Continuous functions Examples

Jason Sass

May 19, 2023

Continuity

Example 1. every constant function is continuous
Let $k \in \mathbb{R}$.
The function given by $f(x) = k$ is continuous.

Proof. Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by $f(x) = k$ for all $x \in \mathbb{R}$.
Let $c \in \mathbb{R}$ be given.
To prove f is continuous, we must prove f is continuous at c.
Let $\epsilon > 0$ be given.
Let $\delta = 1$.
Then $\delta > 0$.
Let $x \in \mathbb{R}$ such that $|x - c| < \delta$.
Since $|f(x) - f(c)| = |k - k| = 0 < \epsilon$, then the conditional if $0 < |x - c| < \delta$, then $|k - k| < \epsilon$ is trivially true.
Therefore, f is continuous at c, as desired. □

Example 2. identity function is continuous
The function given by $f(x) = x$ is continuous.

Proof. Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by $f(x) = x$ for all $x \in \mathbb{R}$.
Let $c \in \mathbb{R}$ be given.
To prove f is continuous, we must prove f is continuous at c.
Let $\epsilon > 0$ be given.
Let $\delta = \epsilon$.
Then $\delta > 0$.
Let $x \in \mathbb{R}$ such that $|x - c| < \delta$.
Then $|x - c| < \delta = \epsilon$, so $|x - c| < \epsilon$, as desired. □

Example 3. square function is continuous
The function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$ is continuous.

Proof. Let $c \in \mathbb{R}$ be given.
To prove f is continuous, we must prove f is continuous at c.
Let $\epsilon > 0$ be given.
Let $\delta = \min\{1, \frac{\epsilon}{1+2|c|}\}$.

Then $\delta \leq 1$ and $\delta \leq \frac{\epsilon}{1+2|c|}$.
Since $|c| \geq 0$, then $2|c| \geq 0$, so $1 + 2|c| \geq 1 > 0$.
Hence, $1 + 2|c| > 0$, so $\frac{\epsilon}{1+2|c|} > 0$.
Thus, $\delta > 0$.
Let $x \in \mathbb{R}$ such that $|x - c| < \delta$.
Since

$$\begin{aligned} |x+c| &= |x-c+2c| \\ &\leq |x-c| + |2c| \\ &= |x-c| + 2|c| \\ &< \delta + 2|c| \\ &\leq 1 + 2|c|, \end{aligned}$$

then $0 \leq |x + c| < 1 + 2|c|$.
Hence,

$$\begin{aligned} |x^2 - c^2| &= |(x-c)(x+c)| \\ &= |x-c||x+c| \\ &< \delta(1+2|c|) \\ &\leq \frac{\epsilon}{1+2|c|} \cdot (1+2|c|) \\ &= \epsilon. \end{aligned}$$

Therefore, $|x^2 - c^2| < \epsilon$, as desired. ∎

Example 4. The function $f : (0, \infty) \to \mathbb{R}$ defined by $f(x) = \frac{1}{x}$ is continuous.

Proof. Let $c \in (0, \infty)$ be given.
Then $c > 0$.
To prove f is continuous, we must prove f is continuous at c.
Let $\epsilon > 0$ be given.
Let $\delta = \min\{\frac{c}{2}, \frac{c^2\epsilon}{2}\}$.
Then $\delta \leq \frac{c}{2}$ and $\delta \leq \frac{c^2\epsilon}{2}$ and $\delta > 0$.
Let $x \in (0, \infty)$ such that $|x - c| < \delta$.
Then $x > 0$ and $0 \leq |x - c| < \delta$.
Since $|x - c| < \delta \leq \frac{c}{2}$, then $|x - c| < \frac{c}{2}$.
Since $\frac{c}{2} > |x - c| \geq |c| - |x| = c - x$, then $\frac{c}{2} > c - x$, so $x > \frac{c}{2} > 0$.
Thus, $0 < \frac{c}{2} < x$, so $0 < \frac{1}{x} < \frac{2}{c}$.

