Continuous functions Exercises

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Continuity

Exercise 1. The function given by $f(x) = x^2$ is continuous at $x = 2$.

Proof. Let $\epsilon > 0$ be given.
Let $\delta = \min\{1, \frac{\epsilon}{5}\}$.
Then $\delta \leq 1$ and $\delta \leq \frac{\epsilon}{5}$ and $\delta > 0$.
Let $x \in \mathbb{R}$ such that $|x - 2| < \delta$.
Then $|x + 2| = |(x - 2) + 4| \leq |x - 2| + 4 < \delta + 4 \leq 5$, so $|x + 2| < 5$.
Therefore, $|x^2 - 4| = |(x - 2)(x + 2)| = |x - 2||x + 2| < 5\delta \leq 5 \cdot \frac{\epsilon}{5} = \epsilon$, so $|x^2 - 4| < \epsilon$, as desired. □

Exercise 2. Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by $f(x) = 3x^2 - 2x + 1$.
Then f is continuous at 2.

Solution. To prove f is continuous at 2, let $\epsilon > 0$ be given.
Let $\delta = \min\{1, \frac{\epsilon}{13}\}$.
Then $\delta \leq 1$ and $\delta \leq \frac{\epsilon}{13}$ and $\delta > 0$.
Let $x \in \mathbb{R}$ such that $|x - 2| < \delta$.
Then $0 \leq |x - 2| < \delta$.
Since $|3x + 4| = |3(x - 2) + 10| \leq 3|x - 2| + 10 < 3\delta + 10 \leq 3 + 10 = 13$, then $|3x + 4| < 13$.
Thus, $|f(x) - f(2)| = |(3x^2 - 2x + 1) - 9| = |3x^2 - 2x - 8| = |(x - 2)(3x + 4)| = |x - 2||3x + 4| < 13\delta \leq \frac{\epsilon}{13} \cdot 13 = \epsilon$.
Therefore, $|f(x) - f(2)| < \epsilon$, as desired. □

Exercise 3. Let $f : [-4, 0] \to \mathbb{R}$ be a function defined by

$$f(x) = \begin{cases} \frac{2x^2 - 18}{x + 3} & \text{if } x \neq 3 \\ -12 & \text{if } x = -3 \end{cases}$$

Then f is continuous at -3.

Solution. To prove f is continuous at $-3 \in [-4, 0]$, let $\epsilon > 0$ be given.

Let $\delta = \frac{\epsilon}{2}$.

Then $\delta > 0$.

Let $x \in [-4, 0]$ such that $|x - (-3)| < \delta$.

Then $0 \leq |x + 3| < \delta$.

Either $x = -3$ or $x \neq -3$.

We consider these cases separately.

Case 1: Suppose $x = -3$.

Then $|f(x) - f(-3)| = |f(-3) - f(-3)| = 0 < \epsilon$.

Hence, the conditional if $|x - (-3)| < \delta$, then $|f(x) - f(-3)| < \epsilon$ is trivially true.

Case 2: Suppose $x \neq -3$.

Then $|f(x) - f(-3)| = |\frac{2x^2-18}{x+3} + 12| = |\frac{2(x-3)(x+3)}{x+3} + 12| = |2(x-3) + 12| = |2x + 6| = 2|x + 3| < 2\delta = \epsilon$, so $|f(x) - f(-3)| < \epsilon$.

Therefore, f is continuous at -3, as desired. □

Exercise 4. Let $f(x) = \frac{x^2+x-6}{x-2}$ be defined for all real numbers $x \neq 2$.

Define f so that f is continuous at 2.

Solution. Since $domf = \mathbb{R} - \{2\}$, then 2 is not in the domain of f, so f is discontinuous at 2.

Define $f(2) = 5$.

Then $domf = \mathbb{R}$.

For $x \neq 2$, observe that

$$\begin{aligned}
\lim_{x\to 2} f(x) &= \lim_{x\to 2} \frac{x^2+x-6}{x-2} \\
&= \lim_{x\to 2} \frac{(x-2)(x+3)}{x-2} \\
&= \lim_{x\to 2}(x+3) \\
&= 2+3 \\
&= 5 \\
&= f(2).
\end{aligned}$$

Since $2 \in \mathbb{R}$ and 2 is an accumulation point of $\mathbb{R}$ and $\lim_{x\to 2} f(x) = f(2)$, then by the characterization of continuity, f is continuous at 2. □

Exercise 5. Let $f : (0, \infty) \to \mathbb{R}$ be the function defined by $f(x) = \frac{1}{x}$.

Let $0 < c < 1$.

If $\delta > 0$ satisfies the ϵ, δ definition of continuity at c for $\epsilon = 1$, then $\delta < \frac{c^2}{1+c}$.

