore, $x \in I$ implies $x \in \{\frac{2k-1}{2} : k \in \mathbb{Z}^+\}$, so $I \subseteq \{\frac{2k-1}{2} : k \in \mathbb{Z}^+\}$.

Let $y \in \{\frac{2k-1}{2} : k \in \mathbb{Z}^+\}$.
Then $y = \frac{2k-1}{2}$ for some $k \in \mathbb{Z}^+$.
Observe that $\sin(\frac{(2k-1)\pi}{2}) = 1$ or $\sin(\frac{(2k-1)\pi}{2}) = -1$, so $\sin(\frac{(2k-1)\pi}{2}) \neq 0$.
Hence, $\sin(\pi y) \neq 0$.
Observe that $\cos(\frac{(2k-1)\pi}{2}) = 0$, so $\cos(\pi y) = 0$.
Observe that

$$\begin{aligned}
\cos(\pi y) - \cot(\pi y) &= \cos(\pi y) - \frac{\cos(\pi y)}{\sin(\pi y)} \\
&= 0 - \frac{0}{\sin(\pi y)} \\
&= 0.
\end{aligned}$$

Thus, $\cos(\pi y) - \cot(\pi y) = 0$, so $\cos(\pi y) = \cot(\pi y)$.

Since $k \in \mathbb{Z}^+$, then $k \geq 1$, so $2k \geq 2$.
Hence, $2k - 1 \geq 1 > 0$, so $2k - 1 > 0$.
Since $k \in \mathbb{Z}$, then $2k - 1 \in \mathbb{Z}$.
Since $y = \frac{2k-1}{2}$ and $2k - 1 \in \mathbb{Z}$ and $2k - 1 > 0$, then $y \in \mathbb{Q}$ and $y > 0$.
Since $y \in \mathbb{Q}$ and $\mathbb{Q} \subset \mathbb{R}$, then $y \in \mathbb{R}$.
Since $y \in \mathbb{R}$ and $y > 0$ and $\cos(\pi y) = \cot(\pi y)$, then $y \in I$.
Therefore, $y \in \{\frac{2k-1}{2} : k \in \mathbb{Z}^+\}$ implies $y \in I$, so $\{\frac{2k-1}{2} : k \in \mathbb{Z}^+\} \subseteq I$.

Since $I \subseteq \{\frac{2k-1}{2} : k \in \mathbb{Z}^+\}$ and $\{\frac{2k-1}{2} : k \in \mathbb{Z}^+\} \subseteq I$, then $I = \{\frac{2k-1}{2} : k \in \mathbb{Z}^+\} = \{\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \frac{7}{2}, \frac{9}{2}, \frac{11}{2}, \frac{13}{2}, ...\}$. $\square$

Exercise 38. Let $J = \{x \in \mathbb{R} : \frac{\sec(x)}{\cos(x) + \tan(x)} = \sin(x)\}$.

Express the set by either interval notation(including $\emptyset$), or by $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$, or $\mathbb{C}$.

Solution. Let $x \in J$.

Then $x \in \mathbb{R}$ and $\frac{\sec(x)}{\cos(x) + \tan(x)} = \sin(x)$.

Observe that

$$\begin{aligned} \sin x &= \frac{\sec(x)}{\cos(x) + \tan(x)} \\ &= \frac{\frac{1}{\cos x}}{\cos x + \frac{\sin x}{\cos x}} \\ &= \frac{1}{\cos^2 x + \sin x} \\ &= \frac{1}{(1 - \sin^2 x) + \sin x}. \end{aligned}$$

Thus, $\cos x \neq 0$ and $\sin x = \frac{1}{(1 - \sin^2 x) + \sin x}$.

Let $y = \sin x$.

Then $y = \frac{1}{1 - y^2 + y}$, so $y - y^3 + y^2 = 1$.

Hence, $y^3 - y^2 - y + 1 = 0$.

Observe that

$$\begin{aligned} 0 &= y^3 - y^2 - y + 1 \\ &= y^2(y-1) - (y-1) \\ &= (y^2 - 1)(y - 1) \\ &= (y-1)(y+1)(y-1). \end{aligned}$$

Thus, $(y-1)(y+1)(y-1) = 0$, so $y = \pm 1$.

Hence, $\sin x = \pm 1$, so either $\sin x = 1$ or $\sin x = -1$.

We consider these cases separately.

Case 1: Suppose $\sin x = 1$.

Then $x = 2n\pi + \frac{\pi}{2}$ for any integer n, so $x \in \{..., -\frac{3\pi}{2}, \frac{\pi}{2}, \frac{5\pi}{2}, \frac{9\pi}{2}, ...\}$.

Case 2: Suppose $\sin x = -1$.

Then $x = 2n\pi - \frac{\pi}{2}$ for any integer n, so $x \in \{..., -\frac{\pi}{2}, \frac{3\pi}{2}, \frac{7\pi}{2}, \frac{11\pi}{2}, ...\}$.

Therefore, in all cases, $x \in \{..., -\frac{-3\pi}{2}, \frac{-\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}, \frac{9\pi}{2}, \frac{11\pi}{2}, ...\}$.

Since $\cos x = 0$ iff $x \in \{..., \frac{-3\pi}{2}, \frac{-\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}, \frac{9\pi}{2}, ...\}$, then $\cos x \neq 0$ iff $x \notin \{..., \frac{-3\pi}{2}, \frac{-\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}, \frac{9\pi}{2}, ...\}$.

Since $\cos x \neq 0$, then $x \notin \{..., \frac{-3\pi}{2}, \frac{-\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}, \frac{9\pi}{2}, ...\}$.

Since $x \in \{..., -\frac{-3\pi}{2}, \frac{-\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}, \frac{9\pi}{2}, \frac{11\pi}{2}, ...\}$, then we must conclude no such real x exists.

Therefore, $J = \emptyset$. $\square$

Exercise 39. Let $K = \{x \in \mathbb{R} : \sqrt{2x+7} \in \mathbb{R}\}$.

Express the set by either interval notation(including $\emptyset$), or by $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$, or $\mathbb{C}$.

Solution. Let $x \in K$.

Then $x \in \mathbb{R}$ and $\sqrt{2x+7} \in \mathbb{R}$, so $\sqrt{2x+7} \geq 0$.

Hence, $2x + 7 \geq 0$, so $x \geq -\frac{7}{2}$.

Thus, $x \in [-\frac{7}{2}, \infty)$.

Therefore, $x \in K$ implies $x \in [-\frac{7}{2}, \infty)$, so $K \subseteq [-\frac{7}{2}, \infty)$.

Let $y \in [-\frac{7}{2}, \infty)$.

Then $y \in \mathbb{R}$ and $y \geq -\frac{7}{2}$, so $2y \geq -7$.

Thus, $2y + 7 \geq 0$.

Since $y \in \mathbb{R}$, then $2y + 7 \in \mathbb{R}$.

Since $2y + 7 \in \mathbb{R}$ and $2y + 7 \geq 0$, then $\sqrt{2y+7} \in \mathbb{R}$.

Since $y \in \mathbb{R}$ and $\sqrt{2y+7} \in \mathbb{R}$, then $y \in K$.

Hence, $y \in [-\frac{7}{2}, \infty)$ implies $y \in K$, so $[-\frac{7}{2}, \infty) \subseteq K$.

Since $K \subseteq [-\frac{7}{2}, \infty)$ and $[-\frac{7}{2}, \infty) \subseteq K$, then $K = [-\frac{7}{2}, \infty)$. $\square$

Exercise 40. Let $L = \{x \in \mathbb{Q} : x^2 + (3 - \sqrt{2})x + 3\sqrt{2} = 0\}$.

Express the set by either interval notation(including $\emptyset$), or by $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$, or $\mathbb{C}$.

Solution. Let $x \in L$.

Then $x \in \mathbb{Q}$ and $x^2 + (3 - \sqrt{2})x + 3\sqrt{2} = 0$.

Since $x \in \mathbb{Q}$, then there exist integers a and b with $b \neq 0$ such that $x = \frac{a}{b}$.

Since $x^2 + (3 - \sqrt{2})x + 3\sqrt{2} = 0$, then by applying the quadratic formula, we obtain

$x = \dfrac{-3+\sqrt{2} \pm \sqrt{11-18\sqrt{2}}}{2}$.

Hence, $a = -3+\sqrt{2}+\sqrt{11-18\sqrt{2}}$ or $a = -3+\sqrt{2}-\sqrt{11-18\sqrt{2}}$, and $b = 2$.

Since either $a = -3+\sqrt{2}+\sqrt{11-18\sqrt{2}}$ or $a = -3+\sqrt{2}-\sqrt{11-18\sqrt{2}}$, then $a \notin \mathbb{Z}$.

But, this contradicts $a \in \mathbb{Z}$.

Thus, no such x exists, so L is empty.

Therefore, $L = \emptyset$. $\square$

Exercise 41. Let $M = \{z \in \mathbb{C} : z\bar{z} = |z|^2\}$.

Express the set by either interval notation(including $\emptyset$), or by $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$, or $\mathbb{C}$.

Solution. Observe that $M \subseteq \mathbb{C}$.

Let $z \in \mathbb{C}$.

Then there exist real numbers a and b such that $z = a + bi$.

Thus, $|z|^2 = a^2 + b^2$ and $\bar{z} = a - bi$.

Observe that

$$\begin{array}{rcl} z\bar{z} & = & (a+bi)(a-bi) \\ & = & a^2 - b^2(i^2) \\ & = & a^2 - b^2(-1) \\ & = & a^2 + b^2 \\ & = & |z|^2. \end{array}$$

Hence, $z\bar{z} = |z|^2$.

Since $z \in \mathbb{C}$ and $z\bar{z} = |z|^2$, then $z \in M$.

Therefore, $z \in \mathbb{C}$ implies $z \in M$, so $\mathbb{C} \subseteq M$.

Since $M \subseteq \mathbb{C}$ and $\mathbb{C} \subseteq M$, then $M = \mathbb{C}$. $\square$

Exercise 42. Let $N = \{z \in \mathbb{C} : Im(z) = 0\}$.

Express the set by either interval notation(including $\emptyset$), or by $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$, or $\mathbb{C}$.

Solution. Let $z \in N$.

Then $z \in \mathbb{C}$ and $Im(z) = 0$.

Since $z \in \mathbb{C}$, then there exist real numbers a and b such that $z = a + bi$.

Since $b = Im(z) = 0$, then $b = 0$, so $z = a + bi = a + 0i = a + 0 = a$.

Since $z = a$ and $a \in \mathbb{R}$, then $z \in \mathbb{R}$.

Hence, $z \in N$ implies $z \in \mathbb{R}$, so $N \subseteq \mathbb{R}$.

Let $r \in \mathbb{R}$.

Then $r = r + 0i$.

Since $r \in \mathbb{R}$ and $0 \in \mathbb{R}$, then $r \in \mathbb{C}$.

Since $r = r + 0i$, then $Im(r) = 0$.

Since $r \in \mathbb{C}$ and $Im(r) = 0$, then $r \in N$.

Hence, $r \in \mathbb{R}$ implies $r \in N$, so $\mathbb{R} \subseteq N$.

Since $N \subseteq \mathbb{R}$ and $\mathbb{R} \subseteq N$, then $N = \mathbb{R}$. $\square$

Exercise 43. Determine if $\{x : x \text{ is an American citizen on July 4, 1976}\}$ is well defined.

Solution. The description of this set is well-defined. It is possible to determine precisely on the given date whether a person is a US citizen by exhibiting proper government documentation of citizenship status.

The universal set is the set of all humans. $\square$

Exercise 44. Determine if $\{x : x \text{ is the digit in the decimal expansion of } \sqrt{2}\ \}$ is well defined.

Solution. The description of this set is well-defined. It is possible to determine whether a given digit is in the decimal expansion using a computer.

The universal set is the set of decimal digits $\{0, 1, 2, ..., 9\}$. $\square$

Exercise 45. Determine if $\{x : x \text{ is an honest man }\}$ is well defined.

Solution. The description of this set is not well-defined. The definition of an 'honest' man is subjective. $\square$

Exercise 46. Determine if $\{x : x \text{ is a month whose name in the English language ends in the letter } r\}$ is well defined.

Solution. The description of this set is well-defined. One can verify whether an English word is a month name and ends with the specified letter.

The universal set is the set of all months $\{January, February, ..., December\}$. $\square$

Exercise 47. Determine if $\{x : x \text{ is a day in the middle of the week}\}$ is well defined.

Solution. The description of this set is not well-defined. The description 'middle of the week' is ambiguous. $\square$

Exercise 48. Determine if $\{x : \sin(2x)\}$ is well defined.

Solution. The description of this set is not well-defined. The description is not meaningful. $\square$

Exercise 49. Determine if $\{x \in \mathbb{N} : x \text{ is a an integral multiple of } 4\}$ is well defined.

Solution. The description of this set is well-defined. It is possible to determine whether a given natural number is an integral multiple of 4.

The universal set is $\mathbb{N}$. $\square$

Exercise 50. Determine if $\{x : x \text{ is an aardling}\}$ is well defined.

Solution. The description of this set is not well-defined. The description is non-sense. $\square$

Exercise 51. Determine if $\{x : \sqrt{\frac{x^2-6x+3}{x^3+4}}\}$.

Solution. The description of this set is not well-defined. The description is just a mathematical expression and doesn't have a meaning. $\square$

Exercise 52. Determine if $\{x \in \mathbb{R} : x + y = 4\}$.

Solution. The description of this set is not well-defined. One cannot determine whether a given real number x is in the set unless a value of y is known. $\square$

Exercise 53. Given the following pattern description of infinite sets, list five additional elements of each:

$A = \{1, \frac{1}{3}, \frac{1}{9}, ...\}$.

Solution. Set A consists of the numbers 3^{-n} for some integer $n = 0, 1, 2, ...$

Observe that $\frac{1}{27} \in A$ and $\frac{1}{81} \in A$ and $\frac{1}{243} \in A$ and $\frac{1}{729} \in A$ and $\frac{1}{2187} \in A$. $\square$

Exercise 54. Given the following pattern description of infinite sets, list five additional elements of each:

$B = \{1, 2, 3, 5, 8, 13, ...\}$.

Solution. Set B consists of the terms of the Fibonacci sequence, where the next term is the sum of the two previous terms.

Observe that $21 \in B$ and $34 \in B$ and $55 \in B$ and $89 \in B$ and $144 \in B$. $\square$

Exercise 55. Given the following pattern description of infinite sets, list five additional elements of each:

$C = \{-1, 2, -4, 8, ...\}$.

Solution. Set C consists of terms of the sequence whose terms are alternating powers of 2, so the n^{th} term is $(-1)^{n+1}2^n$ for some integer $n = 0, 1, 2, ...$

Thus, additional terms are $-2^4, 2^5, -2^6, 2^7, -2^8$.

Observe that $-16 \in C$ and $32 \in C$ and $-64 \in C$ and $128 \in C$ and $-256 \in C$. $\square$

Exercise 56. Given the following pattern description of infinite sets, list five additional elements of each:

$D = \{\pi, 4\pi, 7\pi, 10\pi, ...\}$.

Solution. Set D is an arithmetic progression with common difference 3π.

Additional elements of D include $13\pi, 16\pi, 19\pi, 22\pi, 25\pi$. $\square$

Exercise 57. Given the following pattern description of infinite sets, list five additional elements of each:

$E = \{..., -8, -5, -2, 1, 4, 7, ...\}$.

Solution. Set E is an arithmetic progression with common difference 3.