Observe that

$$\begin{aligned}
|\frac{1}{x} - \frac{1}{c}| &= |\frac{1}{c} - \frac{1}{x}| \\
&= |\frac{x-c}{cx}| \\
&= \frac{1}{c} \cdot \frac{1}{x} \cdot |x-c| \\
&< \frac{1}{c} \cdot \frac{2}{c} \cdot \delta \\
&= \frac{2}{c^2} \cdot \delta \\
&\leq \frac{2}{c^2} \cdot \frac{c^2\epsilon}{2} \\
&= \epsilon.
\end{aligned}$$

Therefore, $|\frac{1}{x} - \frac{1}{c}| < \epsilon$, as desired. □

Proof. To prove f is continuous on its domain, we must prove f is continuous on the interval $(0, \infty)$.

Let $c \in (0, \infty)$ be arbitrary.

We must prove f is continuous at c.

Since every element of a non-degenerate interval is an accumulation point of the interval, then c is an accumulation point of $(0, \infty)$.

Since $c \in (0, \infty)$, then $c > 0$, so c is positive.

Since $\lim_{x \to c} \frac{1}{x} = \frac{1}{c}$ for all positive real c, then $\lim_{x \to c} f(x) = \lim_{x \to c} \frac{1}{x} = \frac{1}{c} = f(c)$.

Since $c \in (0, \infty)$ and $\lim_{x \to c} f(x) = f(c)$, then f is continuous at c, as desired. □

Example 5. absolute value function is continuous

The function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = |x|$ is continuous.

Proof. Let $c \in \mathbb{R}$ be given.

To prove f is continuous, we must prove f is continuous at c.

Let $\epsilon > 0$ be given.

Let $\delta = \epsilon$.

Then $\delta > 0$.

Let $x \in \mathbb{R}$ such that $|x - c| < \delta$.

Then $||x| - |c|| \leq |x - c| < \delta = \epsilon$, so $||x| - |c|| < \epsilon$, as desired. □

Example 6. square root function is continuous

The function $f : [0, \infty) \to \mathbb{R}$ defined by $f(x) = \sqrt{x}$ is continuous.

Proof. Let $c \in [0, \infty)$ be given.

Then $c \geq 0$.

To prove f is continuous, we must prove f is continuous at c.

Either $c > 0$ or $c = 0$.

We consider each case separately.
Case 1: Suppose $c = 0$.
To prove f is continuous at 0, let $\epsilon > 0$ be given.
Let $\delta = \epsilon^2$.
Then $\delta > 0$.
Let $x \geq 0$ such that $|x| < \delta$.
Then $0 \leq x < \delta$, so $0 \leq \sqrt{x} < \sqrt{\delta}$.
Thus, $|\sqrt{x}| = \sqrt{x} < \sqrt{\delta} = \sqrt{\epsilon^2} = |\epsilon| = \epsilon$, so $|\sqrt{x}| < \epsilon$.
Therefore, f is continuous at 0.
Case 2: Suppose $c > 0$.
To prove f is continuous at c, let $\epsilon > 0$ be given.
Let $\delta = \epsilon\sqrt{c}$.
Since $c > 0$, then $\sqrt{c} > 0$.
Since $\epsilon > 0$ and $\sqrt{c} > 0$, then $\delta > 0$.
Let $x \in [0, \infty)$ such that $|x - c| < \delta$.
Since $x \in [0, \infty)$, then $x \geq 0$, so $\sqrt{x} \geq 0$.
Since $\sqrt{x} \geq 0$ and $\sqrt{c} > 0$, then $\sqrt{x} + \sqrt{c} \geq \sqrt{c} > 0$, so $\frac{1}{\sqrt{c}} \geq \frac{1}{\sqrt{x}+\sqrt{c}} > 0$.
Thus, $0 < \frac{1}{\sqrt{x}+\sqrt{c}} \leq \frac{1}{\sqrt{c}}$.
Hence,

$$\begin{aligned}
|\sqrt{x} - \sqrt{c}| &= |(\sqrt{x} - \sqrt{c}) \cdot \frac{\sqrt{x} + \sqrt{c}}{\sqrt{x} + \sqrt{c}}| \\
&= |\frac{x - c}{\sqrt{x} + \sqrt{c}}| \\
&= |x - c| \cdot \frac{1}{\sqrt{x} + \sqrt{c}} \\
&< \delta \cdot \frac{1}{\sqrt{c}} \\
&= \epsilon.
\end{aligned}$$

Therefore, $|\sqrt{x} - \sqrt{c}| < \epsilon$, so f is continuous at c.
Thus, in all cases, f is continuous at c, as desired. □

Example 7. function with a removable discontinuity
Let $f : \mathbb{R} \to \mathbb{R}$ be a function defined by

$$f(x) = \begin{cases} x + 1 & \text{if } x \neq 1 \\ 5 & \text{if } x = 1 \end{cases}$$

Then f is discontinuous at 1.

