Complex Analysis Examples

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Complex Number System $\mathbb{C}$

Example 1. negative square root

Observe that

$$\begin{aligned} \sqrt{2} \cdot i &= \sqrt{2} \cdot \sqrt{-1} \\ &= \sqrt{2 \cdot (-1)} \\ &= \sqrt{-2}. \end{aligned}$$

Example 2. origin of the complex plane

Observe that $0 = 0 + 0i = 0i$ is the origin $(0, 0)$ of the complex plane.

Example 3. imaginary number i

Observe that $i = 0 + 1i = 1i$ is the point $(0, 1)$ of the complex plane.

Example 4. unit of the complex plane

Observe that $1 = 1 + 0i$ is the point $(1, 0)$ of the complex plane.

Example 5. modulus of the imaginary number i

Since $i = 0 + 1i$, then $|i| = \sqrt{0^2 + 1^2} = 1$, so $|i| = 1$.

Example 6. equality of complex numbers

Observe that $z = 0$ iff $z = 0 + 0i$.

Example 7. addition of complex numbers

Let $z_1 = 2 + 3i$ and $z_2 = 4 + 5i$.

The sum is $z_1 + z_2 = (2 + 3i) + (4 + 5i) = 2 + 3i + 4 + 5i = 6 + 8i$.

Example 8. subtraction of complex numbers

Let $z_1 = 2 + 3i$ and $z_2 = 4 + 5i$.

The difference is $z_1 - z_2 = (2 + 3i) - (4 + 5i) = 2 + 3i - 4 - 5i = -2 - 2i$.

Example 9. multiplication of complex numbers

Let $z_1 = 2 + 3i$ and $z_2 = 4 + 5i$.

The product is $z_1 \cdot z_2 = (2+3i)(4+5i) = 8+10i+12i+15(i^2) = 8+22i-15 = -7 + 22i$.

Example 10. complex conjugate

The complex conjugate of $2+3i$ is $2-3i$.

The complex conjugate of $2-3i$ is $2+3i$.

These two complex numbers are mirror images of each other with respect to the x axis of the complex plane.

Example 11. multiplicative inverse of a nonzero complex number

Let $z = 2+3i$.

Since $2 \neq 0$ and $3 \neq 0$, then $2+3i \neq 0+0i$, so $z \neq 0$.

Observe that

$$\begin{aligned}
\frac{1}{z} &= \frac{1}{2+3i} \\
&= \frac{1}{2+3i} \cdot \frac{2-3i}{2-3i} \\
&= \frac{2-3i}{(2+3i)(2-3i)} \\
&= \frac{2-3i}{4-9i^2} \\
&= \frac{2-3i}{4+9} \\
&= \frac{2-3i}{13} \\
&= \frac{2}{13} - \frac{3i}{13}.
\end{aligned}$$

Therefore, the multiplicative inverse of $2+3i$ is $\frac{2}{13} - \frac{3i}{13}$.

Example 12. division of complex numbers

Let $z_1 = 2+3i$ and $z_2 = 4+5i$.

Observe that

$$\begin{aligned}
\frac{z_1}{z_2} &= \frac{2+3i}{4+5i} \\
&= \frac{2+3i}{4+5i} \cdot \frac{4-5i}{4-5i} \\
&= \frac{(2+3i)(4-5i)}{(4+5i)(4-5i)} \\
&= \frac{8-10i+12i+15}{16+25} \\
&= \frac{23+2i}{41} \\
&= \frac{23}{41} + \frac{2i}{41}.
\end{aligned}$$

Therefore, the quotient is $\frac{z_1}{z_2} = \frac{23}{41} + \frac{2i}{41}$.

Example 13. $\mathbb{C}$ is not an ordered field.
$(\mathbb{C}, +, \cdot)$ is not an ordered field.

Proof. Suppose $(\mathbb{C}, +, \cdot)$ is an ordered field.
Then there is a subset P of positive elements of $\mathbb{C}$ and $1 \in P$.
Since $i \in \mathbb{C}$ and $i \neq 0$, then $i^2 \in P$.
Since $i^2 = -1$, then $-1 \in P$.
Hence, we have $1 \in P$ and $-1 \in P$, a violation of trichotomy.
Therefore, $(\mathbb{C}, +, \cdot)$ is not an ordered field. $\square$

Hence, there is no way to order the complex numbers.
Therefore, no order relation can be defined on $\mathbb{C}$.

