Continuous functions Theory

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Continuity

Proposition 1. ***characterization of continuity at a point***

Let $E \subset \mathbb{R}$.

Let $f : E \to \mathbb{R}$ be a function and $c \in E$. Then

1. If c is not an accumulation point of E, then f is continuous at c.

2. If c is an accumulation point of E, then f is continuous at c iff the limit of f at c exists and $\lim_{x \to c} f(x) = f(c)$.

Proof. We prove 1.

Suppose c is not an accumulation point of E.

To prove f is continuous at c, let $\epsilon > 0$ be given.

Since c is not an accumulation point of E, then there exists $\delta > 0$ such that for all $x \in E$, either $x \notin N(c;\delta)$ or $x = c$.

Let $x \in E$ such that $|x - c| < \delta$.

Since $x \in E$, then either $x \notin N(c;\delta)$ or $x = c$.

Since $|x - c| < \delta$, then $x \in N(c;\delta)$.

Hence, $x = c$.

Therefore, $|f(x) - f(c)| = |f(c) - f(c)| = 0 < \epsilon$. ∎

Proof. We prove 2.

Suppose c is an accumulation point of E.

We must prove f is continuous at c iff the limit of f at c exists and $\lim_{x \to c} f(x) = f(c)$.

We first prove if the limit of f at c exists and $\lim_{x \to c} f(x) = f(c)$, then f is continuous at c.

Suppose the limit of f at c exists and $\lim_{x \to c} f(x) = f(c)$.

Then $(\forall \epsilon > 0)(\exists \delta > 0)(\forall x \in E)(0 < |x - c| < \delta \to |f(x) - f(c)| < \epsilon)$.

Hence, $(\forall \epsilon > 0)(\exists \delta > 0)(\forall x \in E)(|x - c| < \delta \to |f(x) - f(c)| < \epsilon)$.

Therefore, f is continuous at c.

Conversely, we prove if f is continuous at c, then the limit of f at c exists and $\lim_{x \to c} f(x) = f(c)$.

Suppose f is continuous at c.

To prove $\lim_{x \to c} f(x) = f(c)$, let $\epsilon > 0$ be given.

Since f is continuous at c, then there exists $\delta > 0$ such that for all $x \in E$, if $|x - c| < \delta$, then $|f(x) - f(c)| < \epsilon$.

Since c is an accumulation point of E, let $x \in E$ such that $0 < |x - c| < \delta$.

Then $x \in E$ and $|x - c| < \delta$, so $|f(x) - f(c)| < \epsilon$, as desired. ☐

Theorem 2. ***sequential characterization of continuity***

Let $E \subset \mathbb{R}$.

Let $f : E \to \mathbb{R}$ *be a function.*

Let $c \in E$.

Then f *is continuous at* c *iff for every sequence* (x_n) *of points in* E *such that* $\lim_{n\to\infty} x_n = c$, $\lim_{n\to\infty} f(x_n) = f(c)$.

Proof. We prove if f is continuous at c, then for every sequence (x_n) of points in E such that $\lim_{n\to\infty} x_n = c$, $\lim_{n\to\infty} f(x_n) = f(c)$.

Suppose f is continuous at c.

Let (x_n) be an arbitrary sequence of points in E such that $\lim_{n\to\infty} x_n = c$.

We must prove $\lim_{n\to\infty} f(x_n) = f(c)$.

Let $\epsilon > 0$ be given.

Since f is continuous at c, then there exists $\delta > 0$ such that for all $x \in E$, if $|x - c| < \delta$, then $|f(x) - f(c)| < \epsilon$.

Since $\lim_{n\to\infty} x_n = c$ and $\delta > 0$, then there exists $N \in \mathbb{N}$ such that if $n > N$, then $|x_n - c| < \delta$.

Let $n \in \mathbb{N}$ such that $n > N$.

Then $|x_n - c| < \delta$.

Since (x_n) is a sequence of points in E, then $x_n \in E$ for all $n \in \mathbb{N}$.

Since $n \in \mathbb{N}$, then $x_n \in E$.

Since $x_n \in E$ and $|x_n - c| < \delta$, then we conclude $|f(x_n) - f(c)| < \epsilon$.

Therefore, $\lim_{n\to\infty} f(x_n) = f(c)$, as desired. ☐

Proof. Conversely, we prove if for every sequence (x_n) of points in E such that $\lim_{n\to\infty} x_n = c$ implies $\lim_{n\to\infty} f(x_n) = f(c)$, then f is continuous at c.

We prove by contrapositive.

Suppose f is not continuous at c.

Then there exists $\epsilon_0 > 0$ such that for each $\delta > 0$ there corresponds $x \in E$ such that $|x - c| < \delta$ and $|f(x) - f(c)| \geq \epsilon_0$.

Let $\delta = \frac{1}{n}$ for each $n \in \mathbb{N}$.

Then for each $n \in \mathbb{N}$, there corresponds $x \in E$ such that $|x - c| < \frac{1}{n}$ and $|f(x) - f(c)| \geq \epsilon_0$.

Thus, there exists a function $g : \mathbb{N} \to \mathbb{R}$ such that $g(n) \in E$ and $|g(n) - c| < \frac{1}{n}$ and $|f(g(n)) - f(c)| \geq \epsilon_0$ for each $n \in \mathbb{N}$, so there exists a sequence (x_n) in $\mathbb{R}$ such that $x_n \in E$ and $|x_n - c| < \frac{1}{n}$ and $|f(x_n) - f(c)| \geq \epsilon_0$ for each $n \in \mathbb{N}$.

Since $x_n \in E$ for each $n \in \mathbb{N}$, then (x_n) is a sequence of points in E.

We prove $\lim_{n\to\infty} x_n = c$.

Let $\epsilon > 0$ be given.

Then $\epsilon \neq 0$, so $\frac{1}{\epsilon} \in \mathbb{R}$.

Hence, by the Archimedean property of $\mathbb{R}$, there exists $N \in \mathbb{N}$ such that $N > \frac{1}{\epsilon}$.

Let $n \in \mathbb{N}$ such that $n > N$.

Then $n > N > \frac{1}{\epsilon}$, so $n > \frac{1}{\epsilon}$.

Hence, $\epsilon > \frac{1}{n}$, so $\frac{1}{n} < \epsilon$.

Since $n \in \mathbb{N}$ and $|x_n - c| < \frac{1}{n}$ for each $n \in \mathbb{N}$, then $|x_n - c| < \frac{1}{n}$.

Thus, $|x_n - c| < \frac{1}{n} < \epsilon$, so $|x_n - c| < \epsilon$.

Therefore, $\lim_{n\to\infty} x_n = c$, as desired.

We prove $\lim_{n\to\infty} f(x_n) \neq f(c)$.

Let $N \in \mathbb{N}$ be given.

Let $n = N + 1$.

Then $n \in \mathbb{N}$ and $n > N$.

Since $n \in \mathbb{N}$ and $|f(x_n) - f(c)| \geq \epsilon_0$ for each $n \in \mathbb{N}$, then $|f(x_n) - f(c)| \geq \epsilon_0$.

Thus, there exists $\epsilon_0 > 0$ such that for each $N \in \mathbb{N}$, there exists $n \in \mathbb{N}$ for which $n > N$ and $|f(x_n) - f(c)| \geq \epsilon_0$.