Proof. To prove f is discontinuous at 1, we must prove $(\exists \epsilon > 0)(\forall \delta > 0)(\exists x \in \mathbb{R})(|x - 1| < \delta \wedge |f(x) - f(1)| \geq \epsilon)$.
Let $\epsilon = 3$.

Let $\delta > 0$ be given.

Let $x = 1 - \frac{\delta}{2}$, the midpoint of $1 - \delta$ and 1.

Then $|x - 1| = |(1 - \frac{\delta}{2}) - 1| = |\frac{-\delta}{2}| = |\frac{\delta}{2}| = \frac{\delta}{2} < \delta$ and $|f(x) - f(1)| = |f(1 - \frac{\delta}{2}) - 5| = |(1 - \frac{\delta}{2} + 1) - 5| = |-3 - \frac{\delta}{2}| = |3 + \frac{\delta}{2}| = 3 + \frac{\delta}{2} > 3 = \epsilon$.

Therefore, $|x - 1| < \delta$ and $|f(x) - f(1)| > \epsilon$, as desired. $\square$

Example 8. function with a jump discontinuity

Let $f : \mathbb{R} \to \mathbb{R}$ be a function defined by

$$f(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$

Then f is discontinuous at 0.

Proof. To prove f is discontinuous at 0, we must prove $(\exists \epsilon > 0)(\forall \delta > 0)(\exists x \in \mathbb{R})(|x| < \delta \wedge |f(x) - f(0)| \geq \epsilon)$.

Let $\epsilon = \frac{1}{2}$.

Let $\delta > 0$ be given.

Let $x = -\frac{\delta}{2}$.

Since $\delta > 0$, then $\frac{\delta}{2} > 0$, so $\frac{-\delta}{2} < 0$.

Thus, $|x| = |\frac{-\delta}{2}| = |\frac{\delta}{2}| = \frac{\delta}{2} < \delta$ and $|f(x) - f(0)| = |f(\frac{-\delta}{2}) - 1| = |0 - 1| = 1 > \frac{1}{2} = \epsilon$.

Therefore, $|x| < \delta$ and $|f(x) - f(0)| > \epsilon$, as desired. $\square$

Example 9. unbounded function, infinite discontinuity

The function $f : (0, \infty) \to \mathbb{R}$ defined by $f(x) = \frac{1}{x}$ is discontinuous at 0 since $0 \notin (0, \infty)$, the domain of f.

Let $r \in \mathbb{R}$ be arbitrary.

Let $g : [0, \infty) \to \mathbb{R}$ be a function defined by

$$g(x) = \begin{cases} \frac{1}{x} & \text{if } x > 0 \\ r & \text{if } x = 0 \end{cases}$$

Then g is discontinuous at 0.

Proof. To prove g is discontinuous at 0, we must prove $(\exists \epsilon > 0)(\forall \delta > 0)(\exists x \in [0, \infty))(|x| < \delta \wedge |g(x) - g(0)| \geq \epsilon)$.

Let $\epsilon = 1$.

Let $\delta > 0$ be given.

Let $x = \min\{\frac{1}{|r|+1}, \frac{\delta}{2}\}$.

Then $x \leq \frac{1}{|r|+1}$ and $x \leq \frac{\delta}{2}$.

Since $|r| \geq 0$, then $|r| + 1 \geq 1 > 0$, so $|r| + 1 > 0$.

Hence, $\frac{1}{|r|+1} > 0$.

Since $\delta > 0$, then $\frac{\delta}{2} > 0$.