Additional elements of E include $10, 13, 16, 19, 22$. $\square$

Exercise 58. Given the following six sets, answer true or false to the below statements.

$A = (-\infty, -7]$
$B = \{4, 8, 12, ..., 96, 100\}$
$C = [-1, 6]$
$D = (-1, 6)$
$E = \{-1, 0, 1, 2, 3, 4, 5, 6\}$
$F = (-\infty, -6)$

a. $-7 \in A$.
b. $6 \in B$.
c. $D \subseteq C$.
d. $C \subseteq D$.
e. $D \subset C$.
f. $E \subseteq D$.
g. $D \subseteq E$.
h. $-6 \in A$.
i. $-6 \in F$.
j. $-\frac{13}{2} \in A$.
k. $-\frac{13}{2} \notin F$.
l. $100 \in B$.
m. $0 \in B$.
n. $-1 \notin D$.

Solution. a. Observe that $A = \{x \in \mathbb{R} : x \leq -7\}$.
Since $-7 \in \mathbb{R}$ and $-7 \leq -7$, then $-7 \in A$, so the statement $-7 \in A$ is true.

b. Observe that B is the set of all multiples of 4 between 4 and 100, so $B = \{4n : n = 1, 2, ..., 25\}$.
Since 6 is not a multiple of 4, then $6 \notin B$, so the statement $6 \in B$ is false.

c. Since D is the open interval $(-1, 6)$ and C is the closed bounded interval $[-1, 6]$, then D is a subset of C, so $D \subseteq C$.
Therefore, $D \subseteq C$ is true.

d. Since $-1 \in C$, but $-1 \notin D$, then C is not a subset of D, so $C \not\subseteq D$.
Therefore, $C \subseteq D$ is false.

e. Since D is a subset of C and C is not a subset of D, then D is a proper subset of C, so $D \subset C$.
Therefore, $D \subset C$ is true.

f. Since $-1 \in E$, but $-1 \notin D$, then E is not a subset of D, so $E \not\subseteq D$.
Therefore, $E \subseteq D$ is false.

g. Since $-\dfrac{1}{2} \in D$, but $-\dfrac{1}{2} \notin E$, then D is not a subset of E, so $D \not\subseteq E$.
Therefore, $D \subseteq E$ is false.

h. Since $-6 \in \mathbb{R}$ and $-6 > -7$, then $-6 \notin A$, so $-6 \in A$ is false.

i. Since $-6 \in \mathbb{R}$ and -6 is not less than -6, then -6 is not an element of F, so $-6 \notin F$.
Therefore, $-6 \in F$ is false.

j. Since $-\dfrac{13}{2} \in \mathbb{R}$ and $-\dfrac{13}{2} > -7$, then $-\dfrac{13}{2} \notin A$, so $-\dfrac{13}{2} \in A$ is false.

k. Since $-\dfrac{13}{2} \in \mathbb{R}$ and $-\dfrac{13}{2} < -6$, then $-\dfrac{13}{2} \in F$, so $-\dfrac{13}{2} \notin F$ is false.

l. Since $100 \in B$, then $100 \in B$ is true.

m. Since $0 < 4$, then $0 \notin B$, so $0 \in B$ is false.

n. Since $-1 \in \mathbb{R}$ and -1 is not greater than -1, then $-1 \notin D$, so $-1 \notin D$ is true. $\square$

Exercise 59. Find all relationships of set equality, subset, and proper subset existing between pairs of the sets.
$A = \{-1, 1, 4\}$
$B = (-1, 4)$
$C = \{x \in \mathbb{R} : x^3 - 4x^2 - x + 4 = 0\}$
$D = [-1, 4]$.

Solution. Since $C = \{x \in \mathbb{R} : x^3 - 4x^2 - x + 4 = 0\}$, then $C = \{x \in \mathbb{R} : (x-1)(x+1)(x-4) = 0\}$, so $C = \{-1, 1, 4\} = A$.

We analyze set relationships between A with B, C, and D.
Observe that $A \neq B$ and $A \not\subseteq B$ and $A \not\subset B$.
Observe that $A = C$ and $A \subseteq C$ and $A \not\subset C$.
Observe that $A \neq D$ and $A \subseteq D$ and $A \subset D$.

We analyze set relationships between B with A, C, and D.
Observe that $B \neq A$ and $B \not\subseteq A$ and $B \not\subset A$.
Observe that $B \neq C$ and $B \not\subseteq C$ and $B \not\subset C$.
Observe that $B \neq D$ and $B \subseteq D$ and $B \subset D$.

We analyze set relationships between C with A, B, and D.
Observe that $C = A$ and $C \subseteq A$ and $C \not\subset A$.
Observe that $C \neq B$ and $C \not\subseteq B$ and $C \not\subset B$.
Observe that $C \neq D$ and $C \subseteq D$ and $C \subset D$.

We analyze set relationships between D with A, B, and C.
Observe that $D \neq A$ and $D \nsubseteq A$ and $D \not\subset A$.
Observe that $D \neq B$ and $D \nsubseteq B$ and $D \not\subset B$.
Observe that $D \neq C$ and $D \nsubseteq C$ and $D \not\subset C$. $\square$

Exercise 60. Find all relationships of set equality, subset, and proper subset existing between pairs of the sets.
$A = \{\emptyset, 0, 1\}$
$B = \{\emptyset, \{\emptyset\}\}$
$C = [0, 1]$
$D = \{\{0, 1\}, \{0\}, \{1\}, \emptyset, \{\emptyset\}\}$

Solution. We analyze set relationships between A with B, C, and D.
Observe that $A \neq B$ and $A \nsubseteq B$ and $A \not\subset B$.
Observe that $A \neq C$ and $A \nsubseteq C$ and $A \not\subset C$.
Observe that $A \neq D$ and $A \nsubseteq D$ and $A \not\subset D$.

We analyze set relationships between B with A, C, and D.
Observe that $B \neq A$ and $B \nsubseteq A$ and $B \not\subset A$.
Observe that $B \neq C$ and $B \nsubseteq C$ and $B \not\subset C$.
Observe that $B \neq D$ and $B \subseteq D$ and $B \subset D$.

We analyze set relationships between C with A, B, and D.
Observe that $C \neq A$ and $C \nsubseteq A$ and $C \not\subset A$.
Observe that $C \neq B$ and $C \nsubseteq B$ and $C \not\subset B$.
Observe that $C \neq D$ and $C \nsubseteq D$ and $C \not\subset D$.

We analyze set relationships between D with A, B, and C.
Observe that $D \neq A$ and $D \nsubseteq A$ and $D \not\subset A$.
Observe that $D \neq B$ and $D \nsubseteq B$ and $D \not\subset B$.
Observe that $D \neq C$ and $D \nsubseteq C$ and $D \not\subset C$. $\square$

Exercise 61. Find all relationships of set equality, subset, and proper subset existing between pairs of the sets.
$A = \{x \in \mathbb{N} : |x| \leq 4\}$
$B = \{-4, -3, -2, -1, 0, 1, 2, 3, 4\}$
$C = \{x \in \mathbb{Z} : |x| < 5\}$

Solution. Observe that $A = \{1, 2, 3, 4\}$ and $C = \{-4, -3, -2, -1, 0, 1, 2, 3, 4\} = B$.

We analyze set relationships between A with B and C.
Observe that $A \neq B$ and $A \subseteq B$ and $A \subset B$.
Observe that $A \neq C$ and $A \subseteq C$ and $A \subset C$.

We analyze set relationships between B with A and C.
Observe that $B \neq A$ and $B \not\subseteq A$ and $B \not\subset A$.
Observe that $B = C$ and $B \subseteq C$ and $B \not\subset C$.

We analyze set relationships between C with A and B.
Observe that $C \neq A$ and $C \not\subseteq A$ and $C \not\subset A$.
Observe that $C = B$ and $C \subseteq B$ and $C \not\subset B$. $\square$

Exercise 62. Explain precisely why the set $\{0, 1, 2\}$ is not an interval.

Solution. Let $S = \{0, 1, 2\}$.

Since $0 \in S$ and $1 \in S$ and $\frac{1}{2} \in \mathbb{R}$ and $0 < \frac{1}{2} < 1$, but $\frac{1}{2} \notin S$, then S is not an interval. $\square$

Exercise 63. Compute $\mathscr{P}(S)$ for the sets below.
i. $S = \{1, 2, 3\}$
ii. $S = \{a, b, c, d\}$
iii. $S = \emptyset$
iv. $S = \{\emptyset\}$
v. $S = \{\emptyset, \{\emptyset\}\}$
vi. $S = \mathscr{P}(\{1, 2\})$

Solution. i. Observe that $\mathscr{P}(S) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$.

ii. Observe that $\mathscr{P}(S) = \{\emptyset,$
$\{a\}, \{b\}, \{c\}, \{d\},$
$\{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\},$
$\{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\},$
$\{a, b, c, d\}\}$.

iii. Observe that $\mathscr{P}(S) = \{\emptyset\}$.

iv. Observe that $\mathscr{P}(S) = \{\emptyset, \{\emptyset\}\}$.

v. Observe that $\mathscr{P}(S) = \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}$.

vi. Let $T = \{1, 2\}$.
Then $\mathscr{P}(T) = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$.
The powerset of the powerset of T consisting of 16 elements is
$\mathscr{P}(\mathscr{P}(T)) = \{\emptyset,$
$\{\emptyset\}, \{\{1\}\}, \{\{2\}\}, \{\{1, 2\}\},$
$\{\emptyset, \{1\}\}, \{\emptyset, \{2\}\}, \{\emptyset, \{1, 2\}\}, \{\{1\}, \{2\}\}, \{\{1\}, \{1, 2\}\}, \{\{2\}, \{1, 2\}\}$
$\{\emptyset, \{1\}, \{2\}\}, \{\emptyset, \{1\}, \{1, 2\}\}, \{\emptyset, \{2\}, \{1, 2\}\}, \{\{1\}, \{2\}, \{1, 2\}\}$
$\{\emptyset, \{1\}, \{2\}, \{1, 2\}\}\}$. $\square$

Example 64. List ten elements of $\mathscr{P}(\{1, 2, 3, ...\})$.

Solution. Since $\{1, 2, 3, ...\} = \mathbb{Z}^+$, then the power set of $\mathbb{Z}^+$ is $\{X : X \subseteq \mathbb{Z}^+\}$ and consists of all subsets of the set of positive integers.

Therefore, an element of the power set of $\mathbb{Z}^+$ is a subset of $\mathbb{Z}^+$.

Below are 10 elements in the power set of $\mathbb{Z}^+$.

1. $\emptyset$
2. $\mathbb{Z}^+$
3. $\{1\}$
4. $\{1, 2, 3, 4, 5\}$
5. $\{2, 4, 6, 8, 10, ...\}$
6. $\{1, 3, 5, 7, 9, ...\}$
7. $\{3, 6, 9, 12, 15, ...\}$
8. $\{4, 8, 12, 16, 20, ...\}$
9. $\{2, 3, 5, 7, 11, 13, ...\}$
10. $\{1, 2, 3, 5, 8, 13, 21, 34, 55, ...\}$ $\square$

Example 65. Is there a finite set X such that the power set of X is infinite?

Solution. If X is a finite set, then there are n elements contained in X, where n is a non-negative integer.

Thus, there are only a finite number of subsets of X, so the power set of X is a finite set.

Therefore, there is no finite set X such that the power set of X is infinite. $\square$

Chapter 1.2 Operations on Sets

Example 66. Subtraction over $\mathbb{R}$ is not commutative.

Observe that $\pi \in \mathbb{R}$ and $\frac{2}{3} \in \mathbb{R}$, but $\pi - \frac{2}{3} \neq \frac{2}{3} - \pi$.

Example 67. Subtraction over $\mathbb{R}$ is not associative.

Observe that $9.5, 8.5, 6.5 \in \mathbb{R}$ and $(9.5 - 8.5) - 6.5 = -5.5$ and $9.5 - (8.5 - 6.5) = 7.5$, so $(9.5 - 8.5) - 6.5 \neq 9.5 - (8.5 - 6.5)$.

Union and Intersection

Example 68. set union and intersection

Let $A = \{1, 3, 5, 7, 9\}$.

Let $B = \{1, 4, 7, 10, 13, 16\}$.

Let $C = \{-5, -3, -1, 1, 3, 5\}$.

Compute $A \cap B$, $A \cup B$, $A \cap C$, $B \cap C$, and $B \cup (A \cap C)$.

Solution. Observe that $A \cap B = \{1, 7\}$.

Observe that $A \cup B = \{1, 3, 4, 5, 7, 9, 10, 13, 16\}$.

Observe that $A \cap C = \{1, 3, 5\}$.

Observe that $B \cap C = \{1\}$.

Observe that

$$\begin{aligned} B \cup (A \cap C) &= \{1, 4, 7, 10, 13, 16\} \cup \{1, 3, 5\} \\ &= \{1, 3, 4, 5, 7, 10, 13, 16\}. \end{aligned}$$

$\square$

Example 69. set union and intersection
Let $D = \{2, 4, 6, 8, 10\}$.
Let $E = (-5, 5)$.
Let $F = [3, \infty)$.
Let $G = \emptyset$.
Compute $E \cap F$, $E \cup F$, $D \cap E$, $D \cup F$, and $D \cup G$.
Compute $C \cap E$ and $A \cap D$ using sets A and C in the previous example.

Solution. Observe that $E \cap F = [3, 5)$.
Observe that $E \cup F = (-5, \infty)$.
Observe that $D \cap E = \{2, 4\}$.
Observe that $D \cup F = \{2\} \cup [3, \infty)$.
Observe that $D \cup G = D \cup \emptyset = D = \{2, 4, 6, 8, 10\}$.
Observe that $C \cap E = \{-3, -1, 1, 3\}$.
Observe that $A \cap D = \emptyset$.

$\square$

Example 70. Compute the solution set for the inequality $|2x+3| \geq 5$, and for the inequality $2x^2 + x - 28 > 0$, and for both inequalities.

Solution. Let A be the solution set for the inequality $|2x + 3| \geq 5$.
Then $A = \{x \in \mathbb{R} : |2x + 3| \geq 5\}$.
Let $x \in A$.
Then $x \in \mathbb{R}$ and $|2x + 3| \geq 5$.
Hence, either $2x + 3 \geq 5$ or $2x + 3 \leq -5$, so either $x \geq 1$ or $x \leq -4$.
Thus, either $x \in [1, \infty)$ or $x \in (-\infty, -4]$, so $x \in (-\infty, -4] \cup [1, \infty)$.
Therefore, A is a subset of $(-\infty, -4] \cup [1, \infty)$.

Let $y \in (-\infty, -4] \cup [1, \infty)$.
Then $y \in \mathbb{R}$ and $y \leq -4$ or $y \geq 1$, so $2y \leq -8$ or $2y \geq 2$.
Hence, $2y + 3 \leq -5$ or $2y + 3 \geq 5$, so $|2y + 3| \geq 5$.
Since $y \in \mathbb{R}$ and $|2y + 3| \geq 5$, then $y \in A$, so $(-\infty, -4] \cup [1, \infty)$ is a subset of A.
Since A is a subset of $(-\infty, -4] \cup [1, \infty)$, and $(-\infty, -4] \cup [1, \infty)$ is a subset of A, then $A = (-\infty, -4] \cup [1, \infty)$. $\square$

Solution. Let B be the solution set for the inequality $2x^2 + x - 28 > 0$.
Then $B = \{x \in \mathbb{R} : 2x^2 + x - 28 > 0\}$.
Let $x \in B$.
Then $x \in \mathbb{R}$ and $x^2 + x - 28 > 0$, so $(2x - 7)(x + 4) > 0$.
Hence, either $2x - 7 > 0$ and $x + 4 > 0$, or $2x - 7 < 0$ and $x + 4 < 0$, so either $x > \frac{7}{2}$ and $x > -4$, or $x < \frac{7}{2}$ and $x < -4$.
Thus, either $x > \frac{7}{2}$ or $x < -4$, so either $x \in (\frac{7}{2}, \infty)$ or $x \in (-\infty, -4)$.
Therefore, $x \in (-\infty, -4) \cup (\frac{7}{2}, \infty)$, so B is a subset of $(-\infty, -4) \cup (\frac{7}{2}, \infty)$.

