Analysis Notes

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Metric Spaces

A metric space is a set of points upon which a notion of distance is defined with the following properties:

1. The distance between points is nonnegative.
2. The distance between points x and y is the same as the distance between y and x.
3. The distance satisfies the triangle inequality.

The metric space concept generalizes the notion of distance between real numbers.

Definition 1. metric

Let M be a nonempty set.

A function $d : M \times M \to \mathbb{R}$ is a **metric on** M iff the following axioms hold:

MS1. $d(x, y) = 0$ iff $x = y$ for all $x, y \in M$.

MS2. $d(x, y) = d(y, x)$ for all $x, y \in M$. (Symmetry)

MS3. $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in M$. (Triangle inequality)

Definition 2. metric space

If d is a metric on a set M, then we say the pair (M, d) is a **metric space**.

Proposition 3. ***distance in a metric space is nonnegative***

If d is a metric on a set M, then $d(x, y) \geq 0$ for all $x, y \in M$.

Let (M, d) be a metric space.

Then d is a metric on the set M.

Therefore, $d : M \times M \to \mathbb{R}$ is a function such that axioms MS1-3 hold.

The function d is called the distance function, or **metric**, and specifies the distance between any two points x and y of the space M.

Therefore, if $x, y \in M$, then $d(x, y)$ is the distance between x and y.

Let $x, y \in M$.

Since $d(x, y) = 0$ iff $x = y$, then $d(x, y) \neq 0$ iff $x \neq y$.

Since d is a metric on M, then $d(x, y) \geq 0$, so either $d(x, y) > 0$ or $d(x, y) = 0$.

Hence, $d(x, y) > 0$ iff $x \neq y$.

Therefore, $d(x, y) > 0$ iff $x \neq y$ for all $x, y \in M$.

If $x = y$, then $d(x, y) = 0$, so the distance between a point and itself is zero.

If $x \neq y$, then $d(x, y) > 0$, so the distance between two distinct points is positive.

Therefore, the distance between any two distinct points of a metric space is positive and the distance between a point and itself is zero.

Example 4. $(\mathbb{R}, d)$ **with Euclidean metric** d

The set of real numbers $\mathbb{R}$ with the distance function $d : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined by $d(x, y) = |x - y|$ for all $(x, y) \in \mathbb{R} \times \mathbb{R}$ is a metric space.

Topology of a metric space

Definition 5. open ball

Let (M, d) be a metric space.

Let r be a positive real number.

Let $p \in M$.

The **open ball with radius** r **and center** p is the set $B_r(p) = \{x \in M : d(x, p) < r\}$.

excludes boundary

Definition 6. closed ball

Let (M, d) be a metric space.

Let r be a positive real number.

Let $p \in M$.

The **closed ball with radius** r **and center** p is the set $\overline{B_r}(p) = \{x \in M : d(x, p) \leq r\}$.

includes boundary

Convergent sequences in a metric space

Definition 7. sequence of points in a metric space

A **sequence of points in a metric space** E is a function $f : \mathbb{N} \to E$.

Let (p_n) be a sequence of points in a metric space E.

Then (p_n) is a function from $\mathbb{N}$ to E that maps $n \mapsto p_n$ for all $n \in \mathbb{N}$.

The points $p_1, p_2, p_3, ..., p_n, ...$ are the terms of the sequence.

p_n is the n^{th} term of the sequence and $p_n \in E$ for all $n \in \mathbb{N}$.

Definition 8. constant sequence in a metric space

Let p be a point of a metric space E.

Let (p_n) be a sequence of points in E such that $p_n = p$ for all $n \in \mathbb{N}$.

Then (p_n) is called a **constant sequence**.

Definition 9. limit of a sequence of points in a metric space

Let (p_n) be a sequence of points in a metric space E.

A point $L \in E$ is a **limit of** (p_n) iff for every positive real ϵ, there exists a natural number N such that $d(p_n, L) < \epsilon$ whenever $n > N$.

In symbols, $L \in E$ is a limit of (p_n) iff
$(\forall \epsilon > 0)(\exists N \in \mathbb{N})(\forall n \in \mathbb{N})(n > N \to d(p_n, L) < \epsilon)$.

Theorem 10. ***uniqueness of a sequence limit***
A sequence of points in a metric space has at most one limit.

Let (p_n) be a sequence of points in a metric space E.
Either a limit of (p_n) exists or does not exist in E, so either (p_n) is convergent or (p_n) is divergent.
If (p_n) has a limit, then the limit is unique.
Thus, if (p_n) is convergent, then its limit is unique.
Therefore, a convergent sequence has a unique limit.
The statement 'a sequence (p_n) converges to a point $L \in E$' is denoted by either $p_n \to L$ or $\lim_{n\to\infty} p_n = L$.
If (p_n) does not have a limit, $\lim_{n\to\infty} p_n$ does not exist.
Therefore, if (p_n) is a divergent sequence, then $\lim_{n\to\infty} p_n$ does not exist.

Let (p_n) be a sequence of points in a metric space E.
We say that (p_n) converges to a point $L \in E$, denoted $\lim_{n\to\infty} p_n = L$, iff for every positive real ϵ, there exists a natural number N such that $d(p_n, L) < \epsilon$ whenever $n > N$.
Therefore $\lim_{n\to\infty} p_n = L$ iff
$(\forall \epsilon > 0)(\exists N \in \mathbb{N})(\forall n \in \mathbb{N})(n > N \to d(p_n, L) < \epsilon)$.

