Complex Analysis Notes

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July 16, 2023

Sets of Numbers

$\mathbb{N} =$ set of all natural numbers $= \{1, 2, 3, ...\}$
$\mathbb{Z} =$ set of all integers $= \{..., -3, -2, -1, 0, 1, 2, 3, ...\}$
$\mathbb{Q} = \{\frac{m}{n} : m, n \in \mathbb{Z} \land n \neq 0\} =$ set of all rational numbers
$\mathbb{R} =$ set of all real numbers
$\mathbb{R}^* = \{x \in \mathbb{R} : x \neq 0\} = (-\infty, 0) \cup (0, \infty) =$ set of all nonzero real numbers
$\mathbb{R}^+ = \{x \in \mathbb{R} : x > 0\} = (0, \infty) =$ set of all positive real numbers
$\mathbb{C} = \{x + yi : x, y \in \mathbb{R}\} =$ set of all complex numbers
$\mathbb{C}^* = \{z \in \mathbb{C} : z \neq 0\} =$ set of all nonzero complex numbers

Number system relationships
$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$.

Complex Number System $\mathbb{C}$

Definition 1. imaginary unit
The **imaginary unit** i is the number whose square is -1.

Therefore $i = \sqrt{-1}$ and $i^2 = -1$.

Definition 2. square root of a negative number
Let $r \in \mathbb{R}^+$.
Define $\sqrt{-r} = \sqrt{r} \cdot i$.

Definition 3. complex number
Let $z = x + yi$, where $x, y \in \mathbb{R}$.
We call z a **complex number** with **real part** x and **imaginary part** y.

Definition 4. complex number parts
Let $z = x + yi$, where $x, y \in \mathbb{R}$.
Define $Re(z) = x$ to denote the real part of z.
Define $Im(z) = y$ to denote the imaginary part of z.

Definition 5. modulus of a complex number
Let $z = x + yi$, where $x, y \in \mathbb{R}$.
The **modulus(absolute value)** of z is $|z| = \sqrt{x^2 + y^2}$.

Let $x, y \in \mathbb{R}$.
Then $|x+yi| = \sqrt{x^2+y^2}$.
The modulus of a complex number $x + yi$ is the distance between the origin and the point (x, y) in the complex plane.
The distance is obtained using the Pythagorean theorem.
The modulus of a complex number is the magnitude of the vector representing $x + yi$.

Since $i = 0 + i$, then $|i| = \sqrt{0^2+1^2} = 1$, so $|i| = 1$.

Definition 6. Polar representation of a complex number
Let $x, y \in \mathbb{R}$.
Let $z = x + yi$ be a complex number with modulus $|z| = \sqrt{x^2+y^2}$.
Let θ be the counter-clockwise angle made with the positive x axis and the line segment from the origin to the point (x, y).
Then $x = |z|\cos\theta$ and $y = |z|\sin\theta$.
The angle θ is the **argument of** z and θ is in the interval $[0, 2\pi)$.

Let $x, y \in \mathbb{R}$.
Let $z = x + yi$ and $|z| = \sqrt{x^2+y^2}$.
Then $x = |z|\cos\theta$ and $y = |z|\sin\theta$.
Observe that

$$\begin{aligned} z &= x + yi \\ &= |z|\cos\theta + (|z|\sin\theta)i \\ &= |z|(\cos\theta + i\sin\theta). \end{aligned}$$

Therefore the polar form of z is $z = |z| \cdot (\cos\theta + i\sin\theta)$.
Observe that $\tan\theta = \frac{y}{x}$ and $|z|^2 = x^2 + y^2$.

In general, a nonzero complex number can be represented by $r \cdot (\cos\theta + i\sin\theta)$ for some $r, \theta \in \mathbb{R}$ and $r > 0$.

Definition 7. equality of complex numbers
Let $z = a + bi$, where $a, b \in \mathbb{R}$.
Let $w = c + di$, where $c, d \in \mathbb{R}$.
Then $z = w$ iff $a = c$ and $b = d$.

Two complex numbers are equal whenever the real parts are equal and the imaginary parts are equal.

The complex number $z = 0$ iff $z = 0 + 0i$ iff $z = (0, 0)$ iff z is the origin of the complex plane.

Two complex numbers in polar form are equal iff they have the same magnitude and their argument(angle) differs by an integer multiple of 2π.

Definition 8. equality of nonzero complex numbers in polar form
Let $z = r_1 \cdot (\cos\theta_1 + i\sin\theta_1)$ for some $r_1, \theta_1 \in \mathbb{R}$ and $r_1 > 0$.
Let $w = r_2 \cdot (\cos\theta_2 + i\sin\theta_2)$ for some $r_2, \theta_2 \in \mathbb{R}$ and $r_2 > 0$.
Then $z = w$ iff $r_1 = r_2$ and $\theta_1 = \theta_2 + 2\pi k$ for some integer k.

Definition 9. addition over $\mathbb{C}$
Let $z = a + bi$, where $a, b \in \mathbb{R}$.
Let $w = c + di$, where $c, d \in \mathbb{R}$.
The **sum** is $z + w = (a + c) + (b + d)i$.

To add complex numbers, add the corresponding real and imaginary parts.

Proposition 10. *Addition is a binary operation on $\mathbb{C}$.*

Since addition is a binary operation on $\mathbb{C}$, then $\mathbb{C}$ is closed under addition.
Let $z, w \in \mathbb{C}$.
Then $z + w \in \mathbb{C}$.

Theorem 11. ***algebraic properties of addition over*** $\mathbb{C}$

1. $z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3$ *for all* $z_1, z_2, z_3 \in \mathbb{C}$. *(associative)*
2. $z_1 + z_2 = z_2 + z_1$ *for all* $z_1, z_2 \in \mathbb{C}$. *(commutative)*
3. $z + 0 = 0 + z = z$ *for all* $z \in \mathbb{C}$. *(additive identity)*
4. $z + (-z) = (-z) + z = 0$ *for all* $z \in \mathbb{C}$. *(additive inverses)*

The additive identity is $0 = 0 + 0i \in \mathbb{C}$.
Let $z = x + yi$ for some $x, y \in \mathbb{R}$.
The additive inverse of z is $-z = -x - yi \in \mathbb{C}$.
Adding complex numbers is similar to vector addition in the complex plane.

Definition 12. subtraction over $\mathbb{C}$
Let $z, w \in \mathbb{C}$.
The **difference** is defined by $z - w = z + (-w)$.

Let $z = a + bi$, where $a, b \in \mathbb{R}$.
Let $w = c + di$, where $c, d \in \mathbb{R}$.
Then $z - w = (a - c) + (b - d)i$.
Subtraction is the inverse of addition.
To subtract complex numbers, subtract the corresponding real and imaginary parts.

Definition 13. multiplication over $\mathbb{C}$
Let $z = a + bi$, where $a, b \in \mathbb{R}$.
Let $w = c + di$, where $c, d \in \mathbb{R}$.
The **product** is $zw = (ac - bd) + (ad + bc)i$.