Let $y \in (-\infty, -4) \cup (\frac{7}{2}, \infty)$.

Then $y \in \mathbb{R}$ and either $y \in (-\infty, -4)$ or $y \in (\frac{7}{2}, \infty)$, so either $y < -4$ or $y > \frac{7}{2}$.

We consider these cases separately.

Case 1: Suppose $y < -4$.

Then $y + 4 < 0$ and $2y < -8$.

Observe that $2y < -8 \Rightarrow 2y - 7 < -15 < 0 \Rightarrow 2y - 7 < 0$, so $2y - 7 < 0$.

Since $y+4 < 0$ and $2y-7 < 0$, then $(y+4)(2y-7) > 0$, so $2y^2+y-28 > 0$.

Since $y \in \mathbb{R}$ and $2y^2 + y - 28 > 0$, then $y \in B$.

Case 2: Suppose $y > \frac{7}{2}$.

Then $2y > 7$ and $y + 4 > \frac{7}{2} + 4$.

Observe that $y + 4 > \frac{7}{2} + 4 = \frac{15}{2} > 0$, so $y + 4 > 0$.

Since $2y > 7$, then $2y - 7 > 0$.

Hence, $y + 4 > 0$ and $2y - 7 > 0$, so $(y + 4)(2y - 7) > 0$.

Therefore, $2y^2 + y - 28 > 0$.

Since $y \in \mathbb{R}$ and $2y^2 + y - 28 > 0$, then $y \in B$.

In all cases, $y \in B$.

Therefore, $y \in (-\infty, -4) \cup (\frac{7}{2}, \infty)$ implies $y \in B$, so $(-\infty, -4) \cup (\frac{7}{2}, \infty)$ is a subset of B.

Since B is a subset of $(-\infty, -4) \cup (\frac{7}{2}, \infty)$, and $(-\infty, -4) \cup (\frac{7}{2}, \infty)$ is a subset of B, then $B = (-\infty, -4) \cup (\frac{7}{2}, \infty)$. $\square$

Solution. Let C be the solution set for both inequalities $|2x + 3| \geq 5$ and $2x^2 + x - 28 > 0$.

Then $C = A \cap B$, where $A = (-\infty, -4] \cup [1, \infty)$ and $B = (-\infty, -4) \cup (\frac{7}{2}, \infty)$.

Therefore, $C = (-\infty, -4) \cup (\frac{7}{2}, \infty)$. $\square$

Complement

Example 71. Let $A = \{1\}$ be a subset of universal set $\mathbb{N}$.

Then $\overline{A} = \{x \in \mathbb{N} : x \notin A\} = \{2, 3, 4, 5, ...\}$.

Example 72. Let $A = \{1\}$ be a subset of universal set $\mathbb{R}$.

Then $\overline{A} = \{x \in \mathbb{R} : x \notin A\} = (-\infty, 1) \cup (1, \infty)$.

Example 73. Let $A = \{1\}$ be a subset of universal set $U = \{1, 2\}$.

Then $\overline{A} = \{x \in U : x \notin A\} = \{2\}$.

Example 74. Let $\mathbb{R}$ be the universal set.
Compute the complements of the sets:
$A = [-1, 1]$
$B = (-\frac{1}{2}, 2]$
$C = (-\infty, 0]$
$D = (0, \infty)$

Solution. The complement of A in $\mathbb{R}$ is the set $\overline{A} = \{x \in \mathbb{R} : x \notin A\} = (-\infty, -1) \cup (1, \infty)$.

The complement of B in $\mathbb{R}$ is the set $\overline{B} = \{x \in \mathbb{R} : x \notin B\} = (-\infty, -\frac{1}{2}] \cup (2, \infty)$.

The complement of C in $\mathbb{R}$ is the set $\overline{C} = \{x \in \mathbb{R} : x \notin C\} = (0, \infty) = D$.

The complement of D in $\mathbb{R}$ is the set $\overline{D} = \{x \in \mathbb{R} : x \notin D\} = (-\infty, 0] = C$.

Observe that $A - B = [-1, -\frac{1}{2}]$ and $B - A = (1, 2]$.
Observe that $A - C = (0, 1]$ and $C - D = (-\infty, 0] = C$.
Observe that $C - B = (-\infty, -\frac{1}{2}]$.
Observe that $A - \emptyset = [-1, 1] = A$.
Observe that $\emptyset - D = \emptyset$.

We formulate these conjectures:
1. $A - B \subseteq A$ for any sets A and B.
2. If A and B are disjoint sets, then $A - B = A$.
3. $A - \emptyset = A$ for any set A.
4. $\emptyset - A = \emptyset$ for any set A. $\square$

Example 75. Let U be the set of all employees of a certain company.
Let $A = \{x \in U : x \text{ is a male}\}$.
Let $B = \{x \in U : x \text{ is 30 years old or less}\}$.
Let $C = \{x \in U : x \text{ is paid } 20,000 \text{ per year or less}\}$.
I. Describe the sets $A \cap C, \overline{A}, A \cup \overline{B}, C \cap \overline{B}$.
II. Describe the sets $A \cap \overline{B}$, $\overline{A \cap B}$, $\overline{A \cup B}$, $A \cup \overline{A}$, and $C \cap \overline{C}$.

Solution. Part I.

Observe that $A \cap C = \{x \in U : x \text{ is a male and } x \text{ is paid 20,000 per year or less}\}$, so $A \cap C$ consists of males who make $20,000$ per year or less.

Observe that $\overline{A} = \{x \in U : x \notin A\} = \{x \in U : x \text{ is not a male}\} = \{x \in U : x \text{ is a female}\}$, so $\overline{A}$ is the set of all female employees.

Observe that $A \cup \overline{B} = \{x : \text{either } x \text{ is a male or } x \text{ is not 30 years old or less}\}$, so $A \cup \overline{B}$ is the set of all male employees together with all employees who are more than 30 years old.

Observe that $C \cap \overline{B} = \{x : x \text{ is paid 20,000 per year or less and } x \text{ is not less than 30 years old or less}\}$, so $C \cap \overline{B}$ is the set of all employees over 30 years old who are paid $20,000$ per year or less.

Part II.

Observe that $A \cap \overline{B}$ is the set of all male employees over 30 years old.

Observe that $\overline{A \cap B}$ is the set of all employees who are either female or more than 30 years old.

Observe that $\overline{A \cup B}$ is the set of all employees who are neither male nor 30 years old or less, so $\overline{A \cup B}$ is the set of all employees who are female or more than 30 years old.

Observe that $A \cup \overline{A}$ is the set of all male employees and all non-male employees, so $A \cup \overline{A}$ is the set of all employees.

Since $C \cap \overline{C}$ is the set of all employees who make both 20,000 per year or less and make more than 20,000 per year, then $C \cap \overline{C}$ consists of zero employees, so $C \cap \overline{C} = \emptyset$. □

Set theoretic difference

Example 76. Let $A = \{1, 3, 5, 7\}$.

Let $B = \{1, 2, 4, 8, 16\}$.

Let $C = \{1, 2, 3, ..., 100\}$.

Is 1 an element of $A - B$?

Compute the following sets $A - B$, $B - A$, $B - C$, and $A - C$.

Solution. Since $1 \in A$ and $1 \in B$, then $1 \notin A - B$.

Observe that $A - B = \{3, 5, 7\}$ and $B - A = \{2, 4, 8, 16\}$.

Observe that $B - C = \emptyset$ and $A - C = \emptyset$.

Since $A - B \neq B - A$, then set difference is not commutative.

It appears that if $A \subseteq B$, then $A - B = \emptyset$(we can prove this to be true).

Observe that $C - A = \{2, 4, 6, 8, 9, 10, 11, 12, ..., 99, 100\} = \{2, 4, 6, 8\} \cup \{9, 10, ..., 100\}$.

Observe that $C - B = \{3, 5, 6, 7, 9, 10, 11, 12, 13, 14, 15, 17, 18, ..., 100\} = \{3, 5, 6, 7, 9, 10, 11, 12, 13, 14, 15\} \cup \{17, 18, ..., 100\}$.

Observe that $A - \overline{B} = \{1\} = A \cap B$. □

Symmetric difference

Example 77. Let $A = \{2, 4, 6, 8, 10\}$.

Let $B = \{6, 8, 10, 12\}$.

Let $C = \{1, 3, 5, 7, 9, 11\}$.

Let $D = \{4, 6, 8\}$.

Compute $A \triangle B$, $A \triangle C$, and $A \triangle D$.

Solution. Observe that

$$\begin{aligned} A \triangle B &= (A - B) \cup (B - A) \\ &= \{2, 4\} \cup \{12\} \\ &= \{2, 4, 12\}. \end{aligned}$$

Therefore, $A \triangle B = \{2, 4, 12\}$.
Observe that

$$\begin{aligned} A \triangle C &= (A - C) \cup (C - A) \\ &= \{2, 4, 6, 8, 10\} \cup \{1, 3, 5, 7, 9, 11\} \\ &= \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\} \\ &= A \cup C. \end{aligned}$$

Therefore, $A \triangle C = A \cup C$.
Observe that

$$\begin{aligned} A \triangle D &= (A - D) \cup (D - A) \\ &= \{2, 10\} \cup \emptyset \\ &= \{2, 10\}. \end{aligned}$$

Therefore, $A \triangle D = \{2, 10\}$.

We compute more examples: $B \triangle A$, $A \triangle (B \triangle C)$, and $(A \triangle B) \triangle C$.
Observe that

$$\begin{aligned} B \triangle A &= (B - A) \cup (A - B) \\ &= \{12\} \cup \{2, 4\} \\ &= \{2, 4, 12\}. \end{aligned}$$

Therefore, $B \triangle A = \{2, 4, 12\}$.
Observe that

$$\begin{aligned} A \triangle (B \triangle C) &= A \triangle [(B - C) \cup (C - B)] \\ &= A \triangle (\{6, 8, 10, 12\} \cup \{1, 3, 5, 7, 9, 11\}) \\ &= A \triangle \{1, 3, 5, 6, 7, 8, 9, 10, 11, 12\} \\ &= \{1, 2, 3, 4, 5, 7, 9, 11, 12\}. \end{aligned}$$

Therefore, $A \triangle (B \triangle C) = \{1, 2, 3, 4, 5, 7, 9, 11, 12\}$.
Observe that

$$\begin{aligned} (A \triangle B) \triangle C &= [(A - B) \cup (B - A)] \triangle C \\ &= (\{2, 4\} \cup \{12\}) \triangle C \\ &= \{2, 4, 12\} \triangle C \\ &= \{1, 2, 3, 4, 5, 7, 9, 11, 12\}. \end{aligned}$$

Therefore, $A \triangle (B \triangle C) = \{1, 2, 3, 4, 5, 7, 9, 11, 12\}$.

We make some conjectures.

1. If A and B are disjoint sets, then $A \triangle B = A \cup B$.
2. If $B \subseteq A$, then $A \triangle B = A - B$ for any sets A and B.
3. $A \triangle B = B \triangle A$ for any sets A and B.
4. $\triangle$ is associative: $A \triangle (B \triangle C) = (A \triangle B) \triangle C$. $\square$

Example 78. Let $W = (-\infty, 3)$.
Let $X = (-3, 5]$.
Let $Y = [4, \infty)$.
Compute $W \triangle X$ and $W \triangle Y$ and $X \triangle Y$.

Solution. Observe that

$$\begin{aligned} W \triangle X &= (W - X) \cup (X - W) \\ &= (-\infty, -3] \cup [3, 5]. \end{aligned}$$

Therefore, $W \triangle X = (-\infty, -3] \cup [3, 5]$.
Observe that

$$\begin{aligned} W \triangle Y &= (W - Y) \cup (Y - W) \\ &= (-\infty, 3) \cup [4, \infty) \\ &= W \cup Y. \end{aligned}$$

Therefore, $W \triangle Y = W \cup Y$.
Observe that

$$\begin{aligned} X \triangle Y &= (X - Y) \cup (Y - X) \\ &= (-3, 4) \cup (5, \infty). \end{aligned}$$

Therefore, $X \triangle Y = (-3, 4) \cup (5, \infty)$. $\square$

Ordered pairs and the cartesian product

Example 79. Let $A - \{1, 2, 3\}$.
Let $B = \{w, x, y, z\}$.
Describe $A \times B$ by the roster method.
Compute $B \times A$, $A \times A$, and $B \times B$.

Solution. Observe that $A \times B = \{(1, w), (1, x), (1, y), (1, z),$
$(2, w), (2, x), (2, y), (2, z),$
$(3, w), (3, x), (3, y), (3, z)\}$.
Observe that $B \times A = \{(w, 1), (w, 2), (w, 3),$
$(x, 1), (x, 2), (x, 3)$
$(y, 1), (y, 2), (y, 3)$
$(z, 1), (z, 2), (z, 3)\}$.
Observe that $A \times A = \{(1, 1), (1, 2), (1, 3),$
$(2, 1), (2, 2), (2, 3)$
$(3, 1), (3, 2), (3, 3)\}$.

Observe that $B \times B = \{(w,w),(w,x),(w,y),(w,z),$
$(x,w),(x,x),(x,y),(x,z),$
$(y,w),(y,x),(y,y),(y,z),$
$(z,w),(z,x),(z,y),(z,z)\}$. $\square$

Example 80. Describe $A \times B$ if $A = B = \mathbb{R}$.

Describe geometrically the subset $I \times J$ of $\mathbb{R} \times \mathbb{R}$, where $I = [3,7]$ and $J = (-2,2)$.

Solution. Observe that $\mathbb{R} \times \mathbb{R}$ is the set of all ordered pairs of real numbers and is the Euclidean plane with x and y axes perpendicular to each other, intersecting at the origin $(0,0)$.

The cartesian product $[3,7] \times (-2,2)$ is the rectangular strip bounded by and including the lines $x = 3$ and $x = 7$ (closed left and right), and bounded by but excluding the lines $y = -2$ and $y = 2$ (open top and bottom). $\square$

Example 81. Let $L = \{x : x \text{ is a student in Math 197}\}$.

Let $M = \{x : x \text{ is a possible final grade in a course}\} = \{A,B,C,D,E\}$.

Describe $L \times M$.

Solution. The cartesian product $L \times M$ is the set of all ordered pairs (a,b), where a is a student in Math 197 and b is the final grade of the student in the course. $\square$

Example 82. Let $A = \{1,2,3\}$.

Describe $A \times \emptyset$ and $\emptyset \times A$.

Solution. The set $A \times \emptyset$ consists of all ordered pairs (a,b) such that $a \in A$ and $b \in \emptyset$.

Since there are no elements in $\emptyset$, then $b \notin \emptyset$, so there is no such pair (a,b) contained in $A \times \emptyset$.