Solution. Suppose $\delta > 0$ and δ satisfies the ϵ, δ definition of continuity at c for $\epsilon = 1$.

Since f is continuous at c for any $c > 0$ and $c > 0$, then f is continuous at c.

Thus, $\delta = \min\{\frac{c}{2}, \frac{c^2\epsilon}{2}\} = \min\{\frac{c}{2}, \frac{c^2}{2}\}$.

Since $c < 1$ and $c > 0$, then $c^2 < c$, so $\frac{c^2}{2} < \frac{c}{2}$.
Hence, $\delta = \frac{c^2}{2}$.
Since $c < 1$, then $1 + c < 2$, so $\frac{1}{2} < \frac{1}{1+c}$.
Since $c^2 > 0$, then $\frac{c^2}{2} < \frac{c^2}{1+c}$, so $\delta < \frac{c^2}{1+c}$, as desired. ☐

Exercise 6. Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by $f(x) = 2x^2 + 3x + 1$.
Then f is continuous.

Solution. Let $c \in \mathbb{R}$ be given.
To prove f is continuous, we must prove f is continuous at c.
Let $\epsilon > 0$ be given.
Let $\delta = \min\{1, \frac{\epsilon}{5+4|c|}\}$.
Then $\delta \leq 1$ and $\delta \leq \frac{\epsilon}{5+4|c|}$ and $\delta > 0$.
Let $x \in \mathbb{R}$ such that $|x - c| < \delta$.
Then $0 \leq |x - c| < \delta$.
Since $|x| = |(x - c) + c| \leq |x - c| + |c| < \delta + |c| \leq 1 + |c|$, then $|x| < 1 + |c|$.
Hence, $|2x + 2c + 3| \leq 2|x| + 2|c| + 3 < 2(1 + |c|) + 2|c| + 3 = 5 + 4|c|$, so $0 \leq |2x + 2c + 3| < 5 + 4|c|$.
Thus,

$$\begin{aligned}
|f(x) - f(c)| &= |(2x^2 + 3x + 1) - (2c^2 + 3c + 1)| \\
&= |2(x^2 - c^2) + 3(x - c)| \\
&= |2(x - c)(x + c) + 3(x - c)| \\
&= |(x - c)[2(x + c) + 3]| \\
&= |(x - c)(2x + 2c + 3)| \\
&= |x - c||2x + 2c + 3| \\
&< \delta \cdot (5 + 4|c|) \\
&\leq \frac{\epsilon}{5 + 4|c|} \cdot (5 + 4|c|) \\
&= \epsilon.
\end{aligned}$$

Therefore, $|f(x) - f(c)| < \epsilon$, as desired. ☐

Exercise 7. Let $f : \mathbb{R} \to \mathbb{R}$ be a function defined by

$$f(x) = \begin{cases} x & \text{if } x \text{ is rational} \\ x^2 & \text{if } x \text{ is irrational} \end{cases}$$

Then f is continuous at 1 and f is discontinuous at 2.

Proof. To prove f is continuous at 1, let $\epsilon > 0$ be given.
Let $\delta = \min\{1, \frac{\epsilon}{3}\}$.
Then $\delta \leq 1$ and $\delta \leq \frac{\epsilon}{3}$ and $\delta > 0$.

Let $x \in \mathbb{R}$ such that $|x-1| < \delta$.
Since $x \in \mathbb{R}$, then either x is rational or x is irrational.
We consider each case separately.
Case 1: Suppose x is rational.
Then $|f(x) - f(1)| = |x - 1| < \delta \leq \frac{\epsilon}{3} < \epsilon$.
Case 2: Suppose x is irrational.
Since $|x+1| = |x-1+2| \leq |x-1| + 2 < \delta + 2 \leq 3$, then $|x+1| < 3$.
Thus,

$$\begin{aligned} |f(x) - f(1)| &= |x^2 - 1| \\ &= |x-1||x+1| \\ &< 3\delta \\ &\leq 3 \cdot \frac{\epsilon}{3} \\ &= \epsilon. \end{aligned}$$

Therefore, in either case, $|f(x) - f(1)| < \epsilon$, so f is continuous at 1. □

Proof. To prove f is discontinuous at 2, we prove $(\exists \epsilon > 0)(\forall \delta > 0)(\exists x \in \mathbb{R})(|x-2| < \delta \wedge |f(x) - f(2)| \geq \epsilon)$.
Let $\epsilon = 2$.
Let $\delta > 0$ be given.
Since $2 < 2 + \delta$ and $\mathbb{R} - \mathbb{Q}$ is dense in $\mathbb{R}$, then there exists $r \in \mathbb{R} - \mathbb{Q}$ such that $2 < r < 2 + \delta$.
Since $r \in \mathbb{R} - \mathbb{Q}$, then $r \in \mathbb{R}$ and $f(r) = r^2$.
Since $2 < r < 2 + \delta$, then $0 < r - 2 < \delta$, so $|r - 2| = r - 2 < \delta$.
Since $r > 2$, then $r^2 > 4$, so $r^2 - 2 > 2 > 0$.
Thus, $|f(r) - f(2)| = |r^2 - 2| = r^2 - 2 > 2 = \epsilon$.
Therefore, there exists $r \in \mathbb{R}$ such that $|r - 2| < \delta$ and $|f(r) - f(2)| > \epsilon$, as desired. □