Therefore, $\lim_{n\to\infty} f(x_n) \neq f(c)$.

Hence, we have shown there exists a sequence (x_n) of points in E such that $\lim_{n\to\infty} x_n = c$ and $\lim_{n\to\infty} f(x_n) \neq f(c)$, as desired. □

Proposition 3. ***restriction of a continuous function is continuous***

Let f be a real valued function of a real variable.

Let g be a restriction of f to a nonempty set $E \subset dom f$.

If f is continuous, then the restriction g is continuous.

Proof. Suppose f is continuous.

To prove g is continuous, we must prove g is continuous on E.

Since $E \neq \emptyset$, let $c \in E$ be arbitrary.

To prove g is continuous at c, let $\epsilon > 0$ be given.

Since $c \in E$ and $E \subset dom f$, then $c \in dom f$.

Since f is continuous and $c \in dom f$, then f is continuous at c.

Thus, there exists $\delta > 0$ such that for all $x \in dom f$, if $|x - c| < \delta$, then $|f(x) - f(c)| < \epsilon$.

Let $x \in E$ such that $|x - c| < \delta$.

Since $x \in E$ and $E \subset dom f$, then $x \in dom f$.

Thus, $x \in dom f$ and $|x - c| < \delta$, so $|f(x) - f(c)| < \epsilon$.

Since g is a restriction of f to E, then $g(x) = f(x)$ for all $x \in E$.

Since $x \in E$ and $c \in E$, then $g(x) = f(x)$ and $g(c) = f(c)$.

Therefore, $|g(x) - g(c)| = |f(x) - f(c)| < \epsilon$, so g is continuous at c, as desired. □

Algebraic properties of continuous functions

Theorem 4. *Let $\lambda \in \mathbb{R}$.*
Let f be a real valued function.
Let $c \in dom f$.
If f is continuous at c, then λf is continuous at c.

Proof. Suppose f is continuous at c.
Since f is a real valued function, then λf is a real valued function.
Since $c \in dom f$ and $dom f = dom(\lambda f)$, then $c \in dom(\lambda f)$.
Either c is an accumulation point of $dom(\lambda f)$ or c is not an accumulation point of $dom(\lambda f)$.
We consider these cases separately.
Case 1: Suppose c is not an accumulation point of $dom(\lambda f)$.
Since $c \in dom(\lambda f)$ and c is not an accumulation point of $dom(\lambda f)$, then λf is continuous at c.
Case 2: Suppose c is an accumulation point of $dom(\lambda f)$.
Since $dom(\lambda f) = dom f$, then c is an accumulation point of $dom f$.
Since $c \in dom f$ and c is an accumulation point of $dom f$ and f is continuous at c, then the limit of f at c exists and $\lim_{x\to c} f(x) = f(c)$.
Observe that

$$\begin{aligned}
(\lambda f)(c) &= \lambda f(c) \\
&= \lambda \lim_{x\to c} f(x) \\
&= \lim_{x\to c} [\lambda f(x)] \\
&= \lim_{x\to c} (\lambda f)(x).
\end{aligned}$$

Since $c \in dom(\lambda f)$ and $\lim_{x\to c}(\lambda f)(x) = (\lambda f)(c)$, then λf is continuous at c.
Therefore, in all cases, λf is continuous at c, as desired. ☐

Corollary 5. ***scalar multiple of a continuous function is continuous***
Let $\lambda \in \mathbb{R}$.
Let f be a real valued function.
If f is continuous, then λf is continuous.

Proof. Suppose f is continuous.
Let $c \in dom(\lambda f)$.
Since $dom(\lambda f) = dom f$, then $c \in dom f$.
Since f is continuous and $c \in dom f$, then f is continuous at c.
Therefore, λf is continuous at c, so λf is continuous. ☐

Theorem 6. *Let f and g be real valued functions.*
Let $c \in dom f \cap dom g$.
If f is continuous at c and g is continuous at c, then $f + g$ is continuous at c.

Proof. Suppose f is continuous at c and g is continuous at c.

Since f and g are real valued functions, then $f+g$ is a real valued function.

Since $c \in dom f \cap dom g$ and $dom f \cap dom g = dom(f+g)$, then $c \in dom(f+g)$.

Either c is an accumulation point of $dom(f+g)$ or c is not an accumulation point of $dom(f+g)$.

We consider these cases separately.

Case 1: Suppose c is not an accumulation point of $dom(f+g)$.

Since $c \in dom(f+g)$ and c is not an accumulation point of $dom(f+g)$, then $f+g$ is continuous at c.

Case 2: Suppose c is an accumulation point of $dom(f+g)$.

Since $dom(f+g) = dom f \cap dom g$, then c is an accumulation point of $dom f \cap dom g$.

Since $c \in dom f \cap dom g$, then $c \in dom f$ and $c \in dom g$.

Since c is an accumulation point of $dom f \cap dom g$ and $dom f \cap dom g$ is a subset of $dom f$, then c is an accumulation point of $dom f$.

Since $c \in dom f$ and c is an accumulation point of $dom f$ and f is continuous at c, then the limit of f at c exists and $\lim_{x \to c} f(x) = f(c)$.

Since c is an accumulation point of $dom f \cap dom g$ and $dom f \cap dom g$ is a subset of $dom g$, then c is an accumulation point of $dom g$.

Since $c \in dom g$ and c is an accumulation point of $dom g$ and g is continuous at c, then the limit of g at c exists and $\lim_{x \to c} g(x) = g(c)$.

Observe that

$$\begin{aligned}
(f+g)(c) &= f(c) + g(c) \\
&= \lim_{x \to c} f(x) + \lim_{x \to c} g(x) \\
&= \lim_{x \to c} [f(x) + g(x)] \\
&= \lim_{x \to c} (f+g)(x).
\end{aligned}$$

Since $c \in dom(f+g)$ and $\lim_{x \to c}(f+g)(x) = (f+g)(c)$, then $f+g$ is continuous at c.

Therefore, in all cases, $f+g$ is continuous at c, as desired. □

Corollary 7. ***sum of continuous functions is continuous***

Let f and g be real valued functions.

If f is continuous and g is continuous, then $f+g$ is continuous.

Proof. Suppose f is continuous and g is continuous.

Let $c \in dom(f+g)$.

Since $dom(f+g) = dom f \cap dom g$, then $c \in dom f \cap dom g$, so $c \in dom f$ and $c \in dom g$.

Since f is continuous and $c \in dom f$, then f is continuous at c.

Since g is continuous and $c \in dom g$, then g is continuous at c.

Therefore, $f+g$ is continuous at c, so $f+g$ is continuous. □

Corollary 8. *Let f and g be real valued functions.*

Let $c \in domf \cap domg$.

If f is continuous at c and g is continuous at c, then $f - g$ is continuous at c.

Proof. Suppose f is continuous at c and g is continuous at c.

Since $c \in domf \cap domg$ and $domg = dom(-g)$, then $c \in domf \cap dom(-g)$.

Since $c \in domf \cap domg$, then $c \in domg$.

Since $c \in domg$ and g is continuous at c, then $-g$ is continuous at c.

Since $c \in domf \cap dom(-g)$ and f is continuous at c and $-g$ is continuous at c, then $f - g = f + (-g)$ is continuous at c, as desired. □

Corollary 9. ***difference of continuous functions is continuous***

Let f and g be real valued functions.