Therefore, $A \times \emptyset = \emptyset$.

The set $\emptyset \times A$ consists of all ordered pairs (a,b) such that $a \in \emptyset$ and $b \in A$.

Since there are no elements in $\emptyset$, then $a \notin \emptyset$, so there is no such pair (a,b) contained in $\emptyset \times A$.

Therefore, $\emptyset \times A = \emptyset$.

We conjecture that $A \times \emptyset = \emptyset \times A = \emptyset$ for any set A. $\square$

Chapter 1.2 Exercises

Exercise 83. Let $U = \{1,2,3,...,9,10\}$.

Let $A = \{1,7,9\}$.

Let $B = \{3,5,6,9,10\}$.

Let $C = \{2,4,8,9\}$.

Compute the following:

a. $B \cup B$

b. $C \cap C$
c. $A \cup \overline{A}$
d. $B \cap \overline{B}$
e. $(A \cup B) \cap A$
f. $(B \cap C) \cup C$
g. $\overline{B}$
h. $\overline{\overline{B}}$
i. $A - A$
j. $B - \overline{B}$
k. $A \triangle A$
l. $C \triangle \overline{C}$

Solution. a. Observe that $B \cup B = \{3, 5, 6, 9, 10\} = B$.
b. Observe that $C \cap C = \{2, 4, 8, 9\} = C$.
c. Observe that $A \cup \overline{A} = \{1, 2, ..., 10\} = U$.
d. Observe that $B \cap \overline{B} = \emptyset$, since B and $\overline{B}$ are disjoint sets.
e. Observe that $(A \cup B) \cap A = \{1, 3, 5, 6, 7, 9, 10\} \cap \{1, 7, 9\} = \{1, 7, 9\}$.
f. Observe that $(B \cap C) \cup C = \{9\} \cup \{2, 4, 8, 9\} = \{2, 4, 8, 9\}$.
g. Observe that $\overline{B} = \overline{\{3, 5, 6, 9, 10\}} = \{1, 2, 4, 7, 8\}$.
h. Observe that $\overline{\overline{B}} = \overline{\{1, 2, 4, 7, 8\}} = \{3, 5, 6, 9, 10\} = B$.
i. Observe that $A - A = \{1, 7, 9\} - \{1, 7, 9\} = \emptyset$.
j. Observe that $B - \overline{B} = \{3, 5, 6, 9, 10\} - \{1, 2, 4, 7, 8\} = \{3, 5, 6, 9, 10\} = B$.
k. Observe that $A \triangle A = (A - A) \cup (A - A) = \emptyset \cup \emptyset = \emptyset$.
l. Observe that

$$\begin{aligned}
C \triangle \overline{C} &= (C - \overline{C}) \cup (\overline{C} - C) \\
&= (\{2, 4, 8, 9\} - \{1, 3, 5, 6, 7, 10\}) \cup (\{1, 3, 5, 6, 7, 10\} - \{2, 4, 8, 9\}) \\
&= \{2, 4, 8, 9\} \cup \{1, 3, 5, 6, 7, 10\} \\
&= \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\} \\
&= U.
\end{aligned}$$

□

Exercise 84. Let $U = \{1, 2, 3, ..., 9, 10\}$.
Let $A = \{1, 7, 9\}$.
Let $B = \{3, 5, 6, 9, 10\}$.
Let $C = \{2, 4, 8, 9\}$.
Compute the following:
a. $A \cap C$
b. $\overline{A \cap C}$
c. $A \cap \overline{C}$
d. $\overline{A} \cup \overline{C}$
e. $C - B$
f. $C \cap \overline{B}$
g. $(A \cup B) \cup C$
h. $A \cap (B \cap C)$.

i. $A \cup (B \cup C)$
j. $(A \cap B) \cap C$
k. $(A \cup B) \cap C$
l. $(A \cup C) \cap (B \cup C)$
m. $(A \cup B) \cup (A \cup \overline{B})$
n. $(A \cap C) \cup (B \cap C)$
o. $A \cup (C \cap \overline{A})$
p. $(A \cap C) \cup (A \cap \overline{C})$
q. $\overline{B \cup C}$
r. $\overline{B} \cup \overline{C}$

Solution. a. Observe that $A \cap C = \{1, 7, 9\} \cap \{2, 4, 8, 9\} = \{9\}$.
b. Observe that $\overline{A \cap C} = \overline{\{9\}} = \{1, 2, 3, 4, 5, 6, 7, 8, 10\}$.
c. Observe that $A \cap \overline{C} = \{1, 7, 9\} \cap \overline{\{2, 4, 8, 9\}} = \{1, 7, 9\} \cap \{1, 3, 5, 6, 7, 10\} = \{1, 7\}$.
d. Observe that $\overline{A} \cup \overline{C} = \overline{\{1, 7, 9\}} \cup \overline{\{2, 4, 8, 9\}} = \{2, 3, 4, 5, 6, 8, 10\} \cup \{1, 3, 5, 6, 7, 10\} = \{1, 2, 3, 4, 5, 6, 7, 8, 10\}$.
e. Observe that $C - B = \{2, 4, 8, 9\} - \{3, 5, 6, 9, 10\} = \{2, 4, 8\}$.
f. Observe that $C \cap \overline{B} = \{2, 4, 8, 9\} \cap \overline{\{3, 5, 6, 9, 10\}} = \{2, 4, 8, 9\} \cap \{1, 2, 4, 7, 8\} = \{2, 4, 8\}$.
g. Observe that

$$\begin{aligned} (A \cup B) \cup C &= (\{1, 7, 9\} \cup \{3, 5, 6, 9, 10\}) \cup \{2, 4, 8, 9\} \\ &= \{1, 3, 5, 6, 7, 9, 10\} \cup \{2, 4, 8, 9\} \\ &= \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\} \\ &= U. \end{aligned}$$

h. Observe that

$$\begin{aligned} A \cap (B \cap C) &= \{1, 7, 9\} \cap (\{3, 5, 6, 9, 10\} \cap \{2, 4, 8, 9\}) \\ &= \{1, 7, 9\} \cap \{9\} \\ &= \{9\}. \end{aligned}$$

i. Observe that

$$\begin{aligned} A \cup (B \cup C) &= \{1, 7, 9\} \cup (\{3, 5, 6, 9, 10\} \cup \{2, 4, 8, 9\}) \\ &= \{1, 7, 9\} \cup \{2, 3, 4, 5, 6, 8, 9, 10\} \\ &= \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\} \\ &= U. \end{aligned}$$

j. Observe that

$$\begin{aligned} (A \cap B) \cap C &= (\{1, 7, 9\} \cap (\{3, 5, 6, 9, 10\}) \cap \{2, 4, 8, 9\} \\ &= \{9\} \cap \{2, 4, 8, 9\} \\ &= \{9\}. \end{aligned}$$

k. Observe that

$$\begin{aligned}(A \cup B) \cap C &= (\{1,7,9\} \cup (\{3,5,6,9,10\}) \cap \{2,4,8,9\} \\ &= \{1,3,5,6,7,9,10\} \cap \{2,4,8,9\} \\ &= \{9\}.\end{aligned}$$

l. Observe that

$$\begin{aligned}(A \cup C) \cap (B \cup C) &= (\{1,2,4,7,8,9\} \cap (\{2,3,4,5,6,8,9,10\} \\ &= \{2,4,8,9\}.\end{aligned}$$

m. Observe that

$$\begin{aligned}(A \cup B) \cup (A \cup \overline{B}) &= \{1,3,5,6,7,9,10\} \cup (\{1,7,9\} \cup \{1,2,4,7,8\}) \\ &= \{1,3,5,6,7,9,10\} \cup \{1,2,4,7,8,9\} \\ &= \{1,2,3,4,5,6,7,8,9,10\} \\ &= U.\end{aligned}$$

n. Observe that $(A \cap C) \cup (B \cap C) = \{9\} \cup \{9\} = \{9\}$.
o. Observe that

$$\begin{aligned}A \cup (C \cap \overline{A}) &= \{1,7,9\} \cup (\{2,4,8,9\} \cap \{2,3,4,5,6,8,10\}) \\ &= \{1,7,9\} \cup \{2,4,8\} \\ &= \{1,2,4,7,8,9\}.\end{aligned}$$

p. Observe that

$$\begin{aligned}(A \cap C) \cup (A \cap \overline{C}) &= \{9\} \cup (\{1,7,9\} \cap \{1,3,5,6,7,10\}) \\ &= \{9\} \cup \{1,7\} \\ &= \{1,7,9\}.\end{aligned}$$

q. Observe that $\overline{B \cup C} = \overline{\{2,3,4,5,6,8,9,10\}} = \{1,7\}$.
r. Observe that

$$\begin{aligned}\overline{B} \cup \overline{C} &= \overline{\{3,5,6,9,10\}} \cup \overline{\{2,4,8,9\}} \\ &= \{1,2,4,7,8\} \cup \{1,3,5,6,7,10\} \\ &= \{1,2,3,4,5,6,7,8,10\}.\end{aligned}$$

□

Exercise 85. Let $U = \{1,2,3,...,9,10\}$.
Let $A = \{1,7,9\}$.
Let $B = \{3,5,6,9,10\}$.
Let $C = \{2,4,8,9\}$.
Compute the following:
a. $\overline{A \cap B \cap C}$

b. $\overline{A} \cup \overline{B} \cup \overline{C}$
c. $(A \cup C) - (A \cap C)$
d. $A \triangle C$
e. $B \triangle U$
f. $A \triangle (B \triangle C)$
g. $(A \triangle B) \triangle C$
h. $C - (B - A)$
i. $(C - B) - A$
j. $(C - B) \cap (C - A)$
k. $C - (B \cup A)$
l. $A \triangle (B \cup C)$

Solution. a. Observe that

$$\begin{aligned} \overline{A \cap B \cap C} &= \overline{\{1,7,9\} \cap \{3,5,6,9,10\} \cap \{2,4,8,9\}} \\ &= \overline{\{9\} \cap \{2,4,8,9\}} \\ &= \overline{\{9\}} \\ &= \{1,2,3,4,5,6,7,8,10\}. \end{aligned}$$

b. Observe that

$$\begin{aligned} \overline{A} \cup \overline{B} \cup \overline{C} &= \overline{\{1,7,9\}} \cup \overline{\{3,5,6,9,10\}} \cup \overline{\{2,4,8,9\}} \\ &= \{2,3,4,5,6,8,10\} \cup \{1,2,4,7,8\} \cup \{1,3,5,6,7,10\} \\ &= \{1,2,3,4,5,6,7,8,10\} \cup \{1,3,5,6,7,10\} \\ &= \{1,2,3,4,5,6,7,8,10\}. \end{aligned}$$

c. Observe that $(A \cup C) - (A \cap C) = \{1,2,4,7,8,9\} - \{9\} = \{1,2,4,7,8\}$.
d. Observe that

$$\begin{aligned} A \triangle C &= (A - C) \cup (C - A) \\ &= \{1,7\} \cup \{2,4,8\} \\ &= \{1,2,4,7,8\}. \end{aligned}$$

e. Observe that

$$\begin{aligned} B \triangle U &= (B - U) \cup (U - B) \\ &= \emptyset \cup (U - B) \\ &= \emptyset \cup \{1,24,7,8\} \\ &= \{1,2,4,7,8\}. \end{aligned}$$

f. Observe that

$$
\begin{aligned}
A \triangle (B \triangle C) &= A \triangle ((B - C) \cup (C - B)) \\
&= \{1,7,9\} \triangle (\{3,5,6,10\} \cup \{2,4,8\}) \\
&= \{1,7,9\} \triangle \{2,3,4,5,6,8,10\} \\
&= (\{1,7,9\} - \{2,3,4,5,6,8,10\}) \cup (\{2,3,4,5,6,8,10\} - \{1,7,9\}) \\
&= \{1,7,9\} \cup \{2,3,4,5,6,8,10\} \\
&= \{1,2,3,4,5,6,7,8,9,10\} \\
&= U.
\end{aligned}
$$

g. Observe that

$$
\begin{aligned}
(A \triangle B) \triangle C &= ((A - B) \cup (B - A)) \triangle C \\
&= (\{1,7\} \cup \{3,5,6,10\}) \triangle \{2,4,8,9\} \\
&= \{1,3,5,6,7,10\} \triangle \{2,4,8,9\} \\
&= \{1,3,5,6,7,10\} \cup \{2,4,8,9\} \\
&= \{1,2,3,4,5,6,7,8,9,10\} \\
&= U.
\end{aligned}
$$

h. Observe that

$$
\begin{aligned}
C - (B - A) &= \{2,4,8,9\} - (\{3,5,6,9,10\} - \{1,7,9\}) \\
&= \{2,4,8,9\} - \{3,5,6,10\} \\
&= \{2,4,8,9\} \\
&= C.
\end{aligned}
$$

i. Observe that

$$
\begin{aligned}
(C - B) - A &= (\{2,4,8,9\} - \{3,5,6,9,10\}) - \{1,7,9\} \\
&= \{2,4,8\} - \{1,7,9\} \\
&= \{2,4,8\}.
\end{aligned}
$$

j. Observe that $(C - B) \cap (C - A) = \{2,4,8\} \cap \{2,4,8\} = \{2,4,8\}$.
k. Observe that $C - (B \cup A) = \{2,4,8,9\} - \{1,3,5,6,7,9,10\} = \{2,4,8\}$.
l. Observe that

$$
\begin{aligned}
A \triangle (B \cup C) &= \{1,7,9\} \triangle \{2,3,4,5,6,8,9,10\} \\
&= \{1,7\} \cup \{2,3,4,5,6,8,10\} \\
&= \{1,2,3,4,5,6,7,8,10\}.
\end{aligned}
$$

$\square$

Exercise 86. Let $U = \{1,2,3,...,9,10\}$.
Let $A = \{1,7,9\}$.