Exercise 8. Let $f : \mathbb{R} \to \mathbb{R}$ be a function defined by

$$f(x) = \begin{cases} 8x & \text{if } x \text{ is rational} \\ 2x^2 + 8 & \text{if } x \text{ is irrational} \end{cases}$$

Then f is continuous at 2 and f is discontinuous at 1.

Proof. To prove f is continuous at 2, let $\epsilon > 0$ be given.
Let $\delta = \min\{1, \frac{\epsilon}{10}\}$.
Then $\delta \leq 1$ and $\delta \leq \frac{\epsilon}{10}$ and $\delta > 0$.
Let $x \in \mathbb{R}$ such that $|x - 2| < \delta$.
Since $x \in \mathbb{R}$, then either x is rational or x is irrational.
We consider each case separately.
Case 1: Suppose x is rational.
Then $|f(x) - f(2)| = |8x - 16| = 8|x - 2| < 8\delta \leq 8 \cdot \frac{\epsilon}{10} < \epsilon$.

Case 2: Suppose x is irrational.

Since $|x+2| = |(x-2)+4| \leq |x-2|+4 < \delta+4 \leq 5$, then $0 \leq |x+2| < 5$.

Thus, $|f(x)-f(2)| = |(2x^2+8)-16| = |2x^2-8| = 2|x^2-4| = 2|x-2||x+2| < 10\delta \leq \epsilon$.

Therefore, in either case, $|f(x)-f(2)| < \epsilon$, so f is continuous at 2. ☐

Proof. To prove f is discontinuous at 1, we prove $(\exists \epsilon > 0)(\forall \delta > 0)(\exists x \in \mathbb{R})(|x-1| < \delta \wedge |f(x)-f(1)| \geq \epsilon)$.

Let $\epsilon = 2$.

Let $\delta > 0$ be given.

Since $1 < 1+\delta$ and $\mathbb{R}-\mathbb{Q}$ is dense in $\mathbb{R}$, then there exists $r \in \mathbb{R}-\mathbb{Q}$ such that $1 < r < 1+\delta$.

Since $r \in \mathbb{R}-\mathbb{Q}$, then $r \in \mathbb{R}$ and $f(r) = 2r^2+8$.

Since $1 < r < 1+\delta$, then $0 < r-1 < \delta$, so $|r-1| = r-1 < \delta$.

Since $r > 1$, then $r^2 > 1$, so $2r^2 > 2$.

Thus, $|f(r)-f(1)| = |(2r^2+8)-8| = |2r^2| = 2r^2 > 2 = \epsilon$.

Therefore, there exists $r \in \mathbb{R}$ such that $|r-1| < \delta$ and $|f(r)-f(1)| > \epsilon$, as desired. ☐

Exercise 9. Let $f : \mathbb{R} \to \mathbb{R}$ be a function defined by

$$f(x) = \begin{cases} x & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$$

Then f is continuous at 0 and f is discontinuous for all $x \neq 0$.

(Therefore, f is continuous only at $x = 0$).

Proof. To prove f is continuous at 0, let $\epsilon > 0$ be given.

Let $\delta = \epsilon$.

Then $\delta > 0$.

Let $x \in \mathbb{R}$ such that $|x| < \delta$.

Since $x \in \mathbb{R}$, then either x is rational or x is irrational.

We consider each case separately.

Case 1: Suppose x is rational.

Then $|f(x)-f(0)| = |x-0| = |x| < \delta = \epsilon$.

Case 2: Suppose x is irrational.

Then $|f(x)-f(0)| = |0-0| = 0 < \epsilon$.

Therefore, in either case, $|f(x)-f(0)| < \epsilon$, so f is continuous at 0. ☐

Proof. Let $c \in \mathbb{R}$ such that $c \neq 0$.

We must prove f is discontinuous at c.

Let $\epsilon = \frac{|c|}{2}$.

Since $c \neq 0$, then $|c| > 0$, so $\epsilon = \frac{|c|}{2} > 0$.

Let $\delta > 0$ be given.

Since $c \in \mathbb{R}$, then either $c \in \mathbb{Q}$ or $c \notin \mathbb{Q}$.

We consider these cases separately.