Proposition 14. *Multiplication is a binary operation on $\mathbb{C}$.*

Since multiplication is a binary operation on $\mathbb{C}$, then $\mathbb{C}$ is closed under multiplication.

Let $z, w \in \mathbb{C}$.

Then $z \cdot w \in \mathbb{C}$.

Theorem 15. ***algebraic properties of multiplication over*** $\mathbb{C}$

1. $z_1 \cdot (z_2 \cdot z_3) = (z_1 \cdot z_2) \cdot z_3$ *for all* $z_1, z_2, z_3 \in \mathbb{C}$. *(associative)*
2. $z_1 \cdot z_2 = z_2 \cdot z_1$ *for all* $z_1, z_2 \in \mathbb{C}$. *(commutative)*
3. $z \cdot 1 = 1 \cdot z = z$ *for all* $z \in \mathbb{C}$. *(multiplicative identity)*
4. $z \cdot 0 = 0 \cdot z = 0$ *for all* $z \in \mathbb{C}$.
5. $z_1 \cdot (z_2 + z_3) = z_1 \cdot z_2 + z_1 \cdot z_3$ *for all* $z_1, z_2, z_3 \in \mathbb{C}$. *(left distributive)*
6. $(z_1 + z_2) \cdot z_3 = z_1 \cdot z_3 + z_2 \cdot z_3$ *for all* $z_1, z_2, z_3 \in \mathbb{C}$. *(right distributive)*

The multiplicative identity is $1 = 1 + 0i \in \mathbb{C}$.

Proposition 16. ***Multiplication of complex numbers in polar form***

Let $z_1 = |z_1| \cdot (\cos\theta_1 + i\sin\theta_1)$ *and* $z_2 = |z_2| \cdot (\cos\theta_2 + i\sin\theta_2)$ *be two complex numbers in polar form.*

The product is $z_1 \cdot z_2 = |z_1| \cdot |z_2| \cdot (\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2))$.

Multiplication of any two complex numbers corresponds to multiplying their moduli and adding their angles.

Definition 17. complex conjugate

A **complex conjugate** of a complex number is another complex number with the an equal real part and an imaginary part equal in magnitude but opposite in sign.

The complex conjugate of a complex number z is denoted $\bar{z}$.

Let $x, y \in \mathbb{R}$.

The complex conjugate of $z = x + yi$ is $\bar{z} = x - yi$.

The complex conjugate of $z = x - yi$ is $\bar{z} = x + yi$.

The complex conjugate of a complex number is its mirror image with respect to the x axis.

Division by zero is not defined.

Hence, if $z = 0$, then $\frac{1}{z} \notin \mathbb{C}$.

Therefore, if $\frac{1}{z} \in \mathbb{C}$, then $z \neq 0$.

Proposition 18. ***Multiplicative inverse of a complex number***

Let $z \in \mathbb{C}$ *and* $z \neq 0$.

The multiplicative inverse of z *is* $\frac{1}{z} \in \mathbb{C}^*$ *and* $\frac{1}{z} = \frac{\bar{z}}{|z|^2}$ *and* $z \cdot \frac{1}{z} = \frac{1}{z} \cdot z = 1$.

Let $z \in \mathbb{C}$.

If $z \neq 0$, then the reciprocal $\frac{1}{z}$ exists and $\frac{1}{z} \in \mathbb{C}$.

Hence, if $z \neq 0$, then $\frac{1}{z} \in \mathbb{C}$.

If $\frac{1}{z} \in \mathbb{C}$, then $z \neq 0$.

Therefore, $\frac{1}{z} \in \mathbb{C}$ iff $z \neq 0$.

Definition 19. division over $\mathbb{C}$

Let $z, w \in \mathbb{C}$ and $w \neq 0$.

The **quotient** is defined by $\frac{z}{w} = z \cdot \frac{1}{w}$.

Let $z = a + bi$, where $a, b \in \mathbb{R}$.

Let $w = c + di$, where $c, d \in \mathbb{R}$ and $w \neq 0$.

Then

$$\begin{aligned}
\frac{a+bi}{c+di} &= \frac{z}{w} \\
&= \frac{z}{w} \cdot \frac{\bar{w}}{\bar{w}} \\
&= \frac{z \cdot \bar{w}}{w \cdot \bar{w}} \\
&= \frac{z \cdot \bar{w}}{|w|^2}.
\end{aligned}$$

Division is the inverse of multiplication.

To divide complex numbers, multiply the quotient by the conjugate of the denominator, as shown above.

Proposition 20. ***Division of complex numbers in polar form***

Let $z_1 = |z_1| \cdot (\cos\theta_1 + i\sin\theta_1)$ and $z_2 = |z_2| \cdot (\cos\theta_2 + i\sin\theta_2)$ be two complex numbers in polar form with $z_2 \neq 0$.

The quotient is $\frac{z_1}{z_2} = \frac{|z_1|}{|z_2|} \cdot (\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2))$.

Division of two complex numbers corresponds to dividing their moduli and subtracting their angles.

Proposition 21. ***Properties of complex modulus***

Let $z \in \mathbb{C}$. Then

1. *$|z| \in \mathbb{R}$ and $|z| \geq 0$.*
2. *$|z| = 0$ iff $z = 0$.*
3. *$|-z| = |z|$.*
4. *$|\bar{z}| = |z|$.*
5. *$|zw| = |z| \cdot |w|$ for all $z, w \in \mathbb{C}$.*
6. *$|\frac{z}{w}| = \frac{|z|}{|w|}$ for all $z, w \in \mathbb{C}$ and $w \neq 0$.*
7. *$|z^n| = |z|^n$ for all $n \in \mathbb{Z}^+$.*
8. *$|z + w| \leq |z| + |w|$ for all $z, w \in \mathbb{C}$ (triangle inequality)*
9. *$|z - w| \geq ||z| - |w||$ for all $z, w \in \mathbb{C}$ (reverse triangle inequality)*

Proposition 22. ***Properties of complex conjugate***

1. *$Re(\bar{z}) = Re(z)$ and $Im(\bar{z}) = -Im(z)$ for all $z \in \mathbb{C}$.*
2. *$\bar{\bar{z}} = z$ for all $z \in \mathbb{C}$. (conjugate of a conjugate)*
3. *$Re(z) = \frac{z+\bar{z}}{2}$ for all $z \in \mathbb{C}$.*
4. *$Im(z) = \frac{z-\bar{z}}{2i}$ for all $z \in \mathbb{C}$.*
5. *$z \cdot \bar{z} = |z|^2$ for all $z \in \mathbb{C}$. (Product of complex conjugates is an absolute square.)*

6. $Re(\alpha \cdot z) = \alpha \cdot Re(z)$ *and* $Im(\alpha \cdot z) = \alpha \cdot Im(z)$ *for all* $\alpha \in \mathbb{R}, z \in \mathbb{C}$. *(scalar multiple)*
7. $\overline{z+w} = \bar{z} + \bar{w}$ *for all* $z, w \in \mathbb{C}$. *(conjugate of sum is sum of conjugates)*
8. $\overline{z-w} = \bar{z} - \bar{w}$ *for all* $z, w \in \mathbb{C}$. *(conjugate of difference is difference of conjugates)*
9. $\overline{z \cdot w} = \bar{z} \cdot \bar{w}$ *for all* $z, w \in \mathbb{C}$. *(conjugate of product is product of conjugates)*
10. $\overline{\frac{z}{w}} = \frac{\bar{z}}{\bar{w}}$ *for all* $z, w \in \mathbb{C}, w \neq 0$. *(conjugate of quotient is quotient of conjugates)*

Theorem 23. ***DeMoivre formula***
For all $\theta \in \mathbb{R}$ *and all* $n \in \mathbb{Z}^+$*, the following identity is true.*
$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)$.