Let $B = \{3, 5, 6, 9, 10\}$.
Let $C = \{2, 4, 8, 9\}$.
Compute the following:
a. $A \times C$
b. $C \times B$
c. $U \times B$
d. $A \times U$
e. $A \times B$
f. $A \times (B \cup C)$
g. $(A \times B) \cup (A \times C)$
h. $\overline{B} \times \overline{C}$
i. $(A \cap B) \times C$
j. $(A \times C) \cap (B \times C)$
k. $B \times (A - C)$
l. $(B \times A) - (B \times C)$

Solution. a. Observe that $A \times C = \{(1, 2), (1, 4), (1, 8), (1, 9),$
$(7, 2), (7, 4), (7, 8), (7, 9),$
$(9, 2), (9, 4), (9, 8), (9, 9)\}$.

b. Observe that $C \times B = \{(2, 3), (2, 5), (2, 6), (2, 9), (2, 10),$
$(4, 3), (4, 5), (4, 6), (4, 9), (4, 10),$
$(8, 3), (8, 5), (8, 6), (8, 9), (8, 10),$
$(9, 3), (9, 5), (9, 6), (9, 9), (9, 10)\}$.

c. Observe that $U \times B = \{(1, 3), (1, 5), (1, 6), (1, 9), (1, 10),$
$(2, 3), (2, 5), (2, 6), (2, 9), (2, 10),$
$(3, 3), (3, 5), (3, 6), (3, 9), (3, 10),$
$(4, 3), (4, 5), (4, 6), (4, 9), (4, 10),$
$(5, 3), (5, 5), (5, 6), (5, 9), (5, 10),$
$(6, 3), (6, 5), (6, 6), (6, 9), (6, 10),$
$(7, 3), (7, 5), (7, 6), (7, 9), (7, 10),$
$(8, 3), (8, 5), (8, 6), (8, 9), (8, 10),$
$(9, 3), (9, 5), (9, 6), (9, 9), (9, 10),$
$(10, 3), (10, 5), (10, 6), (10, 9), (10, 10)\}$.

d. Observe that $A \times U = \{(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (1, 6), (1, 7), (1, 8), (1, 9), (1, 10),$
$(7, 1), (7, 2), (7, 3), (7, 4), (7, 5), (7, 6), (7, 7), (7, 8), (7, 9), (7, 10),$
$(9, 1), (9, 2), (9, 3), (9, 4), (9, 5), (9, 6), (9, 7), (9, 8), (9, 9), (9, 10)\}$.

e. Observe that $A \times B = \{(1, 3), (1, 5), (1, 6), (1, 9), (1, 10),$
$(7, 3), (7, 5), (7, 6), (7, 9), (7, 10),$
$(9, 3), (9, 5), (9, 6), (9, 9), (9, 10)\}$.

f. Observe that $A \times (B \cup C) = \{1, 7, 9\} \times \{2, 3, 4, 5, 6, 8, 9, 10\} =$
$\{(1, 2), (1, 3), (1, 4), (1, 5), (1, 6), (1, 8), (1, 9), (1, 10),$
$(7, 2), (7, 3), (7, 4), (7, 5), (7, 6), (7, 8), (7, 9), (7, 10),$
$(9, 2), (9, 3), (9, 4), (9, 5), (9, 6), (9, 8), (9, 9), (9, 10)\}$.

g. Observe that $(A \times B) \cup (A \times C) =$
$\{(1, 3), (1, 5), (1, 6), (1, 9), (1, 10),$
$(7, 3), (7, 5), (7, 6), (7, 9), (7, 10),$

$(9,3),(9,5),(9,6),(9,9),(9,10)\}\cup$
$\{(1,2),(1,4),(1,8),(1,9),$
$(7,2),(7,4),(7,8),(7,9),$
$(9,2),(9,4),(9,8),(9,9)\} =$
$\{(1,2),(1,3),(1,4),(1,5),(1,6),(1,8),(1,9),(1,10),$
$(7,2),(7,3),(7,4),(7,5),(7,6),(7,8),(7,9),(7,10),$
$(9,2),(9,3),(9,4),(9,5),(9,6),(9,8),(9,9),(9,10)\}.$

h. Observe that $\overline{B} \times \overline{C} = \{1,2,4,7,8\} \times \{1,3,5,6,7,10\} =$
$\{(1,1),(1,3),(1,5),(1,6),(1,7),(1,10),$
$(2,1),(2,3),(2,5),(2,6),(2,7),(2,10),$
$(4,1),(4,3),(4,5),(4,6),(4,7),(4,10),$
$(7,1),(7,3),(7,5),(7,6),(7,7),(7,10),$
$(8,1),(8,3),(8,5),(8,6),(8,7),(8,10)\}.$

i. Observe that $(A \cap B) \times C = \{9\} \times \{2,4,8,9\} =$
$\{(9,2),(9,4),(9,8),(9,9)\}.$

j. Observe that $(A \times C) \cap (B \times C) =$
$\{(1,2),(1,4),(1,8),(1,9),$
$(7,2),(7,4),(7,8),(7,9),$
$(9,2),(9,4),(9,8),(9,9)\}\cap$
$\{(3,2),(3,4),(3,8),(3,9),$
$(5,2),(5,4),(5,8),(5,9),$
$(6,2),(6,4),(6,8),(6,9),$
$(9,2),(9,4),(9,8),(9,9),$
$(10,2),(10,4),(10,8),(10,9)\} =$
$\{(9,2),(9,4),(9,8),(9,9)\}.$

k. Observe that $B \times (A - C) = \{3,5,6,9,10\} \times \{1,7\} =$
$\{(3,1),(3,7),$
$\{(5,1),(5,7),$
$\{(6,1),(6,7),$
$\{(9,1),(9,7),$
$\{(10,1),(10,7)\}.$

l. Observe that $(B \times A) - (B \times C) =$
$\{(3,1),(3,7),(3,9),$
$(5,1),(5,7),(5,9),$
$(6,1),(6,7),(6,9),$
$(9,1),(9,7),(9,9),$
$(10,1),(10,7),(10,9)\}-$
$\{(3,2),(3,4),(3,8),(3,9),$
$(5,2),(5,4),(5,8),(5,9),$
$(6,2),(6,4),(6,8),(6,9),$
$(9,2),(9,4),(9,8),(9,9),$
$(10,2),(10,4),(10,8),(10,9)\} =$
$\{(3,1),(3,7),$
$(5,1),(5,7),$
$(6,1),(6,7),$
$(9,1),(9,7),$

$(10,1),(10,7)\}$.

$\square$

Exercise 87. Let $U = \{1,2,3,...,9,10\}$ and $A = \{2,5,7,9\}$ and $B = \{5,7\}$ and $C = \{2,9\}$ and $D = \{1,4,6,10\}$.

Compute the following:
a. $(D - C) - B$
b. $D - (C - B)$
c. $A \triangle D$
d. $A \cup D$
e. $B \cup (A - B)$
f. $A \cap (B \cup \overline{D})$
g. $(A \cap B) \cup \overline{D}$
h. $B \cup (C - B)$
i. $A \triangle C$

Solution. a. Observe that $(D - C) - B = \{1,4,6,10\} - \{5,7\} = \{1,4,6,10\} = D$.

b. Observe that $D - (C - B) = \{1,4,6,10\} - \{2,9\} = \{1,4,6,10\} = D$.
c. Observe that

$$\begin{aligned} A \triangle D &= (A - D) \cup (D - A) \\ &= \{2,5,7,9\} \cup \{1,4,6,10\} \\ &= \{1,2,4,5,6,7,9,10\}. \end{aligned}$$

d. Observe that $A \cup D = \{2,5,7,9\} \cup \{1,4,6,10\} = \{1,2,4,5,6,7,9,10\}$.
e. Observe that $B \cup (A - B) = \{5,7\} \cup \{2,9\} = \{2,5,7,9\} = A$.
f. Observe that

$$\begin{aligned} A \cap (B \cup \overline{D}) &= \{2,5,7,9\} \cap (\{5,7\} \cup \{2,3,5,7,8,9\}) \\ &= \{2,5,7,9\} \cap \{2,3,5,7,8,9\} \\ &= \{2,5,7,9\} \\ &= A. \end{aligned}$$

g. Observe that

$$\begin{aligned} (A \cap B) \cup \overline{D} &= \{5,7\} \cup \{2,3,5,7,8,9\} \\ &= \{2,3,5,7,8,9\} \\ &= \overline{D}. \end{aligned}$$

h. Observe that $B \cup (C - B) = \{5,7\} \cup \{2,9\} = \{2,5,7,9\} = A$.
i. Observe that

$$\begin{aligned} A \triangle C &= (A - C) \cup (C - A) \\ &= \{5,7\} \cup \emptyset \\ &= \{5,7\} \\ &= B. \end{aligned}$$

$\square$

Exercise 88. Let $U = \mathbb{R}$ and $A = [2, 9)$ and $B = (0, 1]$ and $C = [-1, 4]$.

Express the following in interval notation.

a. $A \cap C$
b. $A \cup C$
c. $A \cap B$
d. $A \cup B$
e. $(A \cup B) - C$
f. $A - C$
g. $C - A$
h. $B \cap C$
i. $B \cup C$
j. $C - \overline{A}$
k. $B - A$
l. $B - C$

Solution. a. Observe that $A \cap C = [2, 9) \cap [-1, 4] = [2, 4]$.
b. Observe that $A \cup C = [2, 9) \cup [-1, 4] = [-1, 9)$.
c. Observe that $A \cap B = [2, 9) \cap (0, 1] = \emptyset$.
d. Observe that $A \cup B = [2, 9) \cup (0, 1] = (0, 1] \cup [2, 9)$.
e. Observe that $(A \cup B) - C = (0, 1] \cup [2, 9) - [-1, 4] = (4, 9)$.
f. Observe that $A - C = [2, 9) - [-1, 4] = (4, 9)$.
g. Observe that $C - A = [-1, 4] - [2, 9) = [-1, 2)$.
h. Observe that $B \cap C = (0, 1] \cap [-1, 4] = (0, 1] = B$.
i. Observe that $B \cup C = (0, 1] \cup [-1, 4] = [-1, 4] = C$.
j. Observe that $C - \overline{A} = [-1, 4] - (-\infty, 2) \cup [9, \infty) = [2, 4]$.
k. Observe that $B - A = (0, 1] - [2, 9) = (0, 1] = B$.
l. Observe that $B - C = (0, 1] - [-1, 4] = \emptyset$. $\square$

Exercise 89. Let $U = \mathbb{R}$ and $A = (-\infty, 6]$ and $B = (-3, \infty)$ and $C = (-4, 1) \cup (3, 7)$.

Compute the following sets.

a. $A \cap B$
b. $A \cup B$
c. $A - B$
d. $A \triangle B$
e. $A \cap C$
f. $B \cap C$
g. $A \cup \overline{C}$
h. $\overline{A \cup C}$
i. $C - A$
j. $\overline{A} \cap \overline{C}$
k. $(A \cap B) \cap C$
l. $(A \triangle B) \triangle C$
m. $A \cap (B \triangle C)$
n. $A \triangle (B \triangle C)$

o. $(A \cap B) \triangle (A \cap C)$

Solution. a. Observe that $A \cap B = (-\infty, 6] \cap (-3, \infty) = (-3, 6]$.
b. Observe that $A \cup B = (-\infty, 6] \cup (-3, \infty) = \mathbb{R}$.
c. Observe that $A - B = (-\infty, 6] - (-3, \infty) = (-\infty, -3]$.
d. Observe that

$$\begin{aligned} A \triangle B &= (A - B) \cup (B - A) \\ &= (-\infty, -3] \cup (6, \infty). \end{aligned}$$

e. Observe that

$$\begin{aligned} A \cap C &= (-\infty, 6] \cap ((-4, 1) \cup (3, 7)) \\ &= (-4, 1) \cup (3, 6]. \end{aligned}$$

f. Observe that

$$\begin{aligned} B \cap C &= (-3, \infty) \cap ((-4, 1) \cup (3, 7)) \\ &= (-3, 1) \cup (3, 6]. \end{aligned}$$

g. Observe that

$$\begin{aligned} A \cup \overline{C} &= (-\infty, 6] \cup ((-\infty, -4] \cup [1, 3] \cup [7, \infty)) \\ &= (-\infty, 6] \cup [7, \infty). \end{aligned}$$

h. Observe that $\overline{A \cup C} = \overline{(-\infty, 7)} = [7, \infty)$.
i. Observe that $C - A = (-4, 1) \cup (3, 7) - (-\infty, 6] = (6, \infty)$.
j. Observe that

$$\begin{aligned} \overline{A} \cap \overline{C} &= (6, \infty) \cap ((-\infty, -4] \cup [1, 3] \cup [7, \infty)) \\ &= [7, \infty). \end{aligned}$$

k. Observe that

$$\begin{aligned} (A \cap B) \cap C &= (-3, 6] \cap ((-4, 1) \cup (3, 7)) \\ &= (-3, 1) \cup (3, 6]. \end{aligned}$$

l. Observe that $(A \triangle B) \triangle C = ((-\infty, -3] \cup (6, \infty)) \triangle C$.
Let $D = (-\infty, -3] \cup (6, \infty)$.
Then

$$\begin{aligned} (A \triangle B) \triangle C &= D \triangle C \\ &= (D - C) \cup (C - D) \\ &= (-\infty, -4] \cup [7, \infty) \cup (C - D) \\ &= (-\infty, -4] \cup [7, \infty) \cup (-3, 1) \cup (3, 6] \\ &= (-\infty, -4] \cup (-3, 1) \cup (3, 6] \cup [7, \infty). \end{aligned}$$

Therefore, $(A \triangle B) \triangle C = (-\infty, -4] \cup (-3, 1) \cup (3, 6] \cup [7, \infty)$.
m. Observe that

$$\begin{aligned} A \cap (B \triangle C) &= A \cap ((B - C) \cup (C - B)) \\ &= A \cap ([1, 3] \cup [7, \infty) \cup (C - B)) \\ &= (-\infty, 6] \cap ([1, 3] \cup [7, \infty) \cup (-4, -3]) \\ &= (-\infty, 6] \cap ([-4, -3] \cup [1, 3] \cup [7, \infty)) \\ &= (-4, -3] \cup [1, 3]. \end{aligned}$$

n. Observe that $A \triangle (B \triangle C) = A \triangle ([-4, -3] \cup [1, 3] \cup [7, \infty))$.
Let $E = [-4, -3] \cup [1, 3] \cup [7, \infty)$.
Then

$$\begin{aligned} A \triangle (B \triangle C) &= A \triangle E \\ &= (A - E) \cup (E - A) \\ &= (-\infty, -4) \cup (-3, 1) \cup (3, 6) \cup (E - A) \\ &= (-\infty, -4] \cup (-3, 1) \cup (3, 6] \cup [7, \infty). \end{aligned}$$

Therefore, $A \triangle (B \triangle C) = (-\infty, -4] \cup (-3, 1) \cup (3, 6] \cup [7, \infty)$.
o. Observe that $(A \cap B) \triangle (A \cap C) = (-3, 6] \triangle ((-4, 1) \cup (3, 6])$.
Let $F = (-3, 6]$ and $G = (-4, 1) \cup (3, 6]$.
Then

$$\begin{aligned} (A \cap B) \triangle (A \cap C) &= F \triangle G \\ &= (F - G) \cup (G - F) \\ &= [1, 3] \cup (-4, -3] \\ &= (-4, -3) \cup [1, 3]. \end{aligned}$$

Therefore, $(A \cap B) \triangle (A \cap C) = (-4, -3) \cup [1, 3]$. $\square$

Exercise 90. Let $U = \mathbb{Z}$ and $A = \{0, 5, 10, 15, ...\}$ and $B = \{..., -10, -5, 0\}$ and $C = \{..., -9, -6, -3, 0, 3, 6, 9, ...\}$ and $D = \{45, 90, 135, 180, ...\}$.

Compute the following sets:
a. $A \cap B$
b. $B \cap D$
c. $D \cap \overline{A}$
d. $(A \cup B) \cap C$
e. $(A \cap C) \cup (B \cap C)$
f. $D - A$
g. $[(A \cup B) \cap C] \cap D$
h. $D \cap A$
i. $(D \cap A) \cup (D \cap \overline{A})$
j. $D \cap C$
k. $D \cap \overline{B}$
l. $B \triangle D$

Solution. a. Observe that $A \cap B = \{0, 5, 10, 15, ...\} \cap \{..., -10, -5, 0\} = \{0\}$.

b. Observe that $B \cap D = \{..., -10, -5, 0\} \cap \{45, 90, 135, 180, ...\} = \emptyset$.

c. Observe that $D \cap \overline{A} = \emptyset$.

d. Since $A \cup B$ is the set of all multiples of 5, and C is the set of all multiples of 3, then $(A \cup B) \cap C$ is the set of all multiples of 15, so $(A \cup B) \cap C = \{..., -45, -30, -15, 0, 15, 30, 45, ...\}$.

e. Since A is the set of all positive multiples of 5 and zero, and C is the set of all multiples of 3, then $A \cap C$ is the set of all positive multiples of 15 and zero, so $A \cap C = \{0, 15, 30, 45, ...\}$.

