# Analysis Theory

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## Metric Spaces

**Proposition 1.** ***distance in a metric space is nonnegative***
*If $d$ is a metric on a set $M$, then $d(x, y) \geq 0$ for all $x, y \in M$.*

*Proof.* Suppose $d$ is a metric on a set $M$.
Let $x, y \in M$ be given.
Since $0 = d(x, x) \leq d(x, y) + d(y, x) = d(x, y) + d(x, y) = 2d(x, y)$, then $0 \leq 2d(x, y)$, so $0 \leq d(x, y)$.
Therefore, $d(x, y) \geq 0$, as desired. □

## Limits

**Theorem 2.** ***uniqueness of a limit of a function***
*Let $(M_1, d_1)$ and $(M_2, d_2)$ be metric spaces.*
*Let $E \subset M_1$.*
*Let $f : E \to M_2$ be a function.*
*Let $a$ be an accumulation point of $E$.*
*Then the limit of $f$ at $a$, if it exists, is unique.*

*Proof.* Suppose a limit of $f$ at $a$ exists.
Then a limit of $f$ at $a$ is an element of $M_2$, so there is at least one limit of $f$ at $a$.
To prove the limit is unique, let $L_1, L_2 \in M_2$ such that $L_1$ is a limit of $f$ at $a$ and $L_2$ is a limit of $f$ at $a$.
We must prove $L_1 = L_2$.
Suppose $L_1 \neq L_2$.
Then $d_2(L_1, L_2) > 0$.
Let $\epsilon = d_2(L_1, L_2)$.
Then $\epsilon > 0$, so $\frac{\epsilon}{2} > 0$.
Since $L_1$ is a limit of $f$ at $a$ and $\frac{\epsilon}{2} > 0$, then there exists $\delta_1 > 0$ such that if $x \neq a$ and $d_1(x, a) < \delta_1$, then $d_2(f(x), L_1) < \frac{\epsilon}{2}$.
Since $L_2$ is a limit of $f$ at $a$ and $\frac{\epsilon}{2} > 0$, then there exists $\delta_2 > 0$ such that if $x \neq a$ and $d_1(x, a) < \delta_2$, then $d_2(f(x), L_2) < \frac{\epsilon}{2}$.
Let $\delta = \min\{\delta_1, \delta_2\}$.

Then $\delta \leq \delta_1$ and $\delta \leq \delta_2$.

Since $\delta_1 > 0$ and $\delta_2 > 0$, then $\delta > 0$.

Since $a$ is an accumulation point of $E$ and $\delta > 0$, then there exists $x \in E$ such that $x \in N'(a;\delta)$.

Since $x \in N'(a;\delta)$, then $x \neq a$ and $x \in N(a;\delta)$, so $d_1(x,a) < \delta$.

Since $d_1(x,a) < \delta \leq \delta_1$, then $d_1(x,a) < \delta_1$.

Since $x \neq a$ and $d_1(x,a) < \delta_1$, then $d_2(f(x),L_1) < \frac{\epsilon}{2}$.

Since $d_1(x,a) < \delta \leq \delta_2$, then $d_1(x,a) < \delta_2$.

Since $x \neq a$ and $d_1(x,a) < \delta_2$, then $d_2(f(x),L_2) < \frac{\epsilon}{2}$.

Observe that

$$\begin{aligned}
d_2(L_1,L_2) &\leq d_2(L_1,f(x)) + d_2(f(x),L_2) \\
&= d_2(f(x),L_1) + d_2(f(x),L_2) \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&= \epsilon.
\end{aligned}$$

Hence, $d_2(L_1,L_2) < \epsilon$.

Thus, we have $d_2(L_1,L_2) = \epsilon$ and $d_2(L_1,L_2) < \epsilon$, a violation of trichotomy of $\mathbb{R}$.

Therefore, $L_1 = L_2$, as desired. □

# Convergent sequences in a metric space

**Theorem 3.** ***uniqueness of a limit of a convergent sequence***

*The limit of a convergent sequence of real numbers is unique.*

*Proof.* Let $(a_n)$ be a convergent sequence of real numbers.

Then a limit of $(a_n)$ exists as a real number.

Thus, there is at least one limit of $(a_n)$.

To prove the limit is unique, let $L_1, L_2 \in \mathbb{R}$ such that $L_1$ is a limit of $(a_n)$ and $L_2$ is a limit of $(a_n)$.

We must prove $L_1 = L_2$.

Suppose $L_1 \neq L_2$.

Then $L_1 - L_2 \neq 0$, so $|L_1 - L_2| > 0$.

Let $\epsilon = \frac{|L_1 - L_2|}{2}$.

Then $\epsilon > 0$.

Since $L_1$ is a limit of $(a_n)$ and $\epsilon > 0$, then there exists $N_1 \in \mathbb{N}$ such that if $n > N_1$ then $|a_n - L_1| < \epsilon$.

Since $L_2$ is a limit of $(a_n)$ and $\epsilon > 0$, then there exists $N_2 \in \mathbb{N}$ such that if $n > N_2$ then $|a_n - L_2| < \epsilon$.

Let $N = \max\{N_1, N_2\}$.

Let $n \in \mathbb{N}$ such that $n > N$.

Since $n > N \geq N_1$, then $n > N_1$.

Hence, $|a_n - L_1| < \epsilon$.

Since $n > N \geq N_2$, then $n > N_2$.

Hence, $|a_n - L_2| < \epsilon$.
Observe that

$$\begin{aligned} |L_1 - L_2| &= |(L_1 - a_n) + (a_n - L_2)| \\ &\leq |L_1 - a_n| + |a_n - L_2| \\ &= |a_n - L_1| + |a_n - L_2| \\ &< \epsilon + \epsilon \\ &= 2\epsilon. \end{aligned}$$

Thus, $|L_1 - L_2| < 2\epsilon$, so $\frac{|L_1 - L_2|}{2} < \epsilon$.
Hence, $\epsilon < \epsilon$, a contradiction.
Therefore, $L_1 = L_2$, as desired. $\square$