If f is continuous and g is continuous, then $f - g$ is continuous.

Proof. Suppose f is continuous and g is continuous.

Let $c \in dom(f - g)$.

Since $dom(f - g) = domf \cap domg$, then $c \in domf \cap domg$, so $c \in domf$ and $c \in domg$.

Since f is continuous and $c \in domf$, then f is continuous at c.

Since g is continuous and $c \in domg$, then g is continuous at c.

Therefore, $f - g$ is continuous at c, so $f - g$ is continuous. □

Theorem 10. *Let f and g be real valued functions.*

Let $c \in domf \cap domg$.

If f is continuous at c and g is continuous at c, then fg is continuous at c.

Proof. Suppose f is continuous at c and g is continuous at c.

Since f and g are real valued functions, then fg is a real valued function.

Since $c \in domf \cap domg$ and $domf \cap domg = dom(fg)$, then $c \in domfg$.

Either c is an accumulation point of $dom(fg)$ or c is not an accumulation point of $dom(fg)$.

We consider these cases separately.

Case 1: Suppose c is not an accumulation point of $dom(fg)$.

Since $c \in dom(fg)$ and c is not an accumulation point of $dom(fg)$, then fg is continuous at c.

Case 2: Suppose c is an accumulation point of $dom(fg)$.

Since $dom(fg) = domf \cap domg$, then c is an accumulation point of $domf \cap domg$.

Since $c \in domf \cap domg$, then $c \in domf$ and $c \in domg$.

Since c is an accumulation point of $domf \cap domg$ and $domf \cap domg$ is a subset of $domf$, then c is an accumulation point of $domf$.

Since $c \in domf$ and c is an accumulation point of $domf$ and f is continuous at c, then the limit of f at c exists and $\lim_{x \to c} f(x) = f(c)$.

Since c is an accumulation point of $domf \cap domg$ and $domf \cap domg$ is a subset of $domg$, then c is an accumulation point of $domg$.

Since $c \in domg$ and c is an accumulation point of $domg$ and g is continuous at c, then the limit of g at c exists and $\lim_{x \to c} g(x) = g(c)$.

Observe that

$$\begin{aligned} (fg)(c) &= f(c)g(c) \\ &= \lim_{x \to c} f(x) \cdot \lim_{x \to c} g(x) \\ &= \lim_{x \to c} [f(x)g(x)] \\ &= \lim_{x \to c} (fg)(x). \end{aligned}$$

Since $c \in dom(fg)$ and $\lim_{x \to c}(fg)(x) = (fg)(c)$, then fg is continuous at c.

Therefore, in all cases, fg is continuous at c, as desired. □

Corollary 11. ***product of continuous functions is continuous***

Let f and g be real valued functions.

If f is continuous and g is continuous, then fg is continuous.

Proof. Suppose f is continuous and g is continuous.

Let $c \in domfg$.

Since $domfg = domf \cap domg$, then $c \in domf \cap domg$, so $c \in domf$ and $c \in domg$.

Since f is continuous and $c \in domf$, then f is continuous at c.

Since g is continuous and $c \in domg$, then g is continuous at c.

Therefore, fg is continuous at c, so fg is continuous. □

Theorem 12. *Let f and g be real valued functions.*

Let $c \in domf \cap domg$.

If f is continuous at c and g is continuous at c and $g(c) \neq 0$, then $\frac{f}{g}$ is continuous at c.

Proof. Suppose f is continuous at c and g is continuous at c and $g(c) \neq 0$.

Since f and g are real valued functions, then $\frac{f}{g}$ is a real valued function.

Since $c \in domf \cap domg$, then $c \in domf$ and $c \in domg$.

Since $dom\frac{f}{g} = domf \cap \{x \in domg : g(x) \neq 0\}$ and $c \in domf$ and $c \in domg$ and $g(c) \neq 0$, then $c \in dom\frac{f}{g}$.

Either c is an accumulation point of $dom\frac{f}{g}$ or c is not an accumulation point of $dom\frac{f}{g}$.

We consider these cases separately.

Case 1: Suppose c is not an accumulation point of $dom\frac{f}{g}$.

Since $c \in dom\frac{f}{g}$ and c is not an accumulation point of $dom\frac{f}{g}$, then $\frac{f}{g}$ is continuous at c.

Case 2: Suppose c is an accumulation point of $dom\frac{f}{g}$.

Let $x \in dom\frac{f}{g}$.

Then $x \in domf \cap \{x \in domg : g(x) \neq 0\}$, so $x \in domf$.

Hence, $dom\frac{f}{g} \subset domf$.

Since c is an accumulation point of $dom\frac{f}{g}$ and $dom\frac{f}{g}$ is a subset of $domf$, then c is an accumulation point of $domf$.

Since $c \in domf$ and c is an accumulation point of $domf$ and f is continuous at c, then the limit of f at c exists and $\lim_{x\to c} f(x) = f(c)$.

Let $x \in dom\frac{f}{g}$.

Then $x \in domf \cap \{x \in domg : g(x) \neq 0\}$, so $x \in \{x \in domg : g(x) \neq 0\}$.

Hence, $x \in domg$, so $dom\frac{f}{g} \subset domg$.

Since c is an accumulation point of $dom\frac{f}{g}$ and $dom\frac{f}{g}$ is a subset of $domg$, then c is an accumulation point of $domg$.

Since $c \in domg$ and c is an accumulation point of $domg$ and g is continuous at c, then the limit of g at c exists and $\lim_{x\to c} g(x) = g(c)$.

Since $g(c) \neq 0$, then $\lim_{x\to c} g(x) \neq 0$.

Let $x \in dom\frac{f}{g}$.

Then $x \in domf \cap \{x \in domg : g(x) \neq 0\}$, so $x \in domf$ and $x \in \{x \in domg : g(x) \neq 0\}$.

Thus, $x \in domf$ and $x \in domg$, so $x \in domf \cap domg$.

Hence, $dom\frac{f}{g} \subset domf \cap domg$.

Since c is an accumulation point of $dom\frac{f}{g}$ and $dom\frac{f}{g}$ is a subset of $domf \cap domg$, then c is an accumulation point of $domf \cap domg$.

Thus,

$$\begin{aligned}
\frac{f}{g}(c) &= \frac{f(c)}{g(c)} \\
&= \frac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)} \\
&= \lim_{x\to c} \frac{f(x)}{g(x)} \\
&= \lim_{x\to c} \frac{f}{g}(x).
\end{aligned}$$

Since $c \in dom\frac{f}{g}$ and c is an accumulation point of $dom\frac{f}{g}$ and $\lim_{x\to c} \frac{f}{g}(x) = \frac{f}{g}(c)$, then $\frac{f}{g}$ is continuous at c.

Therefore, in all cases, $\frac{f}{g}$ is continuous at c, as desired. ☐

Corollary 13. ***quotient of continuous functions is continuous wherever defined***

Let f and g be real valued functions.

If f is continuous and g is continuous, then $\frac{f}{g}$ is continuous for all $x \in domf \cap domg$ such that $g(x) \neq 0$.

Proof. Suppose f is continuous and g is continuous.

Let $c \in domf \cap domg$ such that $g(c) \neq 0$.