Limits

The limit concept expresses the idea that for a given function f, $f(x)$ is arbitrarily close to some point L if x is sufficiently close to some point a.

Definition 11. $\epsilon - \delta$ **definition of a limit of a function at a point**
Let (M_1, d_1) and (M_2, d_2) be metric spaces.
Let $E \subset M_1$.
Let $f : E \to M_2$ be a function.
Let a be an accumulation point of E.
A point $L \in M_2$ **is a limit of** f **at** a iff for every $\epsilon > 0$, there exists $\delta > 0$ such that $d_2(f(x), L) < \epsilon$ whenever $x \neq a$ and $d_1(x, a) < \delta$.

Therefore, $L \in M_2$ is a limit of f at a iff
$(\forall \epsilon > 0)(\exists \delta > 0)(\forall x \in E)(x \neq a \wedge d_1(x, a) < \delta \to d_2(f(x), L) < \epsilon)$.
Therefore, $L \in M_2$ is not a limit of f at a iff
$(\exists \epsilon > 0)(\forall \delta > 0)(\exists x \in E)(x \neq a \wedge d_1(x, a) < \delta \wedge d_2(f(x), L) \geq \epsilon)$.

The logical structure of this definition implies that δ is a function of ϵ, so δ depends on ϵ.

Let (M_1, d_1) and (M_2, d_2) be metric spaces.
Let $E \subset M_1$.
Let $f : E \to M_2$ be a function.
Let a be an accumulation point of E.
Suppose $L \in M_2$ is a limit of f at a.
Then for every $\epsilon > 0$, there exists $\delta > 0$ such that for all $x \in E$, if $x \neq a$ and $d_1(x, a) < \delta$, then $d_2(f(x), L) < \epsilon$.
Observe that

$$\begin{aligned}
(\forall \epsilon > 0)(\exists \delta > 0)(\forall x \in E)(x \neq a \wedge d_1(x, a) < \delta \to d_2(f(x), L) < \epsilon) &\quad \Leftrightarrow \\
(\forall \epsilon > 0)(\exists \delta > 0)(\forall x \in E)(x \neq a \wedge x \in N(a;\delta) \to f(x) \in N(L;\epsilon)) &\quad \Leftrightarrow \\
(\forall \epsilon > 0)(\exists \delta > 0)(\forall x \in E)(x \in N'(a;\delta) \to f(x) \in N(L;\epsilon)) &\quad \Leftrightarrow \\
(\forall \epsilon > 0)(\exists \delta > 0)(\forall x \in E \cap N'(a;\delta))(f(x) \in N(L;\epsilon)). &
\end{aligned}$$

Therefore, $L \in M_2$ is a limit of f at a iff
$(\forall \epsilon > 0)(\exists \delta > 0)(\forall x \in E)(x \neq a \wedge d_1(x, a) < \delta \to d_2(f(x), L) < \epsilon)$
iff
$(\forall \epsilon > 0)(\exists \delta > 0)(\forall x \in E \cap N'(a;\delta))(f(x) \in N(L;\epsilon))$.

Theorem 12. ***uniqueness of a limit of a function***
Let (M_1, d_1) and (M_2, d_2) be metric spaces.
Let $E \subset M_1$.
Let $f : E \to M_2$ be a function.
Let a be an accumulation point of E.
Then the limit of f at a, if it exists, is unique.

Let (M_1, d_1) and (M_2, d_2) be metric spaces.
Let $E \subset M_1$.
Let $f : E \to M_2$ be a function.
Let a be an accumulation point of E.
Suppose a limit of f at a exists.
Since the limit of a function is unique, we denote the limit of f at a by $\lim_{x \to a} f(x)$.
If $L \in M_2$ is the limit of f at a, then we say $\lim_{x \to a} f(x) = L$.
Therefore, $L \in M_2$ is a limit of f at a iff $\lim_{x \to a} f(x) = L$ iff
$(\forall \epsilon > 0)(\exists \delta > 0)(\forall x \in E)(x \neq a \wedge d_1(x, a) < \delta \to d_2(f(x), L) < \epsilon)$
iff
$(\forall \epsilon > 0)(\exists \delta > 0)(\forall x \in E \cap N'(a;\delta))(f(x) \in N(L;\epsilon))$

Let (M_1, d_1) and (M_2, d_2) be metric spaces.
Let $E \subset M_1$.
Let $f : E \to M_2$ be a function.
Let a be an accumulation point of E.

Suppose the limit of f at a exists.
Then there exists $L \in M_2$ such that $\lim_{x \to a} f(x) = L$.
Thus, $(\exists L \in M_2)(\lim_{x \to a} f(x) = L)$.
Observe that $\lim_{x \to a} f(x) = L$ iff
$(\forall \epsilon > 0)(\exists \delta > 0)(\forall x \in E)(x \neq a \wedge d_1(x, a) < \delta \rightarrow d_2(f(x), L) < \epsilon)$
iff
$(\forall \epsilon > 0)(\exists \delta > 0)(\forall x \in E \cap N'(a; \delta))(f(x) \in N(L; \epsilon))$.

Suppose the limit of f at a does not exist.
Then there is no $L \in M_2$ such that $\lim_{x \to a} f(x) = L$.
Thus, $\neg(\exists L \in M_2)(\lim_{x \to a} f(x) = L)$.