Since B is the set of all negative multiples of 5 and zero, and C is the set of all multiples of 3, then $B \cap C$ is the set of all negative multiples of 15 and zero, so $B \cap C = \{0, -15, -30, -45, ...\}$.

Therefore, $(A \cap C) \cup (B \cap C) = \{0, 15, 30, 45, ...\} \cup \{0, -15, -30, -45, ...\} = \{0, \pm 15, \pm 30, \pm 45, ...\}$ is the set of all multiples of 15, so $(A \cap C) \cup (B \cap C) = \{0, \pm 15, \pm 30, \pm 45, ...\}$.

f. Since $D - A = \{45, 90, 135, 180, ...\} - \{0, 5, 10, 15, ...\} = \emptyset$, then $D - A = \emptyset$.

g. Since $(A \cup B) \cap C$ is the set of all multiples of 15, then $(A \cup B) \cap C = \{0, \pm 15, \pm 30, \pm 45, \pm 60, ...\}$.

Observe that

$$\begin{aligned} [(A \cup B) \cap C] \cap D &= \{0, \pm 15, \pm 30, \pm 45, \pm 60, ...\} \cap \{45, 90, 135, 180, ...\} \\ &= \{45, 90, 135, 180, ...\} \\ &= D. \end{aligned}$$

Therefore, $[(A \cup B) \cap C] \cap D = D$ is the set of all positive multiples of 45.

h. Since D is the set of all positive multiples of 45, and A is the set of all positive multiples of 5 and zero, then $D \cap A$ is the set of all positive multiples of 45.

Observe that

$$\begin{aligned} D \cap A &= \{45, 90, 135, 180, ...\} \cap \{0, 5, 10, 15, ...\} \\ &= \{45, 90, 135, 180, ...\} \\ &= D. \end{aligned}$$

Therefore, $D \cap A = D$.

i. Since $D \cap A = D$ and $D \cap \overline{A} = \emptyset$, then $(D \cap A) \cup (D \cap \overline{A}) = D \cup \emptyset = D$.

Therefore, $(D \cap A) \cup (D \cap \overline{A}) = D$.

j. Observe that

$$\begin{aligned} D \cap C &= \{45, 90, 135, 180, ...\} \cap \{..., -9, -6, -3, 0, 3, 6, 9, ...\} \\ &= \{45, 90, 135, 180, ...\} \\ &= D. \end{aligned}$$

Therefore, $D \cap C = D$.

k. Observe that

$$\begin{aligned} D \cap \overline{B} &= \{45, 90, 135, 180, ...\} \cap \overline{\{..., -10, -5, 0\}} \\ &= \{45, 90, 135, 180, ...\} \\ &= D. \end{aligned}$$

Therefore, $D \cap \overline{B} = D$.

l. Since $B - D = B$ and $D - B = D$, then $B \triangle D = (B - D) \cup (D - B) = B \cup D = \{..., -15, -10, -5, 0, 45, 90, 135, 180, ...\}$. □

Exercise 91. Let $U = \{1, 2, 3, 4\}$ and $A = \{1, 3, 4\}$ and $B = \{3\}$ and $C = \{1, 2\}$.

Compute all pairs of disjoint sets among the six sets $A, \overline{A}, B, \overline{B}, C, \overline{C}$.

Solution. Observe that $\overline{A} = U - A = \{2\}$ and $\overline{B} = U - B = \{1, 2, 4\}$ and $\overline{C} = U - C = \{3, 4\}$.

There are $6 \cdot 6 = 36$ possible pairs of sets among the sets $A, \overline{A}, B, \overline{B}, C, \overline{C}$.

We must compute the pairs of disjoints sets out of all these possible pairs.

Observe that $A \cap A = A$ and $A \cap \overline{A} = \emptyset$ and $A \cap B = B$ and $A \cap \overline{B} = \{1, 4\}$ and $A \cap C = \{1\}$ and $A \cap \overline{C} = \overline{C}$.

Observe that $\overline{A} \cap A = \emptyset$ and $\overline{A} \cap \overline{A} = \overline{A}$ and $\overline{A} \cap B = \emptyset$ and $\overline{A} \cap \overline{B} = \overline{A}$ and $\overline{A} \cap C = \overline{A}$ and $\overline{A} \cap \overline{C} = \emptyset$.

Observe that $B \cap A = B$ and $B \cap \overline{A} = \emptyset$ and $B \cap B = B$ and $B \cap \overline{B} = \emptyset$ and $B \cap C = \emptyset$ and $B \cap \overline{C} = B$.

Observe that $\overline{B} \cap A = \{1, 4\}$ and $\overline{B} \cap \overline{A} = \overline{A}$ and $\overline{B} \cap B = \emptyset$ and $\overline{B} \cap \overline{B} = \overline{B}$ and $\overline{B} \cap C = C$ and $\overline{B} \cap \overline{C} = \{4\}$.

Observe that $C \cap A = \{1\}$ and $C \cap \overline{A} = \overline{A}$ and $C \cap B = \emptyset$ and $C \cap \overline{B} = C$ and $C \cap C = C$ and $C \cap \overline{C} = \emptyset$.

Observe that $\overline{C} \cap A = \overline{C}$ and $\overline{C} \cap \overline{A} = \emptyset$ and $\overline{C} \cap B = B$ and $\overline{C} \cap \overline{B} = \{4\}$ and $\overline{C} \cap C = \emptyset$ and $\overline{C} \cap \overline{C} = \overline{C}$.

The disjoint pairs are shown below.

$A \cap \overline{A} = \emptyset$
$\overline{A} \cap \overline{C} = \emptyset$
$B \cap \overline{A} = \emptyset$
$B \cap \overline{B} = \emptyset$
$B \cap C = \emptyset$
$C \cap \overline{C} = \emptyset$ □

Exercise 92. Compute the solution set as a union of intervals of the below inequalities.

a. $|3x - 23| \geq 4$
b. $2x^2 - 4x - 96 \geq 0$
c. $\dfrac{x - 5}{5 - x} < 0$
d. $|4x - 17| > 0$
e. $\dfrac{3x^2 - 27}{(x - 3)(x + 3)} > 0$

Solution. The universal set is $\mathbb{R}$.

Part a.

Let $x \in \mathbb{R}$.

Observe that $|3x - 23| \geq 4$ iff $3x - 23 \geq 4$ or $3x - 23 \leq -4$ iff $x \geq 9$ or $x \leq \frac{19}{3}$ iff $x \in [9, \infty) \cup (-\infty, \frac{19}{3}]$ iff $x \in (-\infty, \frac{19}{3}] \cup [9, \infty)$.

Therefore, the solution set is $(-\infty, \frac{19}{3}] \cup [9, \infty)$.

Part b.

Let $x \in \mathbb{R}$.

Observe that $2x^2 - 4x - 96 = 2(x^2 - 2x - 48) = 2(x - 8)(x + 6) \geq 0$ iff $x = -6$, or $x = 8$, or $x - 8 > 0$ and $x + 6 > 0$, or $x - 8 < 0$ and $x + 6 < 0$ iff $x = -6$, or $x = 8$, or $x > 8$ and $x > -6$, or $x < 8$ and $x < -6$ iff $x = -6$, or $x = 8$, or $x > 8$, or $x < -6$.

Hence, the solutions to the inequality are either $x = -6$ or $x = 8$ or $x > 8$ or $x < -6$, so either $x \leq -6$ or $x \geq 8$.

Therefore, the solution set is $(-\infty, -6] \cup [8, \infty)$.

For part c.

Let $x \in \mathbb{R}$ such that $x \neq 5$.

Since $x - 5 = -(5 - x)$, then $\frac{x-5}{5-x} = \frac{x-5}{-(x-5)} = -1 < 0$ is always true.

Since $x \in \mathbb{R}$ and $x \neq 5$, then either $x < 5$ or $x > 5$, so either $x \in (-\infty, 5)$ or $x \in (5, \infty)$.

Therefore, the solution set is $(-\infty, 5) \cup (5, \infty)$.

For part d.

Let $x \in \mathbb{R}$.

Observe that $|4x - 17| > 0$ iff $4x - 17 > 0$ or $4x - 17 < 0$ iff $x > \frac{17}{4}$ or $x < \frac{17}{4}$ iff $x \in (\frac{17}{4}, \infty)$ or $x \in (-\infty, \frac{17}{4})$ iff $x \in (\frac{17}{4}, \infty) \cup (-\infty, \frac{17}{4})$ iff $x \in (-\infty, \frac{17}{4}) \cup (\frac{17}{4}, \infty)$.

Therefore, the solution set is $(-\infty, \frac{17}{4}) \cup (\frac{17}{4}, \infty)$.

For part e.

Let $x \in \mathbb{R}$.

Observe that $3x^2 - 27 = 3(x^2 - 9) = 3(x - 3)(x + 3)$.

Hence, $\frac{3x^2 - 27}{(x-3)(x+3)} = \frac{3(x-3)(x+3)}{(x-3)(x+3)} = 3 > 0$, if $x \neq 3$ and $x \neq -3$.

Since $x \in \mathbb{R}$ and $x \neq -3$ and $x \neq 3$, then either $x < -3$ or $-3 < x < 3$ or $x > 3$, so $x \in (-\infty, -3) \cup (-3, 3) \cup (3, \infty)$.

Therefore, a solution set is $(-\infty, -3) \cup (-3, 3) \cup (3, \infty)$. $\square$

Exercise 93. Solve simultaneously the pairs of inequalities.
$|x| \geq 1$ and $x^2 - 4 \leq 0$.

Solution. Let $x \in \mathbb{R}$.
Observe that $|x| \geq 1$ iff $x \geq 1$ or $x \leq -1$ iff $x \in [1, \infty)$ or $x \in (-\infty, -1]$ iff $x \in [1, \infty) \cup (-\infty, -1]$ iff $x \in (-\infty, -1] \cup (1, \infty)$.
Therefore, the solution set to $|x| \geq 1$ is the set $A = (-\infty, -1] \cup (1, \infty)$.

Observe that $x^2 - 4 \leq 0$ iff $(x-2)(x+2) \leq 0$ iff either $x = \pm 2$, or $x - 2 > 0$ and $x + 2 < 0$, or $x - 2 < 0$ and $x + 2 > 0$ iff either $x = \pm 2$, or $x > 2$ and $x < -2$, or $x < 2$ and $x > -2$ iff either $x = \pm 2$, or $-2 < x < 2$ iff $-2 \leq x \leq 2$ iff $x \in [-2, 2]$.
Therefore, the solution set to $x^2 - 4 \leq 0$ is the set $B = [-2, 2]$.
To solve the pairs of inequalities simultaneously, we must compute $A \cap B$.
Observe that $A \cap B = [-2, -1] \cup [1, 2]$. $\square$

Exercise 94. Solve simultaneously the pairs of inequalities.
$|4x + 8| \leq 12$ and $x^2 + 6x + 8 > 0$.

Solution. Let $x \in \mathbb{R}$.
Observe that

$$\begin{aligned} |4x+8| \leq 12 &\Leftrightarrow -12 \leq 4x + 8 \leq 12 \\ &\Leftrightarrow -20 \leq 4x \leq 4 \\ &\Leftrightarrow -5 \leq x \leq 1 \\ &\Leftrightarrow x \in [-5, 1]. \end{aligned}$$

Therefore, the solution set to the inequality $|4x + 8| \leq 12$ is the set $A = [-5, 1]$.

Observe that $x^2 + 6x + 8 > 0$ iff $(x+4)(x+2) > 0$ iff $x + 4 > 0$ and $x + 2 > 0$, or $x + 4 < 0$ and $x + 2 < 0$ iff $x > -4$ and $x > -2$, or $x < -4$ and $x < -2$ iff $x > -2$ or $x < -4$ iff $x \in (-2, \infty)$ or $x \in (-\infty, -4)$ iff $x \in (-2, \infty) \cup (-\infty, -4)$ iff $x \in (-\infty, -4) \cup (-2, \infty)$.
Therefore, the solution set to the inequality $x^2 + 6x + 8 > 0$ is the set $B = (-\infty, -4) \cup (-2, \infty)$.
To solve the pairs of inequalities simultaneously, we must compute $A \cap B$.
Observe that $A \cap B = [-5, 1] \cap ((-\infty, -4) \cup (-2, \infty)) = [-5, -4) \cup (-2, 1]$. $\square$

Exercise 95. Let U be the set of all functions having $\mathbb{R}$ as domain and range a subset of $\mathbb{R}$.
Let $A = \{f : f$ is continuous at each $x \in \mathbb{R}\}$.
Let $B = \{f : f$ is differentiable at each $x \in \mathbb{R}\}$.
Let $C = \{f : f'(x) = 2x + 3$ for each $x \in \mathbb{R}\}$.
Let $D = \{f : f$ is a quadratic polynomial$\}$.
Let $E = \{f : f(0) = 0\}$.

Let $F = \{f : f \text{ is a linear polynomial}\}$.
a. List all subset relationships between pairs of these six sets.
b. List all pairs of disjoint sets among these six sets.
c. Describe, as precisely as possible, the following sets:
i. $C \cap E$
ii. $A - B$
iii. $D \cap F$
iv. $A \cap D$
v. $A \cup D$
vi. $C - E$
vii. $F - A$
viii. $F \cap \overline{A}$
ix. $E \cap F$
x. $B \cap E$

Solution. Part a.
TODO: Start here.

□

TODO: Finish exercises in 1.2 section.

Chapter 1.3 Algebraic Properties of Sets

Why an Algebra of Sets?

Example 96. Let $U = \mathbb{R}$.
Let $A = (-\infty, 4) \cup (7, \infty)$.
Let $B = [-2, 11]$.
Compute $(A \cap B) \cup (\overline{A} \cap B) \cup (A \cap \overline{B}) \cup (\overline{A} \cap \overline{B})$.

Solution. Observe that

$$\begin{aligned}
(A \cap B) \cup (\overline{A} \cap B) \cup (A \cap \overline{B}) \cup (\overline{A} \cap \overline{B}) &= (A \cup \overline{A}) \cap B \cup (A \cap \overline{B}) \cup (\overline{A} \cap \overline{B}) \\
&= (U \cap B) \cup (A \cap \overline{B}) \cup (\overline{A} \cap \overline{B}) \\
&= B \cup (A \cap \overline{B}) \cup (\overline{A} \cap \overline{B}) \\
&= B \cup [(A \cup \overline{A}) \cap \overline{B}] \\
&= B \cup (U \cap \overline{B}) \\
&= B \cup \overline{B} \\
&= U \\
&= \mathbb{R}.
\end{aligned}$$

Therefore, $(A \cap B) \cup (\overline{A} \cap B) \cup (A \cap \overline{B}) \cup (\overline{A} \cap \overline{B}) = \mathbb{R}$. □

Elementary Properties of Sets

Example 97. Let $U = \{1, 2, 3, ..., 9, 10\}$.
Let $A = \{1, 3, 5, 7, 9\}$.
Compute $A \cup \overline{A}$ and $A \cap \overline{A}$ and $\overline{\overline{A}}$ and $A \cup A$ and $A \cap A$.