Let z be a nonzero complex number in polar form and $n \in \mathbb{Z}^+$.
Then $z = r(\cos\theta + i\sin\theta)$ for some $r \in \mathbb{R}^+$ and $\theta \in \mathbb{R}$.
Observe that

$$\begin{aligned} z^n &= [r(\cos\theta + i\sin\theta)]^n \\ &= r^n(\cos\theta + i\sin\theta)^n \\ &= r^n(\cos(n\theta) + i\sin(n\theta)). \end{aligned}$$

Proposition 24. ***Arithmetic operations on complex numbers in polar form***
Multiplication
1. $(r_1 e^{i\theta_1})(r_2 e^{i\theta_2}) = (r_1 r_2)e^{i(\theta_1+\theta_2)}$. *(multiply absolute values, add angles)*
Reciprocal
2. $\frac{1}{re^{i\theta}} = (\frac{1}{r})e^{-i\theta}$.
Division
3. $\frac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}} = (\frac{r_1}{r_2})e^{i(\theta_1-\theta_2)}$. *(divide absolute values, subtract angles)*
n^{th} *power*
4. $(re^{i\theta})^n = r^n \cdot e^{in\theta}$ *for any integer* n.
Complex conjugation
5. $\overline{re^{i\theta}} = re^{-i\theta}$.

Theorem 25. $(\mathbb{C}, +, \cdot)$ *is a field.*

Additive identity is $0 = 0 + 0i$.
Let $x, y \in \mathbb{R}$.
Additive inverse of $z = x + yi$ is $-z = -x - yi$.
Multiplicative identity is $1 = 1 + 0i$.
Multiplicative inverse of $z \in \mathbb{C}^*$ is $\frac{1}{z} = \frac{\bar{z}}{|z|^2}$, where $\bar{z}$ is the complex conjugate of z and $|z|$ is the modulus of z.

Example 26. $\mathbb{C}$ **is not an ordered field.**
$(\mathbb{C}, +, \cdot)$ is not an ordered field.

Proof. Suppose $(\mathbb{C}, +, \cdot)$ is an ordered field.
Then there is a subset P of positive elements of $\mathbb{C}$ and $1 \in P$.
Since $i \in \mathbb{C}$ and $i \neq 0$, then $i^2 \in P$.
Since $i^2 = -1$, then $-1 \in P$.
Hence, we have $1 \in P$ and $-1 \in P$, a violation of trichotomy.
Therefore, $(\mathbb{C}, +, \cdot)$ is not an ordered field. $\square$

Hence, there is no way to order the complex numbers.
Therefore, no order relation can be defined on $\mathbb{C}$.

Complex exponential function

Proposition 27. ***existence and uniqueness of complex exponential function***
There is a unique function $f : \mathbb{C} \to \mathbb{C}$ such that $f'(z) = f(z)$ and $f(0) = 1$.

Proposition 28. ***Properties of complex exponential function***
The function $f : \mathbb{C} \to \mathbb{C}$ defined by $f(z) = e^z$ for all $z \in \mathbb{C}$ has the following properties.

1. *The derivative of e^z is e^z.*
2. $e^0 = 1$.
3. $e^{z+w} = e^z \cdot e^w$ *for all* $z, w \in \mathbb{C}$.
4. $(e^z)^n = e^{nz}$ *for all* $z, \in \mathbb{C}$ *and for all* $n \in \mathbb{Z}$.

Theorem 29. ***Euler's formula***
The function $f : \mathbb{C} \to \mathbb{C}$ defined by $f(z) = e^z$ for all $z \in \mathbb{C}$ has the following property:
$e^{i\theta} = \cos\theta + i \sin\theta$ *for all* $\theta \in \mathbb{R}$.

Since $e^{i\pi} = \cos\pi + i \sin\pi = -1 + i(0) = -1$, then $e^{i\pi} = -1$.
Therefore, $e^{i\pi} + 1 = 0$. (**Euler's identity**)

Corollary 30. $|e^{i\theta}| = 1$ *for all* $\theta \in \mathbb{R}$.

REMOVE THE UN-NEEDED ITEMS FROM BELOW TO CLEAN UP OUR complex analysis notes.

Absolute value in an ordered field

The absolute value of an element measures size(magnitude), really it's distance from the origin.

Definition 31. absolute value in an ordered field
Let F be an ordered field.
Let $x \in F$.
The **absolute value** of x, denoted $|x|$, is defined by the rule

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

The absolute value in an ordered field F is a function from F to F.
Observe that $|0| = 0$.
Since $1 > 0$, then $|1| = 1$.

Lemma 32. *Let F be an ordered field. Let $x \in F$.*
1. If $x < 0$, then $\frac{1}{x} < 0$.
2. If $x \neq 0$, then $|\frac{1}{x}| = \frac{1}{|x|}$.

Theorem 33. ***arithmetic operations and absolute value***
Let F be an ordered field. For all $a, b \in F$
1. $|ab| = |a||b|$.
2. if $b \neq 0$, then $|\frac{a}{b}| = \frac{|a|}{|b|}$.
3. $|a|^2 = a^2$.
4. if $a \neq 0$, then $|a^n| = |a|^n$ for all $n \in \mathbb{Z}$.

Theorem 34. ***properties of the absolute value function***
Let $(F, +, \cdot, \leq)$ be an ordered field.
Let $a, k \in F$ and $k > 0$. Then
1. $|a| \geq 0$.
2. $|a| = 0$ iff $a = 0$.
3. $|-a| = |a|$.
4. $-|a| \leq a \leq |a|$.
5. $|a| < k$ iff $-k < a < k$.
6. $|a| > k$ iff $a > k$ or $a < -k$.
7. $|a| = k$ iff $a = k$ or $a = -k$.

Ordered field properties of $\mathbb{R}$

We assume there exists a complete ordered field and call it $\mathbb{R}$.

Axiom 35. $(\mathbb{R}, +, \cdot, \leq)$ ***is a complete ordered field.***
The set of real numbers $\mathbb{R}$ with the operations of addition and multiplication and the relation $\leq$ defined over $\mathbb{R}$ is defined to be a complete ordered field.

Therefore, $(\mathbb{R}, +, \cdot, \leq)$ is defined to be a complete ordered field.