Case 1: Suppose $c \in \mathbb{Q}$.

Since $c < c + \delta$ and $\mathbb{R} - \mathbb{Q}$ is dense in $\mathbb{R}$, then there exists $r \in \mathbb{R} - \mathbb{Q}$ such that $c < r < c + \delta$.

Since $c < r < c + \delta$, then $0 < r - c < \delta$, so $|r - c| = r - c < \delta$.

Observe that $|f(r) - f(c)| = |0 - c| = |c| > \frac{|c|}{2} = \epsilon$.

Therefore, there exists $r \in \mathbb{R}$ such that $|r - c| < \delta$ and $|f(r) - f(c)| > \epsilon$, as desired.

Case 2: Suppose $c \notin \mathbb{Q}$.

Since $c \neq 0$, then either $c > 0$ or $c < 0$.

We consider these cases separately.

Case 2a: Suppose $c > 0$.

Since $c < c + \delta$ and $\mathbb{Q}$ is dense in $\mathbb{R}$, then there exists $r \in \mathbb{Q}$ such that $c < r < c + \delta$, so $0 < r - c < \delta$.

Hence, $|r - c| = r - c < \delta$.

Since $r > c > 0$, then $r > 0$.

Thus, $|f(r) - f(c)| = |r - 0| = |r| = r > c = |c| > \frac{|c|}{2}$.

Case 2b: Suppose $c < 0$.

Since $c - \delta < c$ and $\mathbb{Q}$ is dense in $\mathbb{R}$, then there exists $r \in \mathbb{Q}$ such that $c - \delta < r < c$, so $-\delta < r - c < 0$.

Hence, $|r - c| = -(r - c) < \delta$.

Since $r < c < 0$, then $r < 0$.

Thus, $|f(r) - f(c)| = |r - 0| = |r| = -r > -c = |c| > \frac{|c|}{2}$.

Therefore, there exists $r \in \mathbb{R}$ such that $|r - c| < \delta$ and $|f(r) - f(c)| > \epsilon$, as desired. $\square$

Exercise 10. Let $f : \mathbb{R} \to \mathbb{R}$ be a function defined by

$$f(x) = \begin{cases} 2x & \text{if } x \text{ is rational} \\ x + 3 & \text{if } x \text{ is irrational} \end{cases}$$

Then f is continuous at 3 and discontinuous at $c \neq 3$.

Proof. We prove f is continuous at 3.

Let $\epsilon > 0$ be given.

Let $\delta = \frac{\epsilon}{2}$.

Then $\delta > 0$.

Let $x \in \mathbb{R}$ such that $|x - 3| < \delta$.

Since $x \in \mathbb{R}$, then either x is rational or x is irrational.

We consider these cases separately.

Case 1: Suppose x is rational.

Then $|f(x) - f(3)| = |2x - 6| = |2(x - 3)| = 2|x - 3| < 2\delta = \epsilon$.

Case 2: Suppose x is irrational.

Then $|f(x) - f(3)| = |x + 3 - 6| = |x - 3| < \delta = \frac{\epsilon}{2} < \epsilon$.

Hence, in all cases, $|f(x) - f(3)| < \epsilon$, as desired. $\square$

Lemma 11. *Every real number is an accumulation point of* $\mathbb{R} - \mathbb{Q}$*.*

Proof. Let $p \in \mathbb{R}$ be arbitrary.

To prove p is an accumulation point of $\mathbb{R} - \mathbb{Q}$, let $\delta > 0$ be given.

Then $\delta > p - p$, so $p + \delta > p$.

Since $p < p + \delta$ and $\mathbb{R} - \mathbb{Q}$ is dense in $\mathbb{R}$, then there exists $r \in \mathbb{R} - \mathbb{Q}$ such that $p < r < p + \delta$.

Thus, $0 < r - p < \delta$ and $p < r$.

Since $0 < r - p < \delta$, then $|r - p| = r - p < \delta$, so $r \in N(p; \delta)$.

Since $r > p$, then $r \neq p$, so $r \in N'(p; \delta)$.

Thus, there exists $r \in \mathbb{R} - \mathbb{Q}$ such that $r \in N'(p; \delta)$, so p is an accumulation point of $\mathbb{R} - \mathbb{Q}$, as desired. □

Proof. Let $c \in \mathbb{R}$ such that $c \neq 3$.

We must prove f is discontinuous at c.

Since every real number is an accumulation point of $\mathbb{Q}$ and $c \in \mathbb{R}$, then c is an accumulation point of $\mathbb{Q}$, so there exists a sequence of points in $\mathbb{Q} - \{c\}$ that converges to c.

Let (a_n) be a sequence of points in $\mathbb{Q} - \{c\}$ such that $\lim_{n\to\infty} a_n = c$.