Since $c \in domf \cap domg$, then $c \in domf$ and $c \in domg$.
We must prove $\frac{f}{g}$ is continuous at c.
Since f is continuous and $c \in domf$, then f is continuous at c.
Since g is continuous and $c \in domg$, then g is continuous at c.
Since $g(c) \neq 0$, then $\frac{f}{g}$ is continuous at c, as desired. □

Theorem 14. ***polynomial functions are continuous***
Every polynomial function is continuous.

Proof. Let $p : \mathbb{R} \to \mathbb{R}$ be an arbitrary polynomial function.
To prove p is continuous, let $c \in \mathbb{R}$ be arbitrary.
We must prove p is continuous at c.
Since p is a polynomial function and $c \in \mathbb{R}$, then $\lim_{x \to c} p(x) = p(c)$.
Since $c \in \mathbb{R}$ and c is an accumulation point of $\mathbb{R}$ and $\lim_{x \to c} p(x) = p(c)$, then p is continuous at c, as desired. □

Theorem 15. ***rational functions are continuous wherever defined***
Let r be a rational function defined by $r(x) = \frac{p(x)}{q(x)}$ such that p and q are polynomial functions.
Then r is continuous for all $x \in \mathbb{R}$ such that $q(x) \neq 0$.

Proof. Since p is continuous and q is continuous, then $\frac{p}{q} = r$ is continuous for all $x \in domp \cap domq$ such that $q(x) \neq 0$.
Since $domp \cap domq = \mathbb{R} \cap \mathbb{R} = \mathbb{R}$, then r is continuous for all $x \in \mathbb{R}$ such that $q(x) \neq 0$, as desired. □

Theorem 16. *Let f and g be real valued functions of a real variable.*
If f is continuous at c and g is continuous at $f(c)$, then $g \circ f$ is continuous at c.

Proof. Since f and g are functions, then $g \circ f$ is a function defined by $(g \circ f)(x) = g(f(x))$ for all $x \in dom(g \circ f)$.
Suppose f is continuous at c and g is continuous at $f(c)$.
Since f is continuous at c, then $c \in domf$.
Since g is continuous at $f(c)$, then $f(c) \in domg$.
Since $c \in domf$ and $f(c) \in domg$, then $c \in dom(g \circ f)$.
To prove $g \circ f$ is continuous at c, let $\epsilon > 0$ be given.
Since g is continuous at $f(c)$, then there exists $\delta_1 > 0$ such that for all $x \in domg$, if $|x - f(c)| < \delta_1$, then $|g(x) - g(f(c))| < \epsilon$.
Since f is continuous at c and $\delta_1 > 0$, then there exists $\delta > 0$ such that for all $x \in domf$, if $|x - c| < \delta$, then $|f(x) - f(c)| < \delta_1$.
Let $x \in dom(g \circ f)$ such that $|x - c| < \delta$.
Since $x \in dom(g \circ f)$, then $x \in domf$ and $f(x) \in domg$.
Since $x \in domf$ and $|x - c| < \delta$, then $|f(x) - f(c)| < \delta_1$.
Since $f(x) \in domg$ and $|f(x) - f(c)| < \delta_1$, then $|g(f(x)) - g(f(c))| < \epsilon$.
Therefore, $|(g \circ f)(x) - (g \circ f)(c)| < \epsilon$, so $g \circ f$ is continuous at c. □

Corollary 17. ***composition of continuous functions is continuous***

Let f and g be real valued functions of a real variable.

If f is continuous and g is continuous, then $g \circ f$ is continuous.

Proof. Suppose f is continuous and g is continuous.

Let $c \in dom(g \circ f)$.

Since $dom(g \circ f) = \{x \in domf : f(x) \in domg\}$, then $c \in domf$ and $f(c) \in domg$.

Since f is continuous and $c \in domf$, then f is continuous at c.

Since g is continuous and $f(c) \in domg$, then g is continuous at $f(c)$.

Therefore, $g \circ f$ is continuous at c, so $g \circ f$ is continuous. □

Proposition 18. ***If f is a continuous function, then so is $|f|$.***

Let $f : E \to \mathbb{R}$ be a function.

Let $|f| : E \to \mathbb{R}$ be a function defined by $|f|(x) = |f(x)|$.

If f is continuous, then $|f|$ is continuous.

Proof. Suppose f is continuous.

Let $c \in E$.

Then f is continuous at c.

Let $\epsilon > 0$ be given.

Then there exists $\delta > 0$ such that for all $x \in E$, if $|x - c| < \delta$, then $|f(x) - f(c)| < \epsilon$.

Let $x \in E$ such that $|x - c| < \delta$.

Then $|f(x)-f(c)| < \epsilon$, so $||f|(x)-|f|(c)| = ||f(x)|-|f(c)|| \leq |f(x)-f(c)| < \epsilon$.

Therefore, $|f|$ is continuous at c, so $|f|$ is continuous. □

Proposition 19. ***If f is a continuous function, then so is $\sqrt{f}$.***

Let $f : E \to \mathbb{R}$ be a function such that $f(x) \geq 0$ for all $x \in E$.

Let $\sqrt{f}$ be a function defined by $\sqrt{f}(x) = \sqrt{f(x)}$ for all $x \in E$ such that $f(x) \geq 0$.

If f is continuous, then $\sqrt{f}$ is continuous.

Proof. Suppose f is continuous.

Let $g(x) = \sqrt{x}$.

Then $g : [0, \infty) \to \mathbb{R}$ is a continuous function.

The domain of $\sqrt{f}$ is the set $\{x \in E : f(x) \geq 0\}$.

Since $dom(g \circ f) = \{x \in domf : f(x) \in domg\} = \{x \in E : f(x) \in [0, \infty)\} = \{x \in E : f(x) \geq 0\}$, then the domain of $g \circ f$ is the same as the domain of $\sqrt{f}$, so $dom\sqrt{f} = dom(g \circ f)$.

Let $x \in dom(g \circ f)$.

Then $x \in E$ and $f(x) \geq 0$.

Thus, $\sqrt{f}(x) = \sqrt{f(x)} = g(f(x)) = (g \circ f)(x)$, so $\sqrt{f}(x) = (g \circ f)(x)$.

Hence, $\sqrt{f}(x) = (g \circ f)(x)$ for all $x \in dom(g \circ f)$.

Since $dom\sqrt{f} = dom(g \circ f)$ and $\sqrt{f}(x) = (g \circ f)(x)$ for all $x \in dom(g \circ f)$, then $\sqrt{f} = g \circ f$.

Since f is continuous and g is continuous and the composition of continuous functions is continuous, then $g \circ f$ is continuous.

Therefore, $\sqrt{f}$ is continuous, as desired. □

Continuous functions on compact sets

Lemma 20. ***Bolzano-Weierstrass property of compact sets***

Let $E \subset \mathbb{R}$ be a closed bounded set.

Then every sequence in E has a subsequence (y_n) in E such that $\lim_{n\to\infty} y_n \in E$.

Proof. Let (x_n) be an arbitrary sequence in E.

Then $x_n \in E$ for all $n \in \mathbb{N}$.

Since E is bounded, then there exists $B \in \mathbb{R}$ such that $|x| \leq B$ for all $x \in E$.

Let $n \in \mathbb{N}$ be given.

Then $x_n \in E$, so $|x_n| \leq B$.

Thus, $|x_n| \leq B$ for all $n \in \mathbb{N}$, so (x_n) is bounded in $\mathbb{R}$.