Since $(\mathbb{R}, +, \cdot, \leq)$ is a field, then the field axioms hold for $\mathbb{R}$.

Field axioms of $(\mathbb{R}, +, \cdot, \leq)$
A1. $x + y \in \mathbb{R}$ for all $x, y \in \mathbb{R}$. (closure under addition)
A2. $(x + y) + z = x + (y + z)$ for all $x, y, z \in \mathbb{R}$. (addition is associative)
A3. $x + y = y + x$ for all $x, y \in \mathbb{R}$. (addition is commutative)
A4. $(\exists 0 \in \mathbb{R})(\forall x \in \mathbb{R})(0 + x = x + 0 = x)$. (existence of additive identity)
A5. $(\forall x \in \mathbb{R})(\exists -x \in \mathbb{R})(x + (-x) = -x + x = 0)$. (existence of additive inverses)
M1. $xy \in \mathbb{R}$ for all $x, y \in \mathbb{R}$. (closure under multiplication)

M2. $(xy)z = x(yz)$ for all $x, y, z \in \mathbb{R}$. (multiplication is associative)
M3. $xy = yx$ for all $x, y \in \mathbb{R}$. (multiplication is commutative)
M4. $(\exists 1 \in \mathbb{R})(\forall x \in \mathbb{R})(1 \cdot x = x \cdot 1 = x)$. (existence of multiplicative identity)
M5. $(\forall x \in \mathbb{R}^*)(\exists x^{-1} \in \mathbb{R})(xx^{-1} = x^{-1}x = 1)$. (existence of multiplicative inverses)
D1. $x(y + z) = xy + xz$ for all $x, y, z \in \mathbb{R}$. (multiplication is left distributive over addition)
D2. $(y+z)x = yx+zx$ for all $x, y, z \in \mathbb{R}$. (multiplication is right distributive over addition)
F1. $1 \neq 0$. (multiplicative identity is distinct from additive identity)
The additive identity of $\mathbb{R}$ is 0.
The additive inverse of $x \in \mathbb{R}$ is $-x$.
The multiplicative identity of $\mathbb{R}$ is 1.
The multiplicative inverse of $x \in \mathbb{R}^*$ is $\frac{1}{x} \in \mathbb{R}^*$.

Since $(\mathbb{R}, +, \cdot, \leq)$ is a field, then $\mathbb{R}$ is an integral domain.
Therefore, $xy = 0$ iff $x = 0$ or $y = 0$ for all $x, y \in \mathbb{R}$.
Equivalently, $xy \neq 0$ iff $x \neq 0$ and $y \neq 0$ for all $x, y \in \mathbb{R}$.
Therefore, the product of any two nonzero elements of $\mathbb{R}$ is nonzero.

Since $(\mathbb{R}, +, \cdot, \leq)$ is a field, then $\mathbb{R}$ satisfies the multiplicative cancellation laws.
Therefore, if $xz = yz$ and $z \neq 0$, then $x = y$ for all $x, y, z \in \mathbb{R}$.

Since $(\mathbb{R}, +, \cdot), \leq)$ is an ordered field, then there exists a nonempty subset $\mathbb{R}^+$ of $\mathbb{R}$ such that
OF1. $\mathbb{R}^+$ is closed under addition. $(\forall a, b \in \mathbb{R}^+)(a + b \in \mathbb{R}^+)$.
OF2. $\mathbb{R}^+$ is closed under multiplication. $(\forall a, b \in \mathbb{R}^+)(ab \in \mathbb{R}^+)$.
OF3. For every $r \in \mathbb{R}^+$ exactly one of the following is true:
i. $r \in \mathbb{R}^+$
ii. $r = 0$
iii. $-r \in \mathbb{R}^+$.

Definition 36. Let $\mathbb{R}^+$ be the set of all positive real numbers.
Define the relation $<$ on $\mathbb{R}$ by $a < b$ iff $b - a \in \mathbb{R}^+$ for all $a, b, \in \mathbb{R}$.
Define the relation $>$ on $\mathbb{R}$ by $a > b$ iff $b < a$ for all $a, b \in \mathbb{R}$.
Then $(\mathbb{R}, +, \cdot, <)$ denotes the ordered field $(\mathbb{R}, +, \cdot)$ with the relation $<$ defined over $\mathbb{R}$.

Definition 37. Let $(\mathbb{R}, +, \cdot, <)$ be the ordered field of real numbers.
Define the relation $\leq$ on $\mathbb{R}$ by $a \leq b$ iff either $a < b$ or $a = b$ for all $a, b \in \mathbb{R}$.
Define the relation $\geq$ on $\mathbb{R}$ by $a \geq b$ iff $b \leq a$ for all $a, b \in \mathbb{R}$.
Then $(\mathbb{R}, +, \cdot, \leq)$ denotes the ordered field $(\mathbb{R}, +, \cdot, <)$ with the relation $\leq$ defined over $\mathbb{R}$.

Definition 38. sign of a real number

Let $x \in \mathbb{R}$.
x is **nonzero** iff $x \neq 0$.
x is **positive** iff $x > 0$.
x is **negative** iff $x < 0$.
x is **non-negative** iff $x \geq 0$.
x is **non-positive** iff $x \leq 0$.

$\mathbb{R}^+ = \{x \in \mathbb{R} : x \text{ is positive }\} = \{x \in \mathbb{R} : x > 0\} = (0, \infty)$.
$\mathbb{R}^- = \{x \in \mathbb{R} : x \text{ is negative }\} = \{x \in \mathbb{R} : x < 0\} = (-\infty, 0)$.
$\mathbb{R}^* = \{x \in \mathbb{R} : x \neq 0\} = \mathbb{R}^+ \cup \mathbb{R}^- = (0, \infty) \cup (-\infty, 0)$.
Thus, if $x \in \mathbb{R}^*$ then either x is positive or x is negative.
$\{\mathbb{R}^+, \mathbb{R}^-, \{0\}\}$ is a partition of $\mathbb{R}$.
$\{\mathbb{R}^+, \mathbb{R}^-\}$ is a partition of $\mathbb{R}^*$.
Therefore, $\mathbb{R} = \mathbb{R}^* \cup \{0\} = \mathbb{R}^+ \cup \{0\} \cup \mathbb{R}^-$.
Hence, an element $x \in \mathbb{R}$ is either positive or zero or negative.
Therefore, every real number is either positive, zero, or negative.

Since $(\mathbb{R}, +, \cdot, \leq)$ is an ordered field, the following are true:
1. If $x, y \in \mathbb{R}^+$, then $x + y \in \mathbb{R}^+$. ($\mathbb{R}^+$ is closed under $+$)
2. If $x, y \in \mathbb{R}^+$, then $xy \in \mathbb{R}^+$. ($\mathbb{R}^+$ is closed under $\cdot$)
3. For every $x, y \in \mathbb{R}$, exactly one of the following is true (trichotomy):
$x > y$, $x = y$, $x < y$.
4. If $x < y$ and $y < z$, then $x < z$. ($<$ is transitive)
5. If $x < y$, then $x + z < y + z$. (preserves order for addition)
6. If $x < y$ and $z > 0$, then $xz < yz$. (preserves order for multiplication by a positive element)

Example 39. density of $\mathbb{R}$

Since $(\mathbb{R}, +, \cdot, \leq)$ is an ordered field, then between any two distinct real numbers is another real number.