Thus, $x > 0$, so $x \in [0, \infty)$.
Since $x > 0$ and $x \leq \frac{\delta}{2}$, then $|x| = x \leq \frac{\delta}{2} < \delta$, so $|x| < \delta$.
Since $0 < x \leq \frac{1}{|r|+1}$, then $|r| + 1 \leq \frac{1}{x}$, so $1 \leq \frac{1}{x} - |r|$.
Hence,

$$\begin{aligned} |g(x) - g(0)| &= |\frac{1}{x} - r| \\ &\geq |\frac{1}{x}| - |r| \\ &= \frac{1}{x} - |r| \\ &\geq 1. \end{aligned}$$

Thus, $|g(x) - g(0)| \geq 1 = \epsilon$, so g is discontinuous at 0.
Since r is arbitrary, then g is discontinuous at 0 regardless of how $g(0)$ is defined. □

Example 10. Dirichlet function is discontinuous everywhere
Let $f : \mathbb{R} \to \mathbb{R}$ be a function defined by

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$$

Then f is not continuous at any point in its domain.

Proof. Let $c \in \mathbb{R}$ be arbitrary.
Then either $c \in \mathbb{Q}$ or $c \notin \mathbb{Q}$.
We consider each case separately.
Case 1: Suppose $c \in \mathbb{Q}$.
Then c is rational.
Let $\epsilon = \frac{1}{2}$.
Let $\delta > 0$ be given.
Since $\mathbb{R} - \mathbb{Q}$ is dense in $\mathbb{R}$ and $c < c + \delta$, then there exists $x \in \mathbb{R} - \mathbb{Q}$ such that $c < x < c + \delta$.
Since $c < x < c + \delta$, then $0 < x - c < \delta$, so $|x - c| < \delta$.
Since $x \in \mathbb{R} - \mathbb{Q}$, then x is irrational, so $|f(x) - f(c)| = |0 - 1| = 1 > \frac{1}{2} = \epsilon$.
Therefore, f is discontinuous at c.
Case 2: Suppose $c \notin \mathbb{Q}$.
Then c is irrational.
Let $\epsilon = \frac{1}{2}$.
Let $\delta > 0$ be given.
Since $\mathbb{Q}$ is dense in $\mathbb{R}$ and $c < c + \delta$, then there exists $x \in \mathbb{Q}$ such that $c < x < c + \delta$.
Since $c < x < c + \delta$, then $0 < x - c < \delta$, so $|x - c| < \delta$.
Since $x \in \mathbb{Q}$, then x is rational, so $|f(x) - f(c)| = |1 - 0| = 1 > \frac{1}{2} = \epsilon$.

Therefore, f is discontinuous at c.
Hence, in all cases, f is discontinuous at c.
Since c is arbitrary, then f is discontinuous at c for every real number c.
Therefore, f is discontinuous everywhere in its domain. ☐

Uniform continuity

Example 11. Let $a > 0$.
Let $f : (0, a) \to \mathbb{R}$ be the function given by $f(x) = x^2$.
Then f is uniformly continuous on the interval $(0, a)$.

Proof. To prove f is uniformly continuous on $(0, a)$, let $\epsilon > 0$ be given.
We must prove there exists $\delta > 0$ such that for all $x, y \in (0, a)$, if $|x - y| < \delta$, then $|f(x) - f(y)| < \epsilon$.
Let $\delta = \frac{\epsilon}{2a}$.
Since $\epsilon > 0$ and $a > 0$, then $\delta > 0$.
Let $x, y \in (0, a)$ such that $|x - y| < \delta$.
Then $0 < x < a$ and $0 < y < a$, so $0 < x + y < 2a$.
Hence, $|x + y| = x + y < 2a$, so $|x + y| < 2a$.
Thus,

$$\begin{aligned} |f(x) - f(y)| &= |x^2 - y^2| \\ &= |(x - y)(x + y)| \\ &= |x - y||x + y| \\ &< 2a\delta \\ &= \epsilon. \end{aligned}$$

Therefore, $|f(x) - f(y)| < \epsilon$, so f is uniformly continuous on the interval $(0, a)$. ☐

Example 12. Let $f : \mathbb{R} \to \mathbb{R}$ be the function given by $f(x) = x^2$.
Then f is not uniformly continuous on $\mathbb{R}$.