Solution. Since $\overline{A} = \{2, 4, 6, 8, 10\}$, then $A \cup \overline{A} = \{1, 3, 5, 7, 9, 2, 4, 6, 8, 10\} = U$.
Observe that $A \cap \overline{A} = \emptyset$.
Observe that $\overline{\overline{A}} = \overline{\{2, 4, 6, 8, 10\}} = \{1, 3, 5, 7, 9\} = A$.
Observe that $A \cup A = \{1, 3, 5, 7, 9\} = A$.
Observe that $A \cap A = \{1, 3, 5, 7, 9\} = A$. □

Example 98. Let $U = \{1, 2, 3, ..., 9, 10\}$.
Let $B = \{1, 2, 3\}$.
Compute $B \cup \overline{B}$ and $B \cap \overline{B}$ and $\overline{\overline{B}}$ and $B \cup B$ and $B \cap B$.

Solution. Since $\overline{B} = \{4, 5, 6, 7, 8, 9, 10\}$, then $B \cup \overline{B} = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\} = U$.
Observe that $B \cap \overline{B} = \emptyset$.
Observe that $\overline{\overline{B}} = \overline{\{4, 5, 6, 7, 8, 9, 10\}} = \{1, 2, 3\} = B$.
Observe that $B \cup B = \{1, 2, 3\} = B$.
Observe that $B \cap B = \{1, 2, 3\} = B$. □

Example 99. Let $U = \{1, 2, 3, ..., 9, 10\}$.
Let $C = \{3, 4, 6, 8\}$.
Compute $C \cup \overline{C}$ and $C \cap \overline{C}$ and $\overline{\overline{C}}$ and $C \cup C$ and $C \cap C$.

Solution. Since $\overline{C} = \{1, 2, 5, 7, 9, 10\}$, then $C \cup \overline{C} = \{3, 4, 6, 8, 1, 2, 5, 7, 9, 10\} = U$.
Observe that $C \cap \overline{C} = \emptyset$.
Observe that $\overline{\overline{C}} = \overline{\{1, 2, 5, 7, 9, 10\}} = \{3, 4, 6, 8\} = C$.
Observe that $C \cup C = \{3, 4, 6, 8\} = C$.
Observe that $C \cap C = \{3, 4, 6, 8\} = C$. □

Commutativity and Associativity

Example 100. Let $U = \{1, 2, 3, ..., 9, 10\}$.
Let $A = \{2, 3, 5, 8\}$.
Let $B = \{1, 2, 5, 6, 7, 10\}$.
Let $C = \{2, 3, 4, 9, 10\}$.
Compute:
$(A \cap B) \cap C$ and $A \cap (B \cap C)$
$A \cap B$ and $B \cap A$
$B \cup C$ and $C \cup B$
$A \cup (B \cup C)$ and $(A \cup B) \cup C$.
What conjectures can be made based on these computations and drawing associated Venn diagrams?

Solution. Observe that $(A \cap B) \cap C = \{2, 5\} \cap \{2, 3, 4, 9, 10\} = \{2\}$.

Observe that $A \cap (B \cap C) = \{2, 3, 5, 8\} \cap \{2, 10\} = \{2\}$.

Therefore, $(A \cap B) \cap C = \{2\} = A \cap (B \cap C)$.

We draw some Venn diagrams of both $(A \cap B) \cap C$ and $A \cap (B \cap C)$ and observe more evidence that $(A \cap B) \cap C = A \cap (B \cap C)$ for any sets A, B, and C. We still would have to prove this statement to raise its status from conjecture to theorem.

Observe that $A \cap B = \{2, 5\} = B \cap A$.

We draw some Venn diagrams of both $A \cap B$ and $B \cap A$ and observe more evidence that $A \cap B = B \cap A$ for any sets A and B. This is a conjecture until proved.

Observe that $B \cup C = \{1, 2, 3, 4, 5, 6, 7, 9, 10\} = C \cup B$.

We draw some Venn diagrams of both $B \cup C$ and $C \cup B$ and observe more evidence that $B \cup C = C \cup B$ for any sets B and C. This is a conjecture until proved.

Observe that $A \cup (B \cup C) = \{2, 3, 5, 8\} \cup \{1, 2, 3, 4, 5, 6, 7, 9, 10\} = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\} = U$ and $(A \cup B) \cup C = \{1, 2, 3, 5, 6, 7, 8, 10\} \cup \{2, 3, 4, 9, 10\} = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\} = U$.

Therefore, $A \cup (B \cup C) = U = (A \cup B) \cup C$.

We draw some Venn diagrams of both $A \cup (B \cup C)$ and $(A \cup B) \cup C$ and observe more evidence that $A \cup (B \cup C) = (A \cup B) \cup C$ for any sets A, B, and C. $\square$

Distributivity

Example 101. Let $U = \{1, 2, 3, ..., 9, 10\}$.

Let $A = \{2, 3, 5, 8\}$.

Let $B = \{1, 2, 5, 6, 7, 10\}$.

Let $C = \{2, 3, 4, 9, 10\}$.

Let $D = \{8\}$.

Compute:

$(A \cap B) \cup D$ and $A \cap (B \cup D)$

Solution. Observe that $(A \cap B) \cup D = \{2, 5\} \cup \{8\} = \{2, 5, 8\}$ and $A \cap (B \cup D) = \{2, 3, 5, 8\} \cup \{1, 2, 5, 6, 7, 10, 8\} = \{2, 5, 8\}$.

Therefore, $(A \cap B) \cup D = \{2, 5, 8\} = A \cap (B \cup D)$.

Is $(X \cap Y) \cup Z = X \cap (Y \cup Z)$ for any sets X, Y, and Z?

Let's compute $(A \cap B) \cup C$ and $A \cap (B \cup C)$.

Observe that $(A \cap B) \cup C = \{2, 5\} \cup \{2, 3, 4, 9, 10\} = \{2, 3, 4, 5, 9, 10\}$.

Observe that $A \cap (B \cup C) = \{2, 3, 5, 8\} \cap \{1, 2, 3, 4, 5, 6, 7, 9, 10\} = \{2, 3, 5\}$.

Therefore, $(A \cap B) \cup C = \{2, 3, 4, 5, 9, 10\} \neq \{2, 3, 5\} = A \cap (B \cup C)$, so $(A \cap B) \cup C \neq A \cap (B \cup C)$.

We conclude $(X \cap Y) \cup Z = X \cap (Y \cup Z)$ for any sets X, Y, and Z is false.

We compute $A \cap (B \cup C)$ and $(A \cap B) \cup (A \cap C)$.

Observe that $A \cap (B \cup C) = \{2,3,5,8\} \cap \{1,2,3,4,5,6,7,9,10\} = \{2,3,5\}$ and $(A \cap B) \cup (A \cap C) = \{2,5\} \cup \{2,3\} = \{2,3,5\}$.

Therefore, $A \cap (B \cup C) = \{2,3,5\} = (A \cap B) \cup (A \cap C)$, so $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

We also draw Venn diagrams of both $A \cap (B \cup C)$ and $(A \cap B) \cup (A \cap C)$ and see more evidence of the truth of the conjecture $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ for any sets A, B, and C.

We compute $A \cup (B \cap C)$ and $(A \cup B) \cap (A \cup C)$.

Observe that $A \cup (B \cap C) = \{2,3,5,8\} \cup \{2,10\} = \{2,3,5,8,10\}$ and $(A \cup B) \cap (A \cup C) = \{1,2,3,5,6,7,8,10\} \cap \{2,3,4,5,8,9,10\} = \{2,3,5,8,10\}$.

Therefore, $A \cup (B \cap C) = \{2,3,5,8,10\} = (A \cup B) \cap (A \cup C)$, so $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

Is $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ true in general for any sets A, B, and C?

We draw Venn diagrams for both $A \cup (B \cap C)$ and $(A \cup B) \cap (A \cup C)$ and see more evidence of the truth of the conjecture $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ for any sets A, B, and C. $\square$

De Morgan's laws

Example 102. Let $U = \{1,2,3,...,9,10\}$.

Let $A = \{2,3,5,8\}$.

Let $B = \{1,2,5,6,7,10\}$.

Let $C = \{2,3,4,9,10\}$.

Compute and draw Venn diagrams and formulate any conjectures.

$\overline{A \cup B}$ and $\overline{A} \cup \overline{B}$

$\overline{A \cap B}$ and $\overline{A} \cap \overline{B}$

$\overline{A \cup C}$ and $\overline{A} \cup \overline{C}$

$\overline{A \cap C}$ and $\overline{A} \cap \overline{C}$

Solution. Observe that $\overline{A \cup B} = \overline{\{1,2,3,5,6,7,8,10\}} = \{4,9\}$ and $\overline{A} \cup \overline{B} = \{1,4,6,7,9,10\} \cup \{3,4,8,9\} = \{1,3,4,6,7,8,9,10\}$.

Observe that $\overline{A \cap B} = \overline{\{2,5\}} = \{1,3,4,6,7,8,9,10\}$ and $\overline{A} \cap \overline{B} = \{1,4,6,7,9,10\} \cap \{3,4,8,9\} = \{4,9\}$.

Observe that $\overline{A \cup C} = \overline{\{2,3,4,5,8,10\}} = \{1,6,7\}$ and $\overline{A} \cup \overline{C} = \{1,4,6,7,9,10\} \cup \{1,5,6,7,8\} = \{1,4,5,6,7,8,9,10\}$.

Observe that $\overline{A \cap C} = \overline{\{2,3\}} = \{1,4,5,6,7,8,9,10\}$ and $\overline{A} \cap \overline{C} = \{1,4,6,7,9,10\} \cap \{1,5,6,7,8\} = \{1,6,7\}$.

Observe that $\overline{A \cup B} = \{4,9\} = \overline{A} \cap \overline{B}$, so $\overline{A \cup B} = \overline{A} \cap \overline{B}$.

Observe that $\overline{A \cap B} = \{1,3,4,6,7,8,9,10\} = \overline{A} \cup \overline{B}$, so $\overline{A \cap B} = \overline{A} \cup \overline{B}$.

Observe that $\overline{A \cup C} = \{1,6,7\} = \overline{A} \cap \overline{C}$, so $\overline{A \cup C} = \overline{A} \cap \overline{C}$.

Observe that $\overline{A \cap C} = \{1,4,5,6,7,8,9,10\} = \overline{A} \cup \overline{C}$, so $\overline{A \cap C} = \overline{A} \cup \overline{C}$.

We conjecture $\overline{X \cup Y} = \overline{X} \cap \overline{Y}$ for any sets X and Y, and $\overline{X \cap Y} = \overline{X} \cup \overline{Y}$ for any sets X and Y.

We draw the Venn diagrams of each set and see more evidence that the conjectures $\overline{X \cup Y} = \overline{X} \cap \overline{Y}$ for any sets X and Y, and $\overline{X \cap Y} = \overline{X} \cup \overline{Y}$ for any sets X and Y are true. $\square$

Theorems involving a conditional

Example 103. Let $U = \{1, 2, 3, ..., 9, 10\}$.
Let $A = \{2, 3, 5, 8\}$.
Let $B = \{1, 2, 5, 6, 7, 10\}$.
Let $D = \{8\}$.
Explain why $(A \cap B) \cup D = A \cap (B \cup D)$.

Solution. Observe that $(A \cap B) \cup D = \{2, 5\} \cup \{8\} = \{2, 5, 8\}$ and $A \cap (B \cup D) = \{2, 3, 5, 8\} \cup \{1, 2, 5, 6, 7, 10, 8\} = \{2, 5, 8\}$.
Therefore, $(A \cap B) \cup D = \{2, 5, 8\} = A \cap (B \cup D)$.

We observe that $D \subseteq A$.
We conjecture if $X \subseteq A$ for any subset X of A, then $(A \cap B) \cup X = A \cap (B \cup X)$.
We compute the power set of A to obtain each of the 16 subsets X of A.
For each $X \subseteq A$, we compute $(A \cap B) \cup X$ and $A \cap (B \cup X)$ and see if they are equal.
We use Sage to compute these results for each of the 16 subsets X of A and find that in all cases $(A \cap B) \cup X = A \cap (B \cup X)$.
So we now have more evidence that the conjecture is true: for any sets A and B, if X is any subset of A, then $(A \cap B) \cup X = A \cap (B \cup X)$.
We now prove this conjecture.

Let A and B be arbitrary sets.
Let X be any subset of A.
Then $X \subseteq A$.

We prove $(A \cap B) \cup X$ is a subset of $A \cap (B \cup X)$.
Let $a \in (A \cap B) \cup X$.
Then $a \in A \cap B$ or $a \in X$.
We consider these cases separately.
Case 1: Suppose $a \in A \cap B$.
Then $a \in A$ and $a \in B$.
Since $a \in B$, then $a \in B$ or $a \in X$, so $a \in B \cup X$.
Since $a \in A$ and $a \in B \cup X$, then $a \in A \cap (B \cup X)$.
Case 2: Suppose $a \in X$.
Since $X \subseteq A$, then $a \in A$.
Since $a \in X$, then $a \in B$ or $a \in X$, so $a \in B \cup X$.
Since $a \in A$ and $a \in B \cup X$, then $a \in A \cap (B \cup X)$.

Hence, in all cases, $a \in A \cap (B \cup X)$.

Therefore, $a \in (A \cap B) \cup X$ implies $a \in A \cap (B \cup X)$, so $(A \cap B) \cup X$ is a subset of $A \cap (B \cup X)$.

We prove $A \cap (B \cup X)$ is a subset of $(A \cap B) \cup X$.

Let $b \in A \cap (B \cup X)$.

Then $b \in A$ and $b \in B \cup X$, so $b \in A$, and $b \in B$ or $b \in X$.

Hence, either $b \in A$ and $b \in B$, or $b \in A$ and $b \in X$, so either $b \in A \cap B$ or $b \in X$.

Thus, $b \in (A \cap B) \cup X$.

Therefore, $b \in A \cap (B \cup X)$ implies $b \in (A \cap B) \cup X$, so $A \cap (B \cup X)$ is a subset of $(A \cap B) \cup X$.

Since $(A \cap B) \cup X$ is a subset of $A \cap (B \cup X)$, and $A \cap (B \cup X)$ is a subset of $(A \cap B) \cup X$, then $(A \cap B) \cup X$ equals $A \cap (B \cup X)$, so $(A \cap B) \cup X = A \cap (B \cup X)$.

Therefore, we have proved the statement: for any sets A and B, if X is a subset of A, then $(A \cap B) \cup X = A \cap (B \cup X)$.

In our example, we have $A = \{2, 3, 5, 8\}$ and $B = \{1, 2, 5, 6, 7, 10\}$ and $D = \{8\}$.

Since $D \subseteq A$, then we conclude $(A \cap B) \cup D = A \cap (B \cup D)$. $\square$

Chapter 1.3 Exercises

Exercise 104. Let $U = \{1, 2, 3, ..., 9, 10\}$.

Find specific subsets A, B, C, and/or X of U that contradict the conjectures listed here.

For any subsets A, B, C, and X of U:

a. $A - (B - C) = (A - B) - C$
b. $\overline{A - B} = A - \overline{B}$
c. $A - (\overline{B} \cup C) = (A \cup B) - C$
d. $A \triangle B = A \cup B$
e. $B \cup (A - B) = A$
f. If $A \cap C \subseteq B \cap C$, then $A \subseteq B$
g. If $X \subseteq A$, then $X \cup (A \cap B) = (X \cup A) \cap B$
h. If $A \times B = A \times C$, then $B = C$

Solution. Part a.

Let $A = \{1, 2, 3, 5, 7, 8, 9\}$ and $B = \{2, 5, 8\}$ and $C = \{2, 7\}$.