Therefore, if $a, b \in \mathbb{R}$ and $a < b$, then there exists $r \in \mathbb{R}$ such that $a < r < b$.

Example 40. $\mathbb{R}$ is infinite

Since $(\mathbb{R}, +, \cdot, \leq)$ is an ordered field, then $\mathbb{R}$ contains an infinite number of elements.

Therefore, there are infinitely many real numbers.

Since $(\mathbb{R}, +, \cdot, \leq)$ is an ordered field, then $\leq$ is a total order on $\mathbb{R}$.
Therefore, $(\mathbb{R}, \leq)$ is a total order, so $(\mathbb{R}, \leq)$ is a poset.
Since $\leq$ is a total order over $\mathbb{R}$, then the following are true:
1. $(\forall x \in \mathbb{R})(x \leq x)$ (reflexive)
2. $(\forall x, y \in \mathbb{R})([x \leq y \wedge y \leq x) \rightarrow (x = y)]$ (anti-symmetric)
3. $(\forall x, y, z \in \mathbb{R})(x \leq y \wedge y \leq z \rightarrow x \leq z)$ (transitive)
4. $(\forall x, y \in \mathbb{R})(x \leq y \vee y \leq x)$. (comparable)

Since $\leq$ is a total order on $\mathbb{R}$, then $(\mathbb{R}, \leq)$ is a linearly ordered set.

Since $\mathbb{R}$ is complete, then the Hasse diagram is a straight line with no holes, i.e., the real number line.

Therefore, $(\mathbb{R}, +, \cdot, \leq)$ is a complete linearly ordered field.

Boundedness of sets in an ordered field

Definition 41. upper bound of a subset of an ordered field

Let F be an ordered field.

Let $S \subset F$.

An element $b \in F$ is an **upper bound of** S **in** F iff $(\forall x \in S)(x \leq b)$.

The set S is **bounded above in** F iff S has an upper bound in F.

Therefore, S is bounded above in F iff $(\exists b \in F)(\forall x \in S)(x \leq b)$.

The statement 'S has an upper bound in F' means: $(\exists b \in F)(\forall x \in S)(x \leq b)$.

Observe that

$$\begin{aligned} \neg(\exists b \in F)(\forall x \in S)(x \leq b) &\Leftrightarrow (\forall b \in F)(\exists x \in S)(x \not\leq b) \\ &\Leftrightarrow (\forall b \in F)(\exists x \in S)(x > b). \end{aligned}$$

Therefore, the statement 'S has no upper bound in F' means:

$(\forall b \in F)(\exists x \in S)(x > b)$.

Therefore S has no upper bound in F iff for each $b \in F$ there is some $x \in S$ such that $x > b$.

An element $b \in F$ is not an upper bound for S iff there exists $x \in S$ such that $x > b$.

Definition 42. lower bound of a subset of an ordered field

Let F be an ordered field.

Let $S \subset F$.

An element $b \in F$ is a **lower bound of** S **in** F iff $(\forall x \in S)(b \leq x)$.

The set S is **bounded below in** F iff S has a lower bound in F.

Therefore, S is bounded below in F iff $(\exists b \in F)(\forall x \in S)(b \leq x)$.

The statement 'S has a lower bound in F' means: $(\exists b \in F)(\forall x \in S)(b \leq x)$.

Observe that

$$\begin{aligned} \neg(\exists b \in F)(\forall x \in S)(b \leq x) &\Leftrightarrow (\forall b \in F)(\exists x \in S)(b \not\leq x) \\ &\Leftrightarrow (\forall b \in F)(\exists x \in S)(b > x). \end{aligned}$$

Therefore, the statement 'S has no lower bound in F' means:

$(\forall b \in F)(\exists x \in S)(x < b)$.

Therefore S has no lower bound in F iff for each $b \in F$ there is some $x \in S$ such that $x < b$.

An element $b \in F$ is not a lower bound for S iff there exists $x \in S$ such that $x < b$.

Definition 43. bounded subset of an ordered field

Let F be an ordered field.

Let $S \subset F$.

The set S is **bounded in** F iff there exists $b \in F$ such that $|x| \leq b$ for all $x \in S$.

The set S is **unbounded in** F iff S is not bounded in F.

In symbols, S is bounded in F iff $(\exists b \in F)(\forall x \in S)(|x| \leq b)$.

Therefore, S is unbounded in F iff $(\forall b \in F)(\exists x \in S)(|x| > b)$.

Let S be a subset of an ordered field F.

Suppose S is bounded in F.

Then there exists $B \in F$ such that $|x| \leq B$ for all $x \in S$.

Let $x \in S$.

Then $|x| \leq B$.

Since $|x| \geq 0$ and $B + 1 > B$, then $0 \leq |x| \leq B < B + 1$.

Hence, $|x| < B + 1$ and $0 < B + 1$.

Therefore, there exists $B + 1 > 0$ such that $|x| < B + 1$ for all $x \in S$.

Let $b = B + 1$.

Then there exists $b > 0$ such that $|x| < b$ for all $x \in S$.

Hence, if a set S is bounded in an ordered field F, then there exists $b > 0$ such that $|x| < b$ for all $x \in S$.

Therefore, if a set S is bounded in an ordered field F, then there exists $b > 0$ such that $-b < x < b$ for all $x \in S$.

Theorem 44. *A subset S of an ordered field F is bounded in F iff S is bounded above and below in F.*

Let $S \subset \mathbb{R}$.

Then S is bounded in $\mathbb{R}$ iff S is bounded above and below in $\mathbb{R}$.

Therefore, S is bounded in $\mathbb{R}$ iff S has an upper and lower bound in $\mathbb{R}$.

Observe that

$$\begin{aligned}
(\exists m \in \mathbb{R})(\forall x \in S)(m \leq x) \wedge (\exists M \in \mathbb{R})(\forall x \in S)(x \leq M) &\Rightarrow \\
(\exists m \in \mathbb{R})(\exists M \in \mathbb{R})[(\forall x \in S)(m \leq x) \wedge (\forall x \in S)(x \leq M)] &\Rightarrow \\
(\exists m \in \mathbb{R})(\exists M \in \mathbb{R})[(\forall x \in S)(m \leq x \wedge x \leq M)] &\Rightarrow \\
(\exists m \in \mathbb{R})(\exists M \in \mathbb{R})[(\forall x \in S)(m \leq x \leq M)]&.
\end{aligned}$$

Therefore, S is bounded in $\mathbb{R}$ iff there exist $m, M \in \mathbb{R}$ such that $m \leq x \leq M$ for all $x \in S$.

Since S is bounded in $\mathbb{R}$ iff S is bounded above and below in $\mathbb{R}$, then

S is not bounded in $\mathbb{R}$ iff S is not bounded above in $\mathbb{R}$ or S is not bounded below in $\mathbb{R}$.