Therefore, by the Bolzano-Weierstrass theorem for sequences, (x_n) has a convergent subsequence.

Let (y_n) be a convergent subsequence of (x_n).

Since (y_n) is a subsequence of (x_n), then there exists a strictly increasing function $g : \mathbb{N} \to \mathbb{N}$ such that $y_n = x_{g(n)}$ for all $n \in \mathbb{N}$.

Let $n \in \mathbb{N}$ be given.

Then $g(n) \in \mathbb{N}$ and $y_n = x_{g(n)}$.

Thus, $x_{g(n)} \in E$, so $y_n \in E$.

Hence, $y_n \in E$ for all $n \in \mathbb{N}$, so (y_n) is a sequence in E.

Since (y_n) is convergent, let $c = \lim_{n\to\infty} y_n$.

We must prove $c \in E$.

We prove by contradiction.

Suppose $c \notin E$.

Since E is closed, then c is not an accumulation point of E.

Hence, there exists $\delta > 0$ such that $N'(c;\delta) \cap E = \emptyset$.

Since $\lim_{n\to\infty} y_n = c$ and $\delta > 0$, then there exists $N \in \mathbb{N}$ such that if $n > N$, then $|y_n - c| < \delta$.

Let $n \in \mathbb{N}$ such that $n > N$.

Then $|y_n - c| < \delta$, so $y_n \in N(c;\delta)$.

Since $n \in \mathbb{N}$, then $y_n \in E$.

Since $c \notin E$, then $y_n \neq c$, so $y_n \in N'(c;\delta)$.

Since $y_n \in N'(c;\delta)$ and $y_n \in E$, then $y_n \in N'(c;\delta) \cap E$, so $N'(c;\delta) \cap E \neq \emptyset$.

Hence, we have $N'(c;\delta) \cap E = \emptyset$ and $N'(c;\delta) \cap E \neq \emptyset$, a contradiction.

Therefore, $c \in E$, as desired. □

Theorem 21. ***Boundedness Theorem***

Every real valued function continuous on a closed bounded set is bounded.

Proof. Let $E \subset \mathbb{R}$ be a closed bounded set.

Let $f : E \to \mathbb{R}$ be a continuous function.

We must prove f is bounded.

We prove by contradiction.

Suppose f is not bounded.

Then f is unbounded, so for every real number r, there exists $x \in E$ such that $|f(x)| > r$.

In particular, for every $n \in \mathbb{N}$, there exists $x_n \in E$ such that $|f(x_n)| > n$.

Thus, there exists a sequence (x_n) in E such that $|f(x_n)| > n$ for all $n \in \mathbb{N}$.

Since E is a closed bounded set and (x_n) is a sequence in E, then by the Bolzano-Weierstrass property of compact sets, there exists a subsequence (y_n) in E such that $\lim_{n\to\infty} y_n \in E$.

Let $c = \lim_{n\to\infty} y_n$.

Then $c \in E$.

Since f is continuous on E and $c \in E$, then f is continuous at c.

Since (y_n) is a sequence in E and $\lim_{n\to\infty} y_n = c$, then by the sequential characterization of continuity, we conclude $\lim_{n\to\infty} f(y_n) = f(c)$.

Thus, the sequence $(f(y_n))$ is convergent, so $(f(y_n))$ is bounded.

We prove the sequence $(f(y_n))$ is unbounded.

Let $M \in \mathbb{R}$ be given.

By the Archimedean property of $\mathbb{R}$, there exists $n \in \mathbb{N}$ such that $n > M$.

Since (y_n) is a subsequence of (x_n), then there exists a strictly increasing function $g : \mathbb{N} \to \mathbb{N}$ such that $y_n = x_{g(n)}$ for all $n \in \mathbb{N}$.

Since $g : \mathbb{N} \to \mathbb{N}$ is strictly increasing, then $g(n) \geq n$ for all $n \in \mathbb{N}$.

Since $n \in \mathbb{N}$, then $g(n) \in \mathbb{N}$, so there exists $x_{g(n)} \in E$ such that $|f(x_{g(n)})| > g(n)$.

Observe that

$$\begin{aligned} |f(y_n)| &= |f(x_{g(n)})| \\ &> g(n) \\ &\geq n \\ &> M. \end{aligned}$$

Hence, $|f(y_n)| > M$.

Thus, there exists $n \in \mathbb{N}$ such that $|f(y_n)| > M$, so $(f(y_n))$ is unbounded.

Hence, we have $(f(y_n))$ is bounded and $(f(y_n))$ is unbounded, a contradiction.

Therefore, f is bounded, as desired. ☐

Theorem 22. ***Extreme Value Theorem***

Every real valued function continuous on a nonempty closed bounded set attains a maximum and minimum on the set.

Proof. Let S be a nonempty closed bounded set.

Let $f : S \to \mathbb{R}$ be a continuous function.

To prove there exists a maximum on S, we must prove there exists $c \in S$ such that $f(x) \leq f(c)$ for all $x \in S$.

Let $f(S) = \{f(x) : x \in S\}$.

Since $S \neq \emptyset$, then there exists $s \in S$, so $f(s) \in f(S)$.

Hence, $f(S) \neq \emptyset$.

Since f is continuous on S and the set S is closed and bounded, then by the boundedness theorem, f is bounded.

Hence, $f(S)$ is bounded, so $f(S)$ is bounded above in $\mathbb{R}$.

Since $f(S)$ is not empty and bounded above in $\mathbb{R}$, then by completeness of $\mathbb{R}$, $\sup f(S)$ exists.

Let $M = \sup f(S)$.

Suppose for the sake of contradiction $M \notin f(S)$.

Then there is no $x \in S$ such that $f(x) = M$, so $f(x) \neq M$ for all $x \in S$.

Since M is an upper bound of $f(S)$, then this implies $f(x) < M$ for all $x \in S$, so $M - f(x) > 0$ for all $x \in S$.

Let $g = \frac{1}{M-f}$.

Then $g : S \to \mathbb{R}$ is a function defined by $g(x) = \frac{1}{M-f(x)}$ for all $x \in S$.

Since $M - f(x) > 0$ for all $x \in S$, then $\frac{1}{M-f(x)} > 0$ for all $x \in S$, so $g(x) > 0$ for all $x \in S$.

Since f is continuous on S, then $-f$ is continuous on S, so $M-f$ is continuous on S.

Hence, $\frac{1}{M-f}$ is continuous on S, so g is continuous on S.

Let $g(S) = \{g(x) : x \in S\}$.

Since $S \neq \emptyset$, then there exists $s \in S$, so $g(s) \in g(S)$.

Hence, $g(S) \neq \emptyset$.

Since g is continuous on S and the set S is closed and bounded, then by the boundedness theorem, g is bounded.

Hence, $g(S)$ is bounded, so $g(S)$ is bounded above in $\mathbb{R}$.

Since $g(S)$ is not empty and bounded above in $\mathbb{R}$, then by completeness of $\mathbb{R}$, $\sup g(S)$ exists.

Let $M' = \sup g(S)$.

Since M' is an upper bound of $g(S)$, then $g(x) \leq M'$ for all $x \in S$.

Let $x \in S$.

Then $0 < g(x) \leq M'$, so $0 < \frac{1}{M-f(x)} \leq M'$ and $0 < M'$.