Since every real number is an accumulation point of $\mathbb{R} - \mathbb{Q}$ and $c \in \mathbb{R}$, then c is an accumulation point of $\mathbb{R} - \mathbb{Q}$, so there exists a sequence of points in $\mathbb{R} - \mathbb{Q} - \{c\}$ that converges to c.

Let (b_n) be a sequence of points in $\mathbb{R} - \mathbb{Q} - \{c\}$ such that $\lim_{n\to\infty} b_n = c$.

Suppose f is continuous at c.

Then by the sequential characterization of continuity, $\lim_{n\to\infty} f(a_n) = \lim_{n\to\infty} f(b_n)$.

Thus, $\lim_{n\to\infty}(2a_n) = \lim_{n\to\infty}(b_n + 3)$, so $2\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n + \lim_{n\to\infty} 3$.

Hence, $2c = c + 3$, so $c = 3$.

But, this contradicts the assumption that $c \neq 3$.

Therefore, f is discontinuous at c, as desired. □

Exercise 12. Let $f : \mathbb{R} \to \mathbb{R}$ be the greatest integer function given by $f(x) = \lfloor x \rfloor = \max\{n \in \mathbb{Z} : n \leq x\}$ for all $x \in \mathbb{R}$.

Then f is discontinuous at n for all $n \in \mathbb{Z}$ and f is continuous at c for all $c \in \mathbb{R} - \mathbb{Z}$.

Proof. To prove f is discontinuous at n for all $n \in \mathbb{Z}$, let $n \in \mathbb{Z}$ be given.

We must prove $(\exists \epsilon > 0)(\forall \delta > 0)(\exists x \in \mathbb{R})(|x - n| < \delta \wedge |f(x) - f(n)| \geq \epsilon)$.

Let $\epsilon = \frac{1}{2}$.

Let $\delta > 0$ be given.

Let $M = \max\{n - \delta, n - 1\}$.

Then $n - \delta \leq M$ and $n - 1 \leq M$, and either $M = n - \delta$ or $M = n - 1$.

Since $n - 1 < n$ and $n - \delta < n$, then $M < n$.

Since $\mathbb{R}$ is dense, then there exists $x \in \mathbb{R}$ such that $M < x < n$.

Since $n - \delta \leq M < x < n$, then $n - \delta < x$ and $x < n$, so $n - x < \delta$ and $n - x > 0$.

Thus, $|x - n| = |n - x| = n - x < \delta$.

Since $n - 1 \leq M < x < n$, then $n - 1 < x < n$, so $f(x) = n - 1$.

Hence, $|f(x) - f(n)| = |(n - 1) - n| = 1 > \frac{1}{2} = \epsilon$. □

Proof. To prove f is continuous at c for all $c \in \mathbb{R} - \mathbb{Z}$, let $c \in \mathbb{R} - \mathbb{Z}$ be given.

Then $c \in \mathbb{R}$ and $c \notin \mathbb{Z}$, so there is a unique integer n such that $n < c < n+1$.

Let $\epsilon > 0$ be given.

Let $M = \min\{n+1-c, c-n\}$.

Then $M \leq n+1-c$ and $M \leq c-n$, and either $M = n+1-c$ or $M = c-n$.

Let $\delta = \frac{M}{2}$.

Since $n < c$, then $c - n > 0$.

Since $c < n+1$, then $n+1-c > 0$.

Thus, $M > 0$, so $\frac{M}{2} > 0$.

Hence, $\delta > 0$.

Let $x \in \mathbb{R}$ such that $|x - c| < \delta$.

Then $c - \delta < x < c + \delta$.

Since $\delta = \frac{M}{2} < M \leq n+1-c$, then $\delta < n+1-c$, so $c + \delta < n+1$.

Since $\delta = \frac{M}{2} < M \leq c-n$, then $\delta < c-n$, so $n < c - \delta$.

Hence, $n < c - \delta < x < c + \delta < n+1$, so $n < x < n+1$.

Thus, $|f(x) - f(c)| = |n - n| = 0 < \epsilon$.

Therefore, f is continuous at c, as desired. □

Exercise 13. continuity of a restriction of a function does not necessarily imply continuity of the function

Let $f : \mathbb{R} \to \mathbb{R}$ be a function defined by

$$f(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$

Let g be the restriction of f to $[0, \infty)$.

Then $g(x) = 1$ for all $x \in [0, \infty)$.

Since the constant function given by $h(x) = 1$ is continuous and g is a restriction of h to $[0, \infty)$, then g is continuous, so g is continuous on $[0, \infty)$.

Since $0 \in [0, \infty)$, then g is continuous at 0.

Since 0 is an accumulation point of $\mathbb{R}$, but $\lim_{x \to 0} f(x)$ does not exist, then f is not continuous at 0, so f is not continuous.