Example 14. solve equation $z^n = a + bi$ in $\mathbb{C}$
Compute the solution set of the equation $z^2 = i$.

Solution. Let S be the solution set to the equation $z^2 = i$.
Then $S = \{z \in \mathbb{C} : z^2 = i\}$.
Let $z \in S$.
Then $z \in \mathbb{C}$ and $z^2 = i$.
Since $z \in \mathbb{C}$, then there exist $|z| \in \mathbb{R}$ and $\theta \in \mathbb{R}$ such that $z = |z| cis(\theta)$.
The polar representation of i is $i = 1 \cdot cis(\frac{\pi}{2})$.
Observe that

$$\begin{aligned} 1 \cdot cis(\frac{\pi}{2}) &= i \\ &= z^2 \\ &= (|z| cis(\theta))^2 \\ &= |z|^2 \cdot (cis(\theta))^2 \\ &= |z|^2 \cdot cis(2\theta). \end{aligned}$$

Hence, $1 \cdot cis(\frac{\pi}{2}) = |z|^2 \cdot cis(2\theta)$, so $|z|^2 = 1$, and the angles $\frac{\pi}{2}$ and 2θ differ by an integer multiple of 2π.
Since $|z|^2 = 1$, then $|z| = 1$.
Since the angles $\frac{\pi}{2}$ and 2θ differ by an integer multiple of 2π, then $2\theta - \frac{\pi}{2} = 2n\pi$ for any integer n.
Thus, $2\theta = 2n\pi + \frac{\pi}{2}$, so $\theta = n\pi + \frac{\pi}{4}$ for any integer n.
On the interval $[0, 2\pi)$, $\theta = \frac{\pi}{4}, \frac{5\pi}{4}$.
Since $z = |z| cis(\theta)$, and $|z| = 1$, and either $\theta = \frac{\pi}{4}$ or $\theta = \frac{5\pi}{4}$, then either $z = cis(\frac{\pi}{4})$ or $z = cis(\frac{5\pi}{4})$.
Observe that $cis(\frac{\pi}{4}) = \cos(\frac{\pi}{4}) + i \sin(\frac{\pi}{4}) = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} i$.
Observe that $cis(\frac{5\pi}{4}) = \cos(\frac{5\pi}{4}) + i \sin(\frac{5\pi}{4}) = \frac{-1}{\sqrt{2}} - \frac{1}{\sqrt{2}} i$.

Hence, either $z = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$ or $z = \frac{-1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i$, so $z \in \{\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i, \frac{-1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i\}$.

Therefore, $z \in S$ implies $z \in \{\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i, \frac{-1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i\}$, so $S \subset \{\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i, \frac{-1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i\}$.

We prove $\{\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i, \frac{-1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i\} \subset S$.

Let $\alpha = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$ and $\beta = \frac{-1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i$.

Then $\alpha \in \mathbb{C}$ and $\beta \in \mathbb{C}$.

We must prove $\alpha \in S$ and $\beta \in S$.

We prove $\alpha \in S$.

Observe that

$$\begin{aligned} \alpha^2 &= (\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i)^2 \\ &= \frac{1}{2} + \frac{2i}{2} + \frac{i^2}{2} \\ &= \frac{1}{2} + i - \frac{1}{2} \\ &= i. \end{aligned}$$

Hence, $\alpha^2 = i$.

Since $\alpha \in \mathbb{C}$ and $\alpha^2 = i$, then $\alpha \in S$.

We prove $\beta \in S$.

Observe that

$$\begin{aligned} \beta^2 &= (\frac{-1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i)^2 \\ &= \frac{1}{2} + \frac{2i}{2} + \frac{i^2}{2} \\ &= \frac{1}{2} + i - \frac{1}{2} \\ &= i. \end{aligned}$$

Hence, $\beta^2 = i$.

Since $\beta \in \mathbb{C}$ and $\beta^2 = i$, then $\beta \in S$.

Since $\alpha \in S$ and $\beta \in S$, then $\{\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i, \frac{-1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i\} \subset S$.

Since $S \subset \{\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i, \frac{-1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i\}$ and $\{\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i, \frac{-1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i\} \subset S$, then $S = \{\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i, \frac{-1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i\}$.

Therefore, the solution set to the equation $z^2 = i$ is $\{\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i, \frac{-1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i\}$. $\square$