Hence, $\frac{1}{M'} \leq M - f(x)$, so $f(x) \leq M - \frac{1}{M'}$ for all $x \in S$.

Since $M' > 0$, then $\frac{1}{M'} > 0$.

Thus, there exists $\frac{1}{M'} > 0$ such that $f(x) \leq M - \frac{1}{M'}$ for all $x \in S$.

This contradicts the fact that M is the least upper bound of $f(S)$.

Hence, $M \in f(S)$, so there exists $c \in S$ such that $f(c) = M$.

Since M is an upper bound of $f(S)$, then $f(x) \leq M$ for all $x \in S$.

Therefore, there exists $c \in S$ such that $f(x) \leq f(c)$ for all $x \in S$, so f has a maximum on S. □

Proof. Let S be a nonempty closed bounded set.

Let $f : S \to \mathbb{R}$ be a continuous function.

To prove there exists a minimum on S, we must prove there exists $c \in S$ such that $f(c) \leq f(x)$ for all $x \in S$.

Let $f(S) = \{f(x) : x \in S\}$.

Since $S \neq \emptyset$, then there exists $s \in S$, so $f(s) \in f(S)$.

Hence, $f(S) \neq \emptyset$.

Since f is continuous on S and the set S is closed and bounded, then by the boundedness theorem, f is bounded.

Hence, $f(S)$ is bounded, so $f(S)$ is bounded below in $\mathbb{R}$.

Since $f(S)$ is not empty and bounded below in $\mathbb{R}$, then by completeness of $\mathbb{R}$, $\inf f(S)$ exists.

Let $m = \inf f(S)$.

Suppose for the sake of contradiction $m \notin f(S)$.

Then there is no $x \in S$ such that $f(x) = m$, so $f(x) \neq m$ for all $x \in S$.

Since m is a lower bound of $f(S)$, then this implies $m < f(x)$ for all $x \in S$, so $m - f(x) < 0$ for all $x \in S$.

Let $g = \frac{1}{m-f}$.

Then $g : S \to \mathbb{R}$ is a function defined by $g(x) = \frac{1}{m-f(x)}$ for all $x \in S$.

Since $m - f(x) < 0$ for all $x \in S$, then $\frac{1}{m-f(x)} < 0$ for all $x \in S$, so $g(x) < 0$ for all $x \in S$.

Since f is continuous on S, then $m - f$ is continuous on S, so $\frac{1}{m-f}$ is continuous on S.

Hence, g is continuous on S.

Let $g(S) = \{g(x) : x \in S\}$.

Since $S \neq \emptyset$, then there exists $s \in S$, so $g(s) \in g(S)$.

Hence, $g(S) \neq \emptyset$.

Since g is continuous on S and the set S is closed and bounded, then by the boundedness theorem, g is bounded.

Hence, $g(S)$ is bounded, so $g(S)$ is bounded below in $\mathbb{R}$.

Since $g(S)$ is not empty and bounded below in $\mathbb{R}$, then by completeness of $\mathbb{R}$, $\inf g(S)$ exists.

Let $m' = \inf g(S)$.

Since m' is a lower bound of $g(S)$, then $m' \leq g(x)$ for all $x \in S$.

Let $x \in S$.

Then $m' \leq g(x) < 0$, so $m' \leq \frac{1}{m-f(x)} < 0$ and $m' < 0$.

Since $m - f(x) < 0$, then $m'(m - f(x)) \geq 1$.

Since $m' < 0$, then $m - f(x) \leq \frac{1}{m'}$, so $m - \frac{1}{m'} \leq f(x)$.

Hence, $f(x) \geq m - \frac{1}{m'} = m + \frac{-1}{m'}$, so $f(x) \geq m + \frac{-1}{m'}$ for all $x \in S$.

Since $m' < 0$, then $\frac{-1}{m'} > 0$.
Thus, there exists $\frac{-1}{m'} > 0$ such that $f(x) \geq m + \frac{-1}{m'}$ for all $x \in S$.
This contradicts the fact that m is the greatest lower bound of $f(S)$.

Hence, $m \in f(S)$, so there exists $c \in S$ such that $f(c) = m$.
Since m is a lower bound of $f(S)$, then $m \leq f(x)$ for all $x \in S$.
Therefore, there exists $c \in S$ such that $f(c) \leq f(x)$ for all $x \in S$, so f has a minimum on S. $\square$

Lemma 23. *Let f be a continuous real valued function.*
Let x_0, c, d be real numbers such that $x_0 \in dom f$ and $c < f(x_0) < d$.
Then there exists a positive real number δ such that $c < f(x) < d$ for all $x \in N(x_0; \delta) \cap dom f$.

Proof. Since $c < f(x_0) < d$, then $c < f(x_0)$ and $f(x_0) < d$, so $f(x_0) - c > 0$ and $d - f(x_0) > 0$.
Since f is continuous and $x_0 \in dom f$, then f is continuous at x_0.
Let $\epsilon = \min\{f(x_0) - c, d - f(x_0)\}$.
Then $\epsilon \leq f(x_0) - c$ and $\epsilon \leq d - f(x_0)$ and either $\epsilon = f(x_0) - c$ or $\epsilon = d - f(x_0)$.
Since $f(x_0) - c > 0$ and $d - f(x_0) > 0$, then $\epsilon > 0$.
Since f is continuous at x_0 and $\epsilon > 0$, then there exists $\delta > 0$ such that for all $x \in dom f$, if $|x - x_0| < \delta$, then $|f(x) - f(x_0)| < \epsilon$.
Since $x_0 \in N(x_0; \delta)$ and $x_0 \in dom f$, then $x_0 \in N(x_0; \delta) \cap dom f$, so $N(x_0; \delta) \cap dom f \neq \emptyset$.
Let $x \in N(x_0; \delta) \cap dom f$.
Then $x \in N(x_0; \delta)$ and $x \in dom f$, so $|x - x_0| < \delta$.
Hence, $|f(x) - f(x_0)| < \epsilon$.
Either $f(x) < f(x_0)$ or $f(x) \geq f(x_0)$.
We consider these cases separately.
Case 1: Suppose $f(x) < f(x_0)$.
Then $f(x_0) - f(x) > 0$.
Since $f(x) < f(x_0)$ and $f(x_0) < d$, then $f(x) < d$.
Since $|f(x) - f(x_0)| < \epsilon$ and $\epsilon \leq f(x_0) - c$, then $|f(x) - f(x_0)| < f(x_0) - c$, so $|f(x_0) - f(x)| < f(x_0) - c$.
Hence, $f(x_0) - f(x) < f(x_0) - c$, so $-f(x) < -c$.
Thus, $f(x) > c$.
Therefore, $c < f(x) < d$.
Case 2: Suppose $f(x) \geq f(x_0)$.
Then $f(x) - f(x_0) \geq 0$.
Since $c < f(x_0)$ and $f(x_0) \leq f(x)$, then $c < f(x)$.
Since $|f(x) - f(x_0)| < \epsilon$ and $\epsilon \leq d - f(x_0)$, then $|f(x) - f(x_0)| < d - f(x_0)$.
Hence, $f(x) - f(x_0) < d - f(x_0)$, so $f(x) < d$.
Therefore, $c < f(x) < d$.
Thus, in all cases, $c < f(x) < d$, as desired. $\square$

Theorem 24. ***Intermediate Value Theorem***

Let $a, b \in \mathbb{R}$.