Therefore, g is continuous at 0, but f is not continuous at 0.

Thus, if g is a restriction of f and g is continuous, then f is not necessarily continuous.

Exercise 14. Let $S \subset \mathbb{R}$ and $c \in \mathbb{R}$ and $\alpha > 0$.

Let $I = (c - \alpha, c + \alpha) \subset S$.

Let $f : S \to \mathbb{R}$ be a function.

If the restriction of f to I, denoted f_I, is continuous at c, then f is continuous at c.

Proof. Suppose f_I is continuous at c.

To prove f is continuous at c, let $\epsilon > 0$ be given.

Since f_I is continuous at c, then there exists $\beta > 0$ such that for all $x \in I$, if $|x - c| < \beta$, then $|f_I(x) - f_I(c)| < \epsilon$.

Let $m = \min\{\alpha, \beta\}$.
Then $m \leq \alpha$ and $m \leq \beta$.
Since $\alpha > 0$ and $\beta > 0$, then $m > 0$, so $\frac{m}{2} > 0$.
Let $\delta = \frac{m}{2}$.
Then $\delta > 0$.
Let $x \in S$ such that $|x - c| < \delta$.
Since $|x - c| < \delta = \frac{m}{2} < m \leq \alpha$, then $|x - c| < \alpha$, so $x \in N(c; \alpha) = (c - \alpha, c + \alpha) = I$.
Since $|x - c| < \delta = \frac{m}{2} < m \leq \beta$, then $|x - c| < \beta$.
Since $x \in I$ and $|x - c| < \beta$, then $|f_I(x) - f_I(c)| < \epsilon$.
Therefore, $|f(x) - f(c)| = |f_I(x) - f_I(c)| < \epsilon$, so f is continuous at c, as desired. ☐

Exercise 15. Let $K > 0$.
Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that $|f(x) - f(y)| \leq K|x - y|$ for all $x, y \in \mathbb{R}$.
Then f is continuous on $\mathbb{R}$.

Proof. To prove f is continuous on $\mathbb{R}$, let $c \in \mathbb{R}$ be arbitrary.
To prove f is continuous at c, let $\epsilon > 0$ be given.
Let $\delta = \frac{\epsilon}{K}$.
Since $\epsilon > 0$ and $K > 0$, then $\delta > 0$.
Let $x \in \mathbb{R}$ such that $|x - c| < \delta$.
Since $x \in \mathbb{R}$ and $c \in \mathbb{R}$, then $|f(x) - f(c)| \leq K|x - c| < K\delta = \epsilon$.
Therefore, $|f(x) - f(c)| < \epsilon$, so f is continuous at c, as desired. ☐

Exercise 16. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that $f(x) = x^2$ for all $x \in \mathbb{Q}$.
Compute $f(\sqrt{2})$.

Solution. Since f is continuous on $\mathbb{R}$ and $\sqrt{2} \in \mathbb{R}$, then f is continuous at $\sqrt{2}$.
Hence, by the sequential characterization of continuity, for every sequence (x_n) in $\mathbb{R}$ that converges to $\sqrt{2}$, the sequence $(f(x_n))$ converges to $f(\sqrt{2})$.
Let (x_n) be a sequence of rational numbers defined recursively by $x_1 = 2$ and $x_{n+1} = \frac{x_n}{2} + \frac{1}{x_n}$ for all $n \in \mathbb{N}$.
Then we know $\lim_{n\to\infty} x_n = \sqrt{2}$.
Since (x_n) is a sequence of rational numbers, then $x_n \in \mathbb{Q}$ for all $n \in \mathbb{N}$.
Let $n \in \mathbb{N}$.
Then $x_n \in \mathbb{Q}$.
Since $\mathbb{Q} \subset \mathbb{R}$, then $x_n \in \mathbb{R}$.
Hence, $x_n \in \mathbb{R}$ for all $n \in \mathbb{N}$, so (x_n) is a sequence in $\mathbb{R}$.
Since (x_n) is a sequence in $\mathbb{R}$ and $\lim_{n\to\infty} x_n = \sqrt{2}$, then $\lim_{n\to\infty} f(x_n) = f(\sqrt{2})$.

Hence,

$$\begin{aligned} f(\sqrt{2}) &= \lim_{n\to\infty} f(x_n) \\ &= \lim_{n\to\infty} (x_n x_n) \\ &= (\lim_{n\to\infty} x_n)(\lim_{n\to\infty} x_n) \\ &= \sqrt{2}\cdot\sqrt{2} \\ &= 2. \end{aligned}$$

Therefore, $f(\sqrt{2}) = 2$. ☐

Exercise 17. If $f : \mathbb{Z} \to \mathbb{R}$ is a function, then f is continuous.