Observe that $(A - B) - C = \{1, 3, 7, 9\} - \{2, 7\} = \{1, 3, 9\}$ and $A - (B - C) = \{1, 2, 3, 5, 7, 8\} - \{5, 8\} = \{1, 2, 3, 7, 9\}$.

Therefore, $(A - B) - C = \{1, 3, 9\} \neq \{1, 2, 3, 7, 9\} = A - (B - C)$, so $(A - B) - C \neq A - (B - C)$.

Part b.

Let $A = \{1, 4, 6, 8, 9\}$ and $B = \{1, 2, 7, 8\}$.

Observe that $\overline{A - B} = \overline{\{1, 4, 6, 8, 9\} - \{1, 2, 7, 8\}} = \overline{\{4, 6, 9\}} = \{1, 2, 3, 5, 7, 8, 10\}$.

Observe that $A - \overline{B} = \{1, 4, 6, 8, 9\} - \overline{\{1, 2, 7, 8\}} = \{1, 4, 6, 8, 9\} - \{3, 4, 5, 6, 9, 10\} = \{1, 8\}$.

Therefore, $\overline{A - B} = \{1, 2, 3, 5, 7, 8, 10\} \neq \{1, 8\} = A - \overline{B}$, so $\overline{A - B} \neq A - \overline{B}$.

Part c.

Let $A = \{2, 4, 6, 7, 8, 9, 10\}$ and $B = \{2, 6, 7, 9\}$ and $C = \{1, 2, 6, 8, 10\}$.

Observe that

$$\begin{aligned}
A - (\overline{B} \cup C) &= \{2, 4, 6, 7, 8, 9, 10\} - (\overline{\{2, 6, 7, 9\}} \cup \{1, 2, 6, 8, 10\}) \\
&= \{2, 4, 6, 7, 8, 9, 10\} - (\{1, 3, 4, 5, 8, 10\} \cup \{1, 2, 6, 8, 10\}) \\
&= \{2, 4, 6, 7, 8, 9, 10\} - \{1, 2, 3, 4, 5, 6, 8, 10\} \\
&= \{7, 9\}.
\end{aligned}$$

Observe that

$$\begin{aligned}
(A \cup B) - C &= (\{2, 4, 6, 7, 8, 9, 10\} \cup \{2, 6, 7, 9\}) - \{1, 2, 6, 8, 10\} \\
&= \{2, 4, 6, 7, 8, 9, 10\} - \{1, 2, 6, 8, 10\} \\
&= \{4, 7, 9\}.
\end{aligned}$$

Therefore, $A - (\overline{B} \cup C) = \{7, 9\} \neq \{4, 7, 9\} = (A \cup B) - C$, so $A - (\overline{B} \cup C) \neq (A \cup B) - C$.

Part d.

Let $A = \{2, 4, 6, 7, 9\}$ and $B = \{1, 2, 3, 5, 7, 9\}$.

Observe that

$$\begin{aligned}
A \triangle B &= (\{2, 4, 6, 7, 9\} - \{1, 2, 3, 5, 7, 9\}) \cup (\{1, 2, 3, 5, 7, 9\} - \{2, 4, 6, 7, 9\}) \\
&= \{4, 6\} \cup \{1, 3, 5\} \\
&= \{1, 3, 4, 5, 6\}.
\end{aligned}$$

Observe that $A \cup B = \{2, 4, 6, 7, 9\} \cup \{1, 2, 3, 5, 7, 9\} = \{1, 2, 3, 4, 5, 6, 7, 9\}$.

Therefore, $A \triangle B = \{1, 3, 4, 5, 6\} \neq \{1, 2, 3, 4, 5, 6, 7, 9\} = A \cup B$, so $A \triangle B \neq A \cup B$.

Part e.

Let $A = \{1, 2, 3, 5, 8, 9\}$ and $B = \{2, 3, 8, 10\}$.

Observe that

$$\begin{aligned}
B \cup (A - B) &= \{2, 3, 8, 10\} \cup (\{1, 2, 3, 5, 8, 9\} - \{2, 3, 8, 10\}) \\
&= \{2, 3, 8, 10\} \cup \{1, 5, 9\} \\
&= \{1, 2, 3, 5, 8, 9, 10\} \\
&\neq \{1, 2, 3, 5, 8, 9\} \\
&= A.
\end{aligned}$$

Therefore, $B \cup (A - B) \neq A$.

Part f.

Let $A = \{1, 3, 6, 8\}$ and $B = \{1, 5, 7, 8, 9\}$ and $C = \{1, 5, 8, 9\}$.

Since $A \cap C = \{1, 8\} \subseteq \{1, 5, 8, 9\} = B \cap C$, then $A \cap C \subseteq B \cap C$.

Since $3 \in A$, but $3 \notin B$, then $A \not\subseteq B$.

Hence, $A \cap C \subseteq B \cap C$ and $A \not\subseteq B$, so $A \cap C \subseteq B \cap C$ does not imply $A \subseteq B$.

Therefore, the conjecture 'if $A \cap C \subseteq B \cap C$, then $A \subseteq B$' is false.

Part g.

Let $X = \{1, 2, 3\}$ and $A = \{1, 2, 3, 5, 6, 10\}$ and $B = \{2, 6, 10\}$.

Since $\{1, 2, 3\} \subseteq \{1, 2, 3, 5, 6, 10\}$, then $X \subseteq A$.

Observe that

$$\begin{aligned} X \cup (A \cap B) &= \{1, 2, 3\} \cup (\{1, 2, 3, 5, 6, 10\} \cap \{2, 6, 10\}) \\ &= \{1, 2, 3\} \cup \{2, 6, 10\} \\ &= \{1, 2, 3, 6, 10\}. \end{aligned}$$

Observe that

$$\begin{aligned} (X \cup A) \cap B &= (\{1, 2, 3\} \cup \{1, 2, 3, 5, 6, 10\}) \cap \{2, 6, 10\} \\ &= \{1, 2, 3, 5, 6, 10\} \cap \{2, 6, 10\} \\ &= \{2, 6, 10\}. \end{aligned}$$

Hence, $X \cup (A \cap B) = \{1, 2, 3, 6, 10\} \neq \{2, 6, 10\} = (X \cup A) \cap B$, so $X \cup (A \cap B) \neq (X \cup A) \cap B$.

Thus, $X \subseteq A$ and $X \cup (A \cap B) \neq (X \cup A) \cap B$, so $X \subseteq A$ does not imply $X \cup (A \cap B) = (X \cup A) \cap B$.

Therefore, the conjecture 'if $X \subseteq A$, then $X \cup (A \cap B) = (X \cup A) \cap B$' is false.

Part h.

Let $A = \emptyset$ and $B = \{2, 5\}$ and $C = \{1, 2, 5\}$.

Since $A \times B = \emptyset \times \{2, 5\} = \emptyset = \emptyset \times \{1, 2, 5\} = A \times C$, then $A \times B = A \times C$.

Since $B = \{2, 5\} \neq \{1, 2, 5\} = C$, then $B \neq C$.

Hence, $A \times B = A \times C$ and $B \neq C$, so $A \times B = A \times C$ does not imply $B = C$.

Therefore, the conjecture 'if $A \times B = A \times C$, then $B = C$' is false. $\square$

Exercise 105. Let $U = \{1, 2, 3, ..., 9, 10\}$.

a. Try to find subsets A, B, and X of U with A and B distinct(i.e., $A \neq B$) such that $A \cup X = B \cup X$ and $A \cap X = B \cap X$.

Do not try any more than five combinations of the three sets.

b. Suppose that the three sets described in (a) are impossible to find(i.e. do not exist).

Can you formulate an elegant statement of a theorem asserting this fact?

Note: If $A = B$, then surely $A \cup X = B \cup X$ and $A \cap X = B \cap X$ for any set X.

Solution. Part a.

Let $A = \{1\}$ and $B = \{2\}$ and $X = \{1, 2\}$.

Then $A \neq B$, and $A \cup X = \{1, 2\} = B \cup X$, and $A \cap X = \{1\} \neq \{2\} = B \cap X$.

Let $A = \{1, 2\}$ and $B = \{2\}$ and $X = \{1, 2\}$.

Then $A \neq B$, and $A \cup X = \{1, 2\} = B \cup X$, and $A \cap X = \{1, 2\} \neq \{2\} = B \cap X$.

Let $A = \{1\}$ and $B = \emptyset$ and $X = \{1\}$.

Then $A \neq B$, and $A \cup X = \{1\} \cup \{1\} = \{1\} = \emptyset \cup \{1\} = B \cup X$, and $A \cap X = \{1\} \cap \{1\} = \{1\} \neq \emptyset = \emptyset \cap \{1\} = B \cap X$.

Let $A = \{1\}$ and $B = \emptyset$ and $X = \emptyset$.

Then $A \neq B$, and $A \cup X = \{1\} \cup \emptyset = \{1\} \neq \emptyset = \emptyset \cup \emptyset = B \cup X$, and $A \cap X = \{1\} \cap \emptyset = \emptyset = \emptyset \cap \emptyset = B \cap X$.

Let $A = \emptyset$ and $B = \{1\} = X$.

Then $A \neq B$, and $A \cup X = \emptyset \cup \{1\} = \{1\} = \{1\} \cup \{1\} = B \cup X$, and $A \cap X = \emptyset \cap \{1\} = \emptyset \neq \{1\} = \{1\} \cap \{1\} = B \cap X$.

Part b.

We conjecture that the statement 'there exist subsets A, B, X of universal set U such that $A \neq B$ and $A \cup X = B \cup X$ and $A \cap X = B \cap X$' is false.

Hence, for any subsets A, B, X of universal set U, it's impossible that $A \neq B$ and $A \cup X = B \cup X$ and $A \cap X = B \cap X$.

Thus, for any subsets A, B, X of universal set U, if $A \cup X = B \cup X$ and $A \cap X = B \cap X$, then $A = B$.

We prove the statement: for any subsets A, B, X of universal set U, if $A \cup X = B \cup X$ and $A \cap X = B \cap X$, then $A = B$.

Let A, B, and X be any subsets of a universal set U.

Suppose $A \cup X = B \cup X$ and $A \cap X = B \cap X$.

We must prove $A = B$.

Let $x \in A$.

Then $x \in A$ or $x \in X$, so $x \in A \cup X$.

Since $A \cup X = B \cup X$, then $x \in B \cup X$, so either $x \in B$ or $x \in X$.

We consider these cases separately.

Case 1: Suppose $x \in B$.

Since $x \in A$ and $x \in B$, then $x \in A$ implies $x \in B$, so $A \subseteq B$.

Case 2: Suppose $x \in X$.

Since $x \in A$ and $x \in X$, then $x \in A \cap X$.

Since $A \cap X = B \cap X$, then $x \in B \cap X$, so $x \in B$.

Since $x \in A$ and $x \in B$, then $x \in A$ implies $x \in B$, so $A \subseteq B$.

Therefore, in all cases, $A \subseteq B$.

Let $y \in B$.
Then $y \in B$ or $y \in X$, so $y \in B \cup X$.
Since $B \cup X = A \cup X$, then $y \in A \cup X$, so either $y \in A$ or $y \in X$.
We consider these cases separately.
Case 1: Suppose $y \in A$.
Since $y \in B$ and $y \in A$, then $y \in B$ implies $y \in A$, so $B \subseteq A$.
Case 2: Suppose $y \in X$.
Since $y \in B$ and $y \in X$, then $y \in B \cap X$.
Since $B \cap X = A \cap X$, then $y \in A \cap X$, so $y \in A$.
Since $y \in B$ and $y \in A$, then $y \in B$ implies $y \in A$, so $B \subseteq A$.

Therefore, in all cases, $B \subseteq A$.

Since $A \subseteq B$ and $B \subseteq A$, then $A = B$. $\square$

Exercise 106. Let $U = \{1, 2, 3, ..., 9, 10\}$.

a. Try to find subsets A, B, and X of U with A and B distinct such that $A \cap X = B \cap X$ and $A \cap \overline{X} = B \cap \overline{X}$.

Do not try any more than five combinations of the three sets.

b. Suppose that the three sets described in (a) are impossible to find(i.e. do not exist).

Can you formulate an elegant statement of a theorem asserting this fact?

Note: If $A = B$, then $A \cap X = B \cap X$ and $A \cap \overline{X} = B \cap \overline{X}$ for any set X.

Solution. Part a.

Let $A = \{1, 2\}$ and $B = \{1, 2, 3\}$ and $X = \{1, 2, 7\}$.
Then $A \neq B$ and $A \cap X = \{1, 2\} = B \cap X$.
Observe that

$$\begin{aligned} A \cap \overline{X} &= \{1, 2\} \cap \{3, 4, 5, 6, 8, 9, 10\} \\ &= \emptyset \\ &\neq \{3\} \\ &= \{1, 2, 3\} \cap \{3, 4, 5, 6, 8, 9, 10\} \\ &= B \cap \overline{X}. \end{aligned}$$

Let TODO: Finish this exercise.

$\square$

TODO: Continue working on the 1.3 exercises

Chapter 7.1

TODO: Work on chapter 7 and 8.

Chapter 8.1

Exercise 107. Let $f_1 = \{(2,3),(3,5),(4,7),(5,9\}$.
Analyze f_1.

Solution. Since f_1 is a set of ordered pairs, then f_1 is a relation.
The domain of f_1 is the set $\{2,3,4,5\}$.
The range of f_1 is the set $\{3,5,7,9\}$.
Since no two distinct ordered pairs have the same first element, then f_1 is a function. □

Exercise 108. Let $f_2 = \{(a,z),(b,y),...,(y,b),(z,a)\}$.
Analyze f_2.

Proof. Since f_2 is a set of ordered pairs, then f_2 is a relation.
The domain of f_2 is the set $\{a,b,c,...,y,z\}$, the set of all lower-case letters of the English alphabet.
The range of f_2 is the set $\{a,b,c,...,y,z\}$, the set of all lower-case letters of the English alphabet.
The domain of f_2 equals the range of f_2.
Since no two distinct ordered pairs have the same first element, then f_2 is a function. □

Exercise 109. Let $f_3 = \{(1,1),(2,2),...,(100,100)\}$.
Analyze f_3.

Solution. Since f_3 is a set of ordered pairs, then f_3 is a relation.
The domain of f_3 is the set $\{1,2,3,...,99,100\}$, the set of all integers between 1 and 100, inclusive.
The range of f_3 is the set $\{1,2,3,...,99,100\}$, the set of all integers between 1 and 100, inclusive.
The domain of f_3 equals the range of f_3.
Since no two distinct ordered pairs have the same first element, then f_3 is a function. □

Exercise 110. Let $R_1 = \{(1,a),(1,b),...,(1,z)\}$.
Analyze R_1.

Solution. Since R_1 is a set of ordered pairs, then R_1 is a relation.
The domain of R_1 is the singleton set $\{1\}$.
The range of R_1 is the set $\{a,b,c,...,z\}$, the set of all lower-case letters of the English alphabet.
Since $(1,a) \in R_1$ and $(1,b) \in R_1$ and $a \neq b$, then R_1 is not a function. □

Exercise 111. Let $R_2 = \{(1,1),(1,-1),(4,2),(4,-2),(9,3),(9,-3)\}$.
Analyze R_2.

Solution. Since R_2 is a set of ordered pairs, then R_2 is a relation.
The domain of R_2 is the set $\{1,4,9\}$.
The range of R_2 is the set $\{1,-1,2,-2,3-,-3\}$.
Since $(1,1) \in R_2$ and $(1,-1) \in R_2$, but $1 \neq -1$, then R_2 is not a function.
□