Therefore, S is unbounded in $\mathbb{R}$ iff either S has no upper bound in $\mathbb{R}$ or S has no lower bound in $\mathbb{R}$.

Proposition 45. *Every element of an ordered field is an upper and lower bound of $\emptyset$.*

Let F be an ordered field.
Let $x \in F$.
Then x is an upper and lower bound of $\emptyset$.
Therefore, $\emptyset$ is bounded above and below in F.
Hence, $\emptyset$ is bounded in F.

Example 46. Every rational number is an upper and lower bound for the empty set.
Therefore, $\emptyset$ is bounded above and below in $\mathbb{Q}$.
Hence, $\emptyset$ is bounded in $\mathbb{Q}$.

Example 47. Every real number is an upper and lower bound for the empty set.
Therefore, $\emptyset$ is bounded above and below in $\mathbb{R}$.
Hence, $\emptyset$ is bounded in $\mathbb{R}$.

Proposition 48. ***A subset of a bounded set is bounded.***
Let A be a bounded subset of an ordered field F.
If $B \subset A$, then B is bounded in F.

Proposition 49. ***A union of bounded sets is bounded.***
Let A and B be subsets of an ordered field F.
If A and B are bounded, then $A \cup B$ is bounded.

Definition 50. least upper bound of a subset of an ordered field
Let F be an ordered field and $S \subset F$.
Then $\beta \in F$ is a **least upper bound for S in F** iff β is the least element of the set of all upper bounds of S in F.

Therefore $\beta \in F$ is a **least upper bound of S** iff
1. β is an upper bound for S and
2. $\beta \leq M$ for every upper bound M of S.
$\beta \leq M$ for every upper bound M of S iff
no element of F less than β is an upper bound of S iff
every element of F less than β is not an upper bound of S iff
if $\gamma < \beta$, then γ is not an upper bound of S which means
if $\gamma < \beta$, then there exists $x \in S$ such that $x > \gamma$ which means
for every $\gamma < \beta$, there exists $x \in S$ such that $x > \gamma$ which means
for every $\beta - \gamma > 0$, there exists $x \in S$ such that $x > \beta - (\beta - \gamma)$ which means
for every $\epsilon > 0$, there exists $x \in S$ such that $x > \beta - \epsilon$.
Therefore, $\beta = lub(S)$ iff
1. $(\forall x \in S)(x \leq \beta)$. ($\beta$ is an upper bound of S)
2. $(\forall \epsilon > 0)(\exists x \in S)(x > \beta - \epsilon)$. ($\beta - \epsilon$ is not an upper bound of S).

Theorem 51. ***uniqueness of least upper bound in an ordered field***
A least upper bound of a subset of an ordered field, if it exists, is unique.

Let S be a subset of an ordered field F.
The **least upper bound (lub) of** S is called the **supremum** and is denoted $\sup S$.
Therefore,
1. $(\forall x \in S)(x \leq \sup S)$. ($\sup S$ is an upper bound of S)
2. $(\forall \epsilon > 0)(\exists x \in S)(x > \sup S - \epsilon)$. ($\sup S - \epsilon$ is not an upper bound of S).

Example 52. $\sup(0, 1) = 1$.

Definition 53. greatest lower bound of a subset of an ordered field
Let F be an ordered field and $S \subset F$.
Then $\beta \in F$ is a **greatest lower bound for** S **in** F iff β is the greatest element of the set of all lower bounds of S in F.

Therefore $\beta \in F$ is a **greatest lower bound of** S iff
1. β is a lower bound for S and
2. $M \leq \beta$ for every lower bound M of S.
$M \leq \beta$ for every lower bound M of S iff
no element of F greater than β is a lower bound of S iff
every element of F greater than β is not a lower bound of S iff
if $\gamma > \beta$, then γ is not a lower bound of S which means
if $\gamma > \beta$, then there exists $x \in S$ such that $x < \gamma$ which means
for every $\gamma > \beta$, there exists $x \in S$ such that $x < \gamma$ which means
for every $\gamma - \beta > 0$, there exists $x \in S$ such that $x < \beta + (\gamma - \beta)$ which means
for every $\epsilon > 0$, there exists $x \in S$ such that $x < \beta + \epsilon$.
Therefore, $\beta = glb(S)$ iff
1. $(\forall x \in S)(\beta \leq x)$. ($\beta$ is a lower bound of S)
2. $(\forall \epsilon > 0)(\exists x \in S)(x < \beta + \epsilon)$. ($\beta + \epsilon$ is not a lower bound of S).

Theorem 54. ***uniqueness of greatest lower bound in an ordered field***
A greatest lower bound of a subset of an ordered field, if it exists, is unique.

Let S be a subset of an ordered field F.
The **greatest lower bound (glb) of** S is called the **infimum** and is denoted $\inf S$.
Therefore,
1. $(\forall x \in S)(\inf S \leq x)$. ($\inf S$ is a lower bound of S)
2. $(\forall \epsilon > 0)(\exists x \in S)(x < \inf S + \epsilon)$. ($\inf S + \epsilon$ is not a lower bound of S)

Example 55. $\inf(0, 1) = 0$.

Proposition 56. *1. There is no least upper bound of $\emptyset$ in an ordered field.*
2. There is no greatest lower bound of $\emptyset$ in an ordered field.

Let F be an ordered field.
Then $\sup \emptyset$ does not exist in F and $\inf \emptyset$ does not exist in F.
Therefore, $\sup \emptyset$ does not exist in $\mathbb{Q}$ and $\inf \emptyset$ does not exist in $\mathbb{Q}$ and $\sup \emptyset$ does not exist in $\mathbb{R}$ and $\inf \emptyset$ does not exist in $\mathbb{R}$.

Let $S \subset F$.

If $S = \emptyset$, then $\sup S$ does not exist, so if $\sup S$ exists, then $S \neq \emptyset$.

If $S = \emptyset$, then $\inf S$ does not exist, so if $\inf S$ exists, then $S \neq \emptyset$.

Theorem 57. ***approximation property of suprema and infima***

Let S be a subset of an ordered field F.

1. If $\sup S$ exists, then $(\forall \epsilon > 0)(\exists x \in S)(\sup S - \epsilon < x \leq \sup S)$.

2. If $\inf S$ exists, then $(\forall \epsilon > 0)(\exists x \in S)(\inf S \leq x < \inf S + \epsilon)$.

If $\sup S$ exists, then there is some element of S arbitrarily close to $\sup S$.

If $\inf S$ exists, then there is some element of S arbitrarily close to $\inf S$.

Proposition 58. *Let S be a subset of an ordered field F.*

If $\sup S$ and $\inf S$ exist, then $\inf S \leq \sup S$.

Proposition 59. *Let S be a subset of an ordered field F.*

Let $-S = \{-s : s \in S\}$.

1. If $\inf S$ exists, then $\sup(-S) = -\inf S$.

2. If $\sup S$ exists, then $\inf(-S) = -\sup S$.

Lemma 60. *Let S be a subset of an ordered field F.*

Let $k \in F$.