Proof. Suppose $f : \mathbb{Z} \to \mathbb{R}$ is a function.

To prove f is continuous, let $n \in \mathbb{Z}$.

Since there are no accumulation points of $\mathbb{Z}$, then n is not an accumulation point of $\mathbb{Z}$.

Since $n \in \mathbb{Z}$, then by the characterization of continuity, f is continuous at n.

Therefore, f is continuous on $\mathbb{Z}$, so f is continuous. ☐

Exercise 18. Let $E \subset \mathbb{R}$.

Let $f : E \to \mathbb{R}$ be a function.

If $c \in E$ and c is not an accumulation point of E, then for every sequence (x_n) of points in E such that (x_n) converges to c, the sequence $(f(x_n))$ converges to $f(c)$.

Proof. Suppose $c \in E$ and c is not an accumulation point of E.

Then f is continuous at c.

Therefore, by the sequential characterization of continuity, for every sequence (x_n) of points in E such that (x_n) converges to c, the sequence $(f(x_n))$ converges to $f(c)$. ☐

Exercise 19. Using the sequential characterization of continuity prove the function $f : (0, \infty) \to \mathbb{R}$ defined by $f(x) = \frac{1}{x}$ is continuous.

Proof. To prove f is continuous on its domain, we must prove f is continuous on the interval $(0, \infty)$.

Let $c \in (0, \infty)$ be arbitrary.

Then $c > 0$, so $c \neq 0$.

To prove f is continuous at c using the sequential characterization of continuity, let (x_n) be an arbitrary sequence of real numbers in $(0, \infty)$ such that $\lim_{n\to\infty} x_n = c$.

We must prove $\lim_{n\to\infty} f(x_n) = f(c)$.

Since $\lim_{n\to\infty} x_n = c \neq 0$, then

$$\begin{aligned} f(c) &= \frac{1}{c} \\ &= \frac{\lim_{n\to\infty} 1}{\lim_{n\to\infty} x_n} \\ &= \lim_{n\to\infty} \frac{1}{x_n} \\ &= \lim_{n\to\infty} f(x_n). \end{aligned}$$

Therefore, $\lim_{n\to\infty} f(x_n) = f(c)$, as desired. □

Exercise 20. Show that the sequence (a_n) defined by $a_n = \sqrt[n]{e^{n+1}}$ for all $n \in \mathbb{N}$ is convergent.

Solution. We see intuitively that the sequence converges to e. □

Proof. To prove (a_n) is convergent, we prove $\lim_{n\to\infty} \sqrt[n]{e^{n+1}} = e$.

We first prove $\lim_{n\to\infty} e^{\frac{1}{n}} = 1$.

Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by $f(x) = e^x$.

We assume f is continuous on $\mathbb{R}$.

Since f is continuous at 0, then by the sequential characterization of continuity, if (x_n) is a sequence of points in $\mathbb{R}$ that converges to 0, then the sequence $(f(x_n))$ converges to $f(0)$.

Since $(\frac{1}{n})$ is a sequence of real numbers that converges to 0, then the sequence $(f(\frac{1}{n}))$ converges to $f(0)$.

Thus, $1 = e^0 = f(0) = \lim_{n\to\infty} f(\frac{1}{n}) = \lim_{n\to\infty} e^{\frac{1}{n}}$.

Observe that

$$\begin{aligned} \lim_{n\to\infty} \sqrt[n]{e^{n+1}} &= \lim_{n\to\infty} e^{\frac{n+1}{n}} \\ &= \lim_{n\to\infty} e^{1+\frac{1}{n}} \\ &= \lim_{n\to\infty} ee^{\frac{1}{n}} \\ &= e \lim_{n\to\infty} e^{\frac{1}{n}} \\ &= e \cdot 1 \\ &= e. \end{aligned}$$

□

Exercise 21. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function.

Let $S = \{x \in \mathbb{R} : f(x) = 0\}$.

Let (x_n) be a sequence in S such that $\lim_{n\to\infty} x_n = c$.

Then $c \in S$.

Proof. Since (x_n) is a sequence in S, then $x_n \in S$ for each $n \in \mathbb{N}$.

Let $n \in \mathbb{N}$.

Then $x_n \in S$.

Since $S \subset \mathbb{R}$, then $x_n \in \mathbb{R}$ for each $n \in \mathbb{N}$, so (x_n) is a sequence in $\mathbb{R}$.

Since $\lim_{n\to\infty} x_n = c$, then $c \in \mathbb{R}$.

To prove $c \in S$, we must prove $f(c) = 0$.

Since f is continuous on $\mathbb{R}$ and $c \in \mathbb{R}$, then f is continuous at c.

Since (x_n) is a sequence in $\mathbb{R}$ and $\lim_{n\to\infty} x_n = c$, then by the sequential characterization of continuity, we conclude $\lim_{n\to\infty} f(x_n) = f(c)$.