Let f be a real valued function continuous on the closed interval $[a, b]$.

For every real number k such that $f(a) < k < f(b)$, there exists $c \in (a, b)$ such that $f(c) = k$.

Proof. Let k be an arbitrary real number such that $f(a) < k < f(b)$.

Then $f(a) < k$ and $k < f(b)$.

Let $S = \{x \in [a, b] : f(x) < k\}$.

Since $a \in [a, b]$ and $f(a) < k$, then $a \in S$, so $S \neq \emptyset$.

Let $x \in S$.

Then $x \in [a, b]$, so $a \leq x \leq b$.

Hence, $x \leq b$, so b is an upper bound of S.

Thus, S is bounded above in $\mathbb{R}$.

Since S is not empty and bounded above in $\mathbb{R}$, then by the completeness of $\mathbb{R}$, $\sup S$ exists.

Let $c = \sup S$.

Since c is the least upper bound of S and b is an upper bound of S, then $c \leq b$.

Since $a \in S$ and c is an upper bound of S, then $a \leq c$.

Hence, $a \leq c \leq b$, so $c \in [a, b]$.

Since f is a function and $c \in [a, b]$, then $f(c) \in \mathbb{R}$.

We must prove $f(c) = k$.

Suppose for the sake of contradiction $f(c) < k$.

Then $f(c) < k < f(b)$, so $f(c) < f(b)$.

Hence, $f(c) \neq f(b)$.

Since f is a function, then $c \neq b$.

Since $c \leq b$ and $c \neq b$, then $c < b$.

Since $f(c) < k$, then by the previous lemma, there exists $\delta > 0$ such that $f(x) < k$ for all $x \in N(x_0; \delta) \cap [a, b]$.

Let $m = \min\{b, c + \delta\}$.

Then $m \leq b$ and $m \leq c + \delta$ and either $m = b$ or $m = c + \delta$.

Since $c < b$ and $c < c + \delta$, then $c < m$.

Let $x = \frac{c+m}{2}$.

Then $2x = c + m$.

Since $c < m$, then $2c < c + m < 2m$, so $2c < 2x < 2m$.

Hence, $c < x < m$, so $c < x$ and $x < m$.

Since $x < m \leq c + \delta$, then $x < c + \delta$, so $x - c < \delta$.

Since $x > c$, then $x - c > 0$, so $|x - c| = x - c < \delta$.

Thus, $x \in N(c; \delta)$.

Since $a \leq c < x < m \leq b$, then $a < x < b$, so $x \in [a, b]$.

Since $x \in N(c; \delta)$ and $x \in [a, b]$, then $x \in N(c; \delta) \cap [a, b]$, so $f(x) < k$.

Since $x \in [a, b]$ and $f(x) < k$, then $x \in S$.

Thus, there exists $x \in S$ such that $x > c$.

This contradicts the fact that c is an upper bound of S.

Therefore, $f(c)$ cannot be less than k.

Suppose for the sake of contradiction $f(c) > k$.
Since $k < f(c)$, then by the previous lemma, there exists $\delta > 0$ such that $k < f(x)$ for all $x \in N(c;\delta) \cap [a,b]$.
Since $f(c) > k$, then $c \notin S$.
Since c is an upper bound of S, then this implies $x < c$ for all $x \in S$.
Since c is the least upper bound of S and $\delta > 0$, then there exists $x \in S$ such that $x > c - \delta$.
Since $x \in S$, then $x \in [a,b]$ and $f(x) < k$ and $x < c$.
Since $x < c$, then $c - x > 0$.
Since $x > c - \delta$, then $\delta > c - x$.
Thus, $|x - c| = |c - x| = c - x < \delta$, so $x \in N(c;\delta)$.
Since $x \in N(c;\delta)$ and $x \in [a,b]$, then $x \in N(c;\delta) \cap [a,b]$, so $k < f(x)$.
Thus, we have $f(x) < k$ and $f(x) > k$, a contradiction.
Therefore, $f(c)$ cannot be greater than k.

Since $f(c) \in \mathbb{R}$ and $f(c)$ cannot be less than k and $f(c)$ cannot be greater than k, then $f(c) = k$.

Since $c = b$ implies $f(c) = f(b)$, then $f(c) \neq f(b)$ implies $c \neq b$.
Since $f(c) = k < f(b)$, then $f(c) \neq f(b)$, so $c \neq b$.
Since $c \leq b$ and $c \neq b$, then $c < b$.
Since $c = a$ implies $f(c) = f(a)$, then $f(c) \neq f(a)$ implies $c \neq a$.
Since $f(c) = k > f(a)$, then $f(c) \neq f(a)$, so $c \neq a$.
Since $c \geq a$ and $c \neq a$, then $c > a$.
Therefore, $a < c < b$, so $c \in (a,b)$.
Thus, there exists $c \in (a,b)$ such that $f(c) = k$, as desired. ☐

Lemma 25. *Let I be an interval and $a \in I$ and $b \in I$ and $a < b$.*
Then $[a,b] \subset I$.

Proof. Let $x \in [a,b]$.
Then $a \leq x \leq b$, so either $x = a$ or $x = b$ or $a < x < b$.
We consider these cases separately.
Case 1: Suppose $x = a$.
Since $a \in I$, then $x \in I$.
Case 2: Suppose $x = b$.
Since $b \in I$, then $x \in I$.
Case 3: Suppose $a < x < b$.
Since $a \in I$ and $b \in I$ and $a < x < b$ and I is an interval, then $x \in I$.
Therefore, in all cases, $x \in I$, so $[a,b] \subset I$, as desired. ☐

Theorem 26. ***intervals are preserved by continuous functions***
Let f be a real valued function continuous on an interval I.
Then $f(I)$ is an interval.

Proof. Either f is constant or not.

We consider these cases separately.

Case 1: Suppose f is constant.

Then there exists $k \in \mathbb{R}$ such that $f(x) = k$ for all $x \in I$, so $f(I) = \{k\}$.

Thus, $f(I)$ is a singleton set.

Since a singleton set is an interval, then $f(I)$ is an interval.

Case 2: Suppose f is not constant.

Then there exist at least two distinct elements of $f(I)$.

Thus, there exist real numbers $y_1 \in f(I)$ and $y_2 \in f(I)$ such that $y_1 \neq y_2$.

Let $y_1, y_2 \in \mathbb{R}$ such that $y_1 \in f(I)$ and $y_2 \in f(I)$ and $y_1 \neq y_2$.

Since $y_1 \neq y_2$, then either $y_1 < y_2$ or $y_1 > y_2$.

Without loss of generality, assume $y_1 < y_2$.

By the density of $\mathbb{R}$, there exists a real number y_0 such that $y_1 < y_0 < y_2$.

Since $y_1 \in f(I)$, then there exists $x_1 \in I$ such that $f(x_1) = y_1$.

Since $y_2 \in f(I)$, then there exists $x_2 \in I$ such that $f(x_2) = y_2$.

Thus, $f(x_1) < y_0 < f(x_2)$ and $f(x_1) < f(x_2)$.

Since $f(x_1) < f(x_2)$, then $f(x_1) \neq f(x_2)$.

Since f is a function, then this implies $x_1 \neq x_2$.

Since I is an interval and $x_1 \in I$ and $x_2 \in I$ and $x_1 \neq x_2$, then by the previous lemma, the closed, bounded interval with endpoints x_1 and x_2 is a subset of I.