Let $K = \{k\}$.

Let $k + S = \{k + s : s \in S\}$.

Let $K + S = \{k + s : k \in K, s \in S\}$. Then

1. $\sup K = k$.

2. $\inf K = k$.

3. $k + S = K + S$.

Proposition 61. ***additive property of suprema and infima***

Let A and B be subsets of an ordered field F.

Let $A + B = \{a + b : a \in A, b \in B\}$.

1. If $\sup A$ and $\sup B$ exist, then $\sup(A + B) = \sup A + \sup B$.

2. If $\inf A$ and $\inf B$ exist, then $\inf(A + B) = \inf A + \inf B$.

Corollary 62. *Let S be a subset of an ordered field F.*

Let $k \in F$.

Let $k + S = \{k + s : s \in S\}$.

1. If $\sup S$ exists, then $\sup(k + S) = k + \sup S$.

2. If $\inf S$ exists, then $\inf(k + S) = k + \inf S$.

Corollary 63. *Let A and B be subsets of an ordered field F.*

Let $A - B = \{a - b : a \in A, b \in B\}$.

If $\sup A$ and $\inf B$ exist, then $\sup(A - B) = \sup A - \inf B$.

Proposition 64. ***comparison property of suprema and infima***

Let A and B be subsets of an ordered field F such that $A \subset B$.

1. If $\sup A$ and $\sup B$ exist, then $\sup A \leq \sup B$.

2. If $\inf A$ and $\inf B$ exist, then $\inf B \leq \inf A$.

Proposition 65. ***scalar multiple property of suprema and infima***
Let S be a subset of an ordered field F.
Let $k \in F$.
Let $kS = \{ks : s \in S\}$.
1. If $k > 0$ and $\sup S$ exists, then $\sup(kS) = k \sup S$.
2. If $k > 0$ and $\inf S$ exists, then $\inf(kS) = k \inf S$.
3. If $k < 0$ and $\inf S$ exists, then $\sup(kS) = k \inf S$.
4. If $k < 0$ and $\sup S$ exists, then $\inf(kS) = k \sup S$.

Proposition 66. ***sufficient conditions for existence of supremum and infimum in an ordered field***
Let S be a subset of an ordered field F.
1. If $\max S$ exists, then $\sup S = \max S$.
2. If $\min S$ exists, then $\inf S = \min S$.

Proposition 67. *Let S be a subset of an ordered field F.*
Let $-S = \{-s : s \in S\}$.
1. If $\min S$ exists, then $\max(-S) = -\min S$.
2. If $\max S$ exists, then $\min(-S) = -\max S$.

Lemma 68. *Let A and B be nonempty subsets of an ordered field F.*
Then $u \in F$ is an upper bound of $A \cup B$ iff u is an upper bound of A and B.

Proposition 69. *Let A and B be subsets of an ordered field F.*
If $\sup A$ and $\sup B$ exist, then $\sup(A \cup B) = \max\{\sup A, \sup B\}$.

Let A and B be subsets of an ordered field F.
If $\max A$ and $\max B$ exist in F, then $\sup A = \max A$ and $\sup B = \max B$.
Thus, $\sup(A \cup B) = \max\{\max A, \max B\}$.

Lemma 70. *Let A and B be subsets of an ordered field F.*
If $\max A$ and $\max B$ exist in F, then $\max(A \cup B) = \max\{\max A, \max B\}$.

Theorem 71. *Every nonempty finite subset of an ordered field has a maximum.*

Let S be a nonempty finite subset of an ordered field F.
Then $\max S$ exists.
Since $S \subset F$ and $\max S$ exists, then $\sup S = \max S$.

Example 72. Every nonempty finite subset of $\mathbb{R}$ has a maximum.

Complete ordered fields

Definition 73. complete ordered field
An ordered field F is **complete** iff every nonempty subset of F that is bounded above in F has a least upper bound in F. Otherwise, F is said to be **incomplete**.

Axiom 74. $\mathbb{R}$ ***is Dedekind complete.***

Every nonempty subset of $\mathbb{R}$ *that is bounded above in* $\mathbb{R}$ *has a least upper bound in* $\mathbb{R}$.

Let S be a nonempty subset of $\mathbb{R}$ that is bounded above in $\mathbb{R}$.
Then S has a least upper bound in $\mathbb{R}$.
Therefore $\sup S$ is the least upper bound of S in $\mathbb{R}$.
Hence $\sup S \in \mathbb{R}$ and
1. $(\forall x \in S)(x \leq \sup S)$.
2. If b is any upper bound of S, then $\sup S \leq b$.
Equivalently,
1. $(\forall x \in S)(x \leq \sup S)$.
2. $(\forall \epsilon > 0)(\exists x \in S)(x > \sup S - \epsilon)$.

Theorem 75. ***greatest lower bound property in a complete ordered field***

Every nonempty subset of a complete ordered field F that is bounded below in F has a greatest lower bound in F.

Example 76. Every nonempty set of real numbers that is bounded below in $\mathbb{R}$ has a greatest lower bound in $\mathbb{R}$.

Let S be a nonempty subset of $\mathbb{R}$ that is bounded below in $\mathbb{R}$.
Then S has a greatest lower bound in $\mathbb{R}$.
Therefore $\inf S$ is the greatest lower bound of S in $\mathbb{R}$.
Hence $\inf S \in \mathbb{R}$ and
1. $(\forall x \in S)(\inf S \leq x)$.
2. If b is any lower bound of S, then $b \leq \inf S$.
Equivalently,
1. $(\forall x \in S)(\inf S \leq x)$.
2. $(\forall \epsilon > 0)(\exists x \in S)(x < \inf S + \epsilon)$.

Proposition 77. *There is no rational number x such that $x^2 = 2$.*

Example 78. $\mathbb{Q}$ **is not a complete ordered field.**

The set $\{q \in \mathbb{Q} : q^2 < 2\}$ is bounded above in $\mathbb{Q}$, but does not have a least upper bound in $\mathbb{Q}$.

Therefore, $\mathbb{Q}$ is not a complete ordered field.
Since $\mathbb{Q}$ is not complete, then the Hasse diagram of $\mathbb{Q}$ is linear with 'holes'.
Thus, $\mathbb{Q}$ is incomplete and the number line for $\mathbb{Q}$ has holes, while $\mathbb{R}$ is complete and the number line for $\mathbb{R}$ does not have any holes.
Rework this section.

Proposition 79. *Let A and B be subsets of* $\mathbb{R}$ *such that* $\sup A$ *and* $\sup B$ *exist in* $\mathbb{R}$.

If $A \cap B \neq \emptyset$, then $\sup(A \cap B) \leq \min\{\sup A, \sup B\}$.

Moreover, if A and B are bounded intervals such that $A \cap B \neq \emptyset$, then $\sup(A \cap B) = \min\{\sup A, \sup B\}$.

Archimedean ordered fields

Definition 80. Archimedean ordered field
An ordered field F is **Archimedean ordered** iff $(\forall a \in F, b > 0)(\exists n \in \mathbb{N})(nb > a)$.