Proof. To prove f is not uniformly continuous on $\mathbb{R}$, we prove $(\exists \epsilon > 0)(\forall \delta > 0)(\exists x, y \in \mathbb{R})(|x - y| < \delta \wedge |f(x) - f(y)| \geq \epsilon)$.
Let $\epsilon = 1$.
Let $\delta > 0$ be given.
Let $x = \frac{2}{\delta} - \frac{\delta}{4}$.
Let $y = x + \frac{\delta}{2}$.
Since $|x - y| = |y - x| = |\frac{\delta}{2}| = \frac{\delta}{2} < \delta$, then $|x - y| < \delta$.
Since $2 > 1$ and $\delta > 0$, then $\frac{2}{\delta} > \frac{1}{\delta}$.
Hence, $x = \frac{2}{\delta} - \frac{\delta}{4} > \frac{1}{\delta} - \frac{\delta}{4}$, so $x > \frac{1}{\delta} - \frac{\delta}{4}$.
Thus, $x + \frac{\delta}{4} > \frac{1}{\delta}$, so $2x + \frac{\delta}{2} > \frac{2}{\delta}$.
Therefore, $x + (x + \frac{\delta}{2}) > \frac{2}{\delta}$, so $|x + y| = x + y > \frac{2}{\delta} > 0$.
Observe that

$$
\begin{aligned}
|f(x)-f(y)| &= |x^2-y^2| \\
&= |(x-y)(x+y)| \\
&= |x-y||x+y| \\
&> \frac{\delta}{2}\cdot\frac{2}{\delta} \\
&= 1.
\end{aligned}
$$

Therefore, $|f(x)-f(y)| > 1 = \epsilon$, so f is not uniformly continuous on $\mathbb{R}$. □

Example 13. Let $f : (1,\infty) \to \mathbb{R}$ be a function defined by $f(x) = \frac{1}{x}$.
Then f is uniformly continuous on the interval $(1,\infty)$.

Proof. To prove f is uniformly continuous on $(1,\infty)$, let $\epsilon > 0$ be given.
Let $\delta = \epsilon$.
Then $\delta > 0$.
Let $x, y \in (1,\infty)$ such that $|x-y| < \delta$.
Then $x > 1$ and $y > 1$, so $xy > 1 > 0$.
Hence, $1 > \frac{1}{xy} > 0$, so $0 < \frac{1}{xy} < 1$.
Observe that

$$
\begin{aligned}
|f(x)-f(y)| &= |\frac{1}{x}-\frac{1}{y}| \\
&= |\frac{y-x}{xy}| \\
&= |\frac{x-y}{xy}| \\
&= \frac{1}{xy}|x-y| \\
&< \delta \\
&= \epsilon.
\end{aligned}
$$

Therefore, $|f(x)-f(y)| < \epsilon$, as desired. □

Example 14. Let $f : (0,1) \to \mathbb{R}$ be the function given by $f(x) = \frac{1}{x}$.
Then f is not uniformly continuous on the interval $(0,1)$.

Proof. To prove f is not uniformly continuous on $(0,\infty)$, we prove $(\exists\epsilon > 0)(\forall\delta > 0)(\exists x, y \in (0,1))(|x-y| < \delta \wedge |f(x)-f(y)| \geq \epsilon)$.
Let $\epsilon = 1$.
Let $\delta > 0$ be given.
Let $\alpha = \min\{1,\delta\}$.
Then $\alpha \leq 1$ and $\alpha \leq \delta$ and $\alpha > 0$.
Let $x = \frac{\alpha}{3}$ and $y = \frac{\alpha}{2}$.
Since $0 < \alpha \leq 1 < 3$, then $0 < \alpha < 3$, so $0 < \frac{\alpha}{3} < 1$.
Hence, $\frac{\alpha}{3} \in (0,1)$, so $x \in (0,1)$.

Since $0 < \alpha \leq 1 < 2$, then $0 < \alpha < 2$, so $0 < \frac{\alpha}{2} < 1$.

Hence, $\frac{\alpha}{2} \in (0, 1)$, so $y \in (0, 1)$.

Since $|x - y| = |y - x| = |\frac{\alpha}{2} - \frac{\alpha}{3}| = \frac{\alpha}{6} < \alpha \leq \delta$, then $|x - y| < \delta$.

Since $\alpha \leq 1$ and $\alpha > 0$, then $1 \leq \frac{1}{\alpha}$.

Since $|f(x) - f(y)| = |f(\frac{\alpha}{3}) - f(\frac{\alpha}{2})| = |\frac{3}{\alpha} - \frac{2}{\alpha}| = \frac{1}{\alpha} \geq 1 = \epsilon$, then $|f(x) - f(y)| \geq \epsilon$, as desired. $\square$