Since f is continuous on I, then the restriction of f to the closed, bounded interval with endpoints x_1 and x_2 is continuous.

Since $f(x_1) < y_0 < f(x_2)$, then by IVT, there exists x_0 between x_1 and x_2 such that $f(x_0) = y_0$.

Since $x_1 \in I$ and $x_2 \in I$ and x_0 is between x_1 and x_2 and I is an interval, then $x_0 \in I$.

Thus, there exists $x_0 \in I$ such that $f(x_0) = y_0$, so $y_0 \in f(I)$.

Since $y_1 \in f(I)$ and $y_2 \in f(I)$ and $y_1 < y_0 < y_2$ implies $y_0 \in f(I)$, then $f(I)$ is an interval. ☐

Uniform continuity

Proposition 27. ***uniform continuity implies continuity***

Let $E \subset \mathbb{R}$.

Let $f : E \to \mathbb{R}$ be a function.

If f is uniformly continuous on E, then f is continuous on E.

Proof. Suppose f is uniformly continuous on E.

Let c be an arbitrary element of E.

Let $\epsilon > 0$ be given.

Since f is uniformly continuous on E, then there exists $\delta > 0$ such that for all $x, y \in E$, if $|x - y| < \delta$, then $|f(x) - f(y)| < \epsilon$.

Let $x \in E$ such that $|x - c| < \delta$.

Since $x \in E$ and $c \in E$ and $|x - c| < \delta$, then $|f(x) - f(c)| < \epsilon$.

Therefore, f is continuous at c, so f is continuous on E. ☐

Lemma 28. *Let $E \subset \mathbb{R}$.*

Let $f : E \to \mathbb{R}$ be a function.

If f is not uniformly continuous on E, then there exist $\epsilon_1 > 0$ and sequences (x_n) and (y_n) in E such that $\lim_{n\to\infty}(x_n - y_n) = 0$ and $|f(x_n) - f(y_n)| \geq \epsilon_1$ for all $n \in \mathbb{N}$.

Proof. Suppose f is not uniformly continuous on E.

Then there exists $\epsilon_1 > 0$ such that for every $\delta > 0$ there are $x, y \in E$ such that $|x - y| < \delta$ and $|f(x) - f(y)| \geq \epsilon_1$.

Let $\delta = \frac{1}{n}$ for all $n \in \mathbb{N}$.

Then for each $n \in \mathbb{N}$, there is $x \in E$ and there is $y \in E$ with $|x - y| < \frac{1}{n}$ and $|f(x) - f(y)| \geq \epsilon_1$.

Hence, there is a function $g : \mathbb{N} \to E$ and there is $y \in E$ with $|g(n) - y| < \frac{1}{n}$ and $|f(g(n)) - f(y)| \geq \epsilon_1$ for each $n \in \mathbb{N}$.

Thus, there is a sequence (x_n) in E and there is $y \in E$ with $|x_n - y| < \frac{1}{n}$ and $|f(x_n) - f(y)| \geq \epsilon_1$ for each $n \in \mathbb{N}$.

Since for each $n \in \mathbb{N}$ there is $y \in E$ such that $|x_n - y| < \frac{1}{n}$ and $|f(x_n) - f(y)| \geq \epsilon_1$, then there is a function $h : \mathbb{N} \to E$ such that $|x_n - h(n)| < \frac{1}{n}$ and $|f(x_n) - f(h(n))| \geq \epsilon_1$ for each $n \in \mathbb{N}$.

Thus, there is a sequence (y_n) in E such that $|x_n - y_n| < \frac{1}{n}$ and $|f(x_n) - f(y_n)| \geq \epsilon_1$ for each $n \in \mathbb{N}$.

We prove $\lim_{n\to\infty}(x_n - y_n) = 0$.

Let $\epsilon > 0$ be given.

Then $\frac{1}{\epsilon} > 0$, so by the Archimedean property of $\mathbb{R}$, there exists $N \in \mathbb{N}$ such that $N > \frac{1}{\epsilon}$.

Hence, $\epsilon > \frac{1}{N}$.

Let $n \in \mathbb{N}$ such that $n > N$.

Since $n \in \mathbb{N}$ and $|x_n - y_n| < \frac{1}{n}$ for all $n \in \mathbb{N}$, then $|x_n - y_n| < \frac{1}{n}$.

Since $n > N > 0$, then $0 < \frac{1}{n} < \frac{1}{N}$.

Thus, $|x_n - y_n| < \frac{1}{n} < \frac{1}{N} < \epsilon$, so $|x_n - y_n| < \epsilon$.

Therefore, there exist $\epsilon_1 > 0$ and sequences (x_n) and (y_n) in E such that $\lim_{n\to\infty}(x_n - y_n) = 0$ and $|f(x_n) - f(y_n)| \geq \epsilon_1$ for each $n \in \mathbb{N}$. □

Theorem 29. ***Heine-Cantor Uniform Continuity Theorem***

Let $E \subset \mathbb{R}$.

Let $f : E \to \mathbb{R}$ be a function.

If f is continuous on E and E is a closed bounded set, then f is uniformly continuous on E.

Proof. Suppose f is continuous on E and E is a closed bounded set.

We prove by contradiction.

Suppose f is not uniformly continuous on E.

By the previous lemma, there exist $\epsilon_1 > 0$ and sequences (x_n) and (y_n) in E such that $\lim_{n\to\infty}(x_n - y_n) = 0$ and $|f(x_n) - f(y_n)| \geq \epsilon_1$ for all $n \in \mathbb{N}$.

Since (x_n) is a sequence in E and E is a closed bounded set, then by the Bolzano-Weierstrass property of compact sets, there is a subsequence of (x_n) in E that converges to some point in E.

Let (s_n) be a subsequence of (x_n) in E that converges to some point in E.

Then $s_n \in E$ for all $n \in \mathbb{N}$ and there exists $c \in E$ such that $\lim_{n\to\infty} s_n = c$.

Since (y_n) is a sequence in E and E is a closed bounded set, then by the Bolzano-Weierstrass property of compact sets, there is a subsequence of (y_n) in E that converges to some point in E.

Let (t_n) be a subsequence of (y_n) in E that converges to some point in E.

The $t_n \in E$ for all $n \in \mathbb{N}$ and there exists $d \in E$ such that $\lim_{n\to\infty} t_n = d$.

Since (s_n) is a subsequence of (x_n), then there exists a strictly increasing function $a : \mathbb{N} \to \mathbb{N}$ such that $s_n = x_{a(n)}$ for all $n \in \mathbb{N}$.

Since (t_n) is a subsequence of (y_n), then there exists a strictly increasing function $b : \mathbb{N} \to \mathbb{N}$ such that $t_n = y_{b(n)}$ for all $n \in \mathbb{N}$.

Let $n \in \mathbb{N}$.

Then $a(n) \in \mathbb{N}$ and $s_n = x_{a(n)}$ and $b(n) \in \mathbb{N}$ and $t_n = y_{b(n)}$.

Since $a(n) \in \mathbb{N}$, then $|x_{a(n)} - y_n| < \frac{1}{n}$ and $|f(x_n) - f(y_n)| \geq \epsilon_1$.

Since f is continuous on E and $c \in E$, then f is continuous at c.

Hence, $\square$