Let F be an Archimedean ordered field.
Then regardless of how small b is and how large a is, a sufficient number of repeated additions of b to itself will exceed a.
Equivalently, an ordered field F is Archimedean ordered iff $(\forall a \in F, b > 0)(\exists n \in \mathbb{N})(n > \frac{a}{b})$.

Theorem 81. ***Archimedean property of*** $\mathbb{Q}$
The field $(\mathbb{Q}, +, \cdot, \leq)$ *is Archimedean ordered.*

Therefore, for all $q \in \mathbb{Q}, \epsilon > 0$ there exists $n \in \mathbb{N}$ such that $n\epsilon > q$.

Theorem 82. ***Archimedean property of*** $\mathbb{R}$
A complete ordered field is necessarily Archimedean ordered.

Since $(\mathbb{R}, +, \cdot, \leq)$ is a complete ordered field, then $(\mathbb{R}, +, \cdot, \leq)$ is Archimedean ordered.
Therefore, for all $x \in \mathbb{R}, \epsilon > 0$ there exists $n \in \mathbb{N}$ such that $n\epsilon > x$.

Theorem 83. $\mathbb{N}$ ***is unbounded in an Archimedean ordered field.***
Let F *be an Archimedean ordered field.*
Then for every $x \in F$*, there exists* $n \in \mathbb{N}$ *such that* $n > x$.

Since $(\mathbb{R}, +, \cdot, \leq)$ is Archimedean ordered, then for every real number x, there exists a natural number n such that $n > x$.
In symbols, $(\forall x \in \mathbb{R})(\exists n \in \mathbb{N})(n > x)$.
Therefore, $\mathbb{N}$ is unbounded in $\mathbb{R}$.

Proposition 84. *Let* F *be an Archimedean ordered field.*
For every positive $\epsilon \in F$*, there exists* $n \in \mathbb{N}$ *such that* $\frac{1}{n} < \epsilon$.

Since $\mathbb{R}$ is Archimedean ordered, then for every positive real ϵ, there exists $n \in \mathbb{N}$ such that $\frac{1}{n} < \epsilon$.
In symbols, $(\forall \epsilon > 0)(\exists n \in \mathbb{N})(\frac{1}{n} < \epsilon)$.

Example 85. Let $S = \{\frac{1}{n} : n \in \mathbb{N}\}$.
Then $\max S = \sup S = 1$ and $\min S$ does not exist and $\inf S = 0$.

Lemma 86. ***Each real number lies between two consecutive integers***
For each real number x *there is a unique integer* n *such that* $n \leq x < n + 1$.

In symbols, $(\forall x \in \mathbb{R})(\exists! n \in \mathbb{Z})(n \leq x < n + 1)$.
Let $x \in \mathbb{R}$.
Then there is a unique integer n such that $n \leq x < n + 1$.

Theorem 87. $\mathbb{Q}$ ***is dense in*** $\mathbb{R}$

For every $a, b \in \mathbb{R}$ with $a < b$, there exists $q \in \mathbb{Q}$ such that $a < q < b$.

Therefore, between any two distinct real numbers is a rational number.
Hence, if $a < b$, then there exists $q \in \mathbb{Q}$ in the open interval (a, b).
Therefore, there is a rational number in every nonempty open interval.

Corollary 88. ***between any two distinct real numbers is a nonzero rational number***

For every $a, b \in \mathbb{R}$ with $a < b$, there exists $q \in \mathbb{Q}$ such that $q \neq 0$ and $a < q < b$.

Existence of square roots in $\mathbb{R}$

Definition 89. square root of a real number

Let $r \in \mathbb{R}$.
A square root of r is a real number x such that $x^2 = r$.

Proposition 90. *A square root of a negative real number does not exist in $\mathbb{R}$.*

Proposition 91. *Zero is the unique square root of* 0.

Lemma 92. *Let F be an ordered field.*
Let $a, b \in F$.
If $0 < a < b$, then $0 < a^2 < ab < b^2$.

Lemma 93. *Let F be an ordered field.*
Let $a \in F$.
If $|a| < \epsilon$ for all $\epsilon > 0$, then $a = 0$.

Theorem 94. ***existence and uniqueness of positive square roots***

Let $r \in \mathbb{R}$.
A unique positive square root of r exists in $\mathbb{R}$ iff $r > 0$.

Definition 95. nonnegative square root of a real number

Let $x \in \mathbb{R}$ such that $x \geq 0$.
The nonnegative square root of x is denoted $\sqrt{x}$.

Therefore, $\sqrt{x} \geq 0$ and $(\sqrt{x})^2 = x$.

Let $x \in \mathbb{R}$.
Then $\sqrt{x} > 0$ iff $x > 0$.

Proposition 96. *Let $x \in \mathbb{R}$.*
Then $\sqrt{x} \in \mathbb{R}$ iff $x \geq 0$.

Proposition 97. *Let $x \in \mathbb{R}$.*
Then $\sqrt{x} \geq 0$ iff $x \geq 0$.

Let $x \in \mathbb{R}$.
If $x > 0$, then $\sqrt{x} > 0$ and $-\sqrt{x} < 0$ and $(\sqrt{x})^2 = x$ and $(-\sqrt{x})^2 = x$.

Proposition 98. *Let $a, b \in \mathbb{R}$ with $a \geq 0$ and $b \geq 0$.*
Then $\sqrt{a} = \sqrt{b}$ iff $a = b$.

Proposition 99. *Let $a, b \in \mathbb{R}$.*
If $a \geq 0$ and $b \geq 0$, then $\sqrt{ab} = \sqrt{a}\sqrt{b}$.

Proposition 100. *Let $x \in \mathbb{R}$. Then*
1. $\sqrt{x} = 0$ *iff* $x = 0$.
2. $\sqrt{x^2} = |x|$.

Let $x \in \mathbb{R}$.
Since $\sqrt{x} = 0$ iff $x = 0$, then $\sqrt{0} = 0$.
Since $\sqrt{x^2} = |x|$, then either $\sqrt{x^2} = x$ or $\sqrt{x^2} = -x$.

Lemma 101. *Let $x \in \mathbb{R}$.*
If $x > 0$, then $\sqrt{\frac{1}{x}} = \frac{1}{\sqrt{x}}$.

Proposition 102. *Let $a, b \in \mathbb{R}$.*
If $a \geq 0$ and $b > 0$, then $\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$.

Lemma 103. *Let $a, b \in \mathbb{R}$.*
If $0 < a \leq b$, then $0 < a^2 \leq b^2$.

Proposition 104. *Let $a, b \in \mathbb{R}$.*
Then $0 < a < b$ iff $0 < \sqrt{a} < \sqrt{b}$.

Corollary 105. *Let $x \in \mathbb{R}$.*
1. *If $0 < x < 1$, then $0 < x^2 < x < \sqrt{x} < 1$.*
2. *If $x > 1$, then $1 < \sqrt{x} < x < x^2$.*