Analysis Exercises

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Metric Spaces

Exercise 1. Let (M, d) be a metric space.
Let $a, b \in M$ with $a \neq b$.
Then there exist positive real numbers ϵ and δ such that $N(a;\epsilon) \cap N(b;\delta) = \emptyset$.

Proof. Let $\epsilon = \delta = \frac{d(a,b)}{2}$.
Since $a \neq b$, then $d(a, b) > 0$, so $\frac{d(a,b)}{2} > 0$.
Hence, $\epsilon = \delta > 0$.
Suppose for the sake of contradiction $N(a;\epsilon) \cap N(b;\delta) \neq \emptyset$.
Then there exists x such that $x \in N(a;\epsilon) \cap N(b;\delta)$.
Hence, $x \in N(a;\epsilon)$ and $x \in N(b;\delta)$, so $d(x, a) < \epsilon$ and $d(x, b) < \delta$.
Observe that

$$\begin{aligned} d(a,b) &\leq d(a,x) + d(x,b) \\ &= d(x,a) + d(x,b) \\ &< \epsilon + \delta \\ &= \epsilon + \epsilon \\ &= 2\epsilon \\ &= 2 \cdot \frac{d(a,b)}{2} \\ &= d(a,b). \end{aligned}$$

Thus, $d(a, b) < d(a, b)$, a contradiction.
Therefore, $N(a;\epsilon) \cap N(b;\delta) = \emptyset$. ☐