Since $x_n \in S$ for each $n \in \mathbb{N}$, then $f(x_n) = 0$ for each $n \in \mathbb{N}$, so the sequence $(f(x_n))$ is the constant sequence 0.

Thus, $0 = \lim_{n\to\infty} f(x_n) = f(c)$, so $f(c) = 0$, as desired. □

Exercise 22. Let $f : E \to \mathbb{R}$ be a function.

Let $c \in E$.

If f is continuous at c, then there exists $M > 0$ and $\delta > 0$ such that $|f(x)| < M$ for all $x \in N(c; \delta) \cap E$.

Proof. Suppose f is continuous at c.

Let $\epsilon = 1$ be given.

Then there exists $\delta > 0$ such that for all $x \in E$, if $|x - c| < \delta$, then $|f(x) - f(c)| < 1$.

Let $M = 1 + |f(c)|$.

Since $1 > 0$ and $|f(c)| \geq 0$, then $M > 0$.

Let $x \in N(c; \delta) \cap E$.

Then $x \in N(c; \delta)$ and $x \in E$.

Since $x \in N(c; \delta)$, then $|x - c| < \delta$.

Since $x \in E$ and $|x - c| < \delta$, then $|f(x) - f(c)| < 1$.

Thus, $|f(x)| = |f(x) - f(c) + f(c)| \leq |f(x) - f(c)| + |f(c)| < 1 + |f(c)| = M$, so $|f(x)| < M$. □

Lemma 23. *Let $f : \mathbb{R} \to \mathbb{R}$ be a function continuous at $c \in \mathbb{R}$ and $f(c) > 0$.*

Then there exists $\delta > 0$ such that if $x \in N(c; \delta)$, then $f(x) > 0$.

Proof. Since $f(c) > 0$, then $\frac{f(c)}{2} > 0$.

Since f is continuous at c, then there exists $\delta > 0$ such that for all $x \in \mathbb{R}$, if $|x - c| < \delta$, then $|f(x) - f(c)| < \frac{f(c)}{2}$.

Let $x \in N(c; \delta)$.

Then $x \in \mathbb{R}$ and $|x - c| < \delta$, so $|f(x) - f(c)| < \frac{f(c)}{2}$.

Hence, $\frac{-f(c)}{2} < f(x) - f(c) < \frac{f(c)}{2}$, so $\frac{-f(c)}{2} < f(x) - f(c)$.

Thus, $0 < \frac{f(c)}{2} < f(x)$, so $0 < f(x)$.

Therefore, $f(x) > 0$, as desired. □

Exercise 24. Let f and g be real valued functions continuous on $\mathbb{R}$.

Let $S = \{x \in \mathbb{R} : f(x) \geq g(x)\}$.

If (x_n) is a sequence in S such that $\lim_{n\to\infty} x_n = c$, then $c \in S$.

Proof. Let (x_n) be a sequence in S such that $\lim_{n\to\infty} x_n = c$.

Since (x_n) is in S, then $x_n \in S$ for all $n \in \mathbb{N}$, so $f(x_n) \geq g(x_n)$ for all $n \in \mathbb{N}$.

Suppose for the sake of contradiction $c \notin S$.

Then $c \in \mathbb{R}$ and $f(c) < g(c)$, so $f(c) - g(c) < 0$.

Let $h = f - g$.

Then $h : \mathbb{R} \to \mathbb{R}$ is a function defined by $h(x) = (f - g)(x) = f(x) - g(x)$ for all $x \in \mathbb{R}$.

Thus, $h(c) = f(c) - g(c) < 0$, so $h(c) < 0$.

Since f and g are continuous functions and $h = f - g$, then h is continuous, so h is continuous at c.

Since (x_n) is an arbitrary sequence in $\mathbb{R}$ such that $\lim_{n\to\infty} x_n = c$, then by the sequential characterization of continuity, we have $\lim_{n\to\infty} h(x_n) = h(c)$, so the sequence $(h(x_n))$ is convergent.

Since $f(x_n) \geq g(x_n)$ for all $n \in \mathbb{N}$, then $h(x_n) = f(x_n) - g(x_n) \geq 0$ for all $n \in \mathbb{N}$.

Since $0 \leq h(x_n)$ for all $n \in \mathbb{N}$, then 0 is a lower bound of $(h(x_n))$.

Since $(h(x_n))$ is a convergent sequence in $\mathbb{R}$ and 0 is a lower bound of $(h(x_n))$, then $0 \leq \lim_{n\to\infty} h(x_n)$, so $0 \leq h(c)$.

Thus, we have $h(c) \geq 0$ and $h(c) < 0$, a contradiction.

Therefore, $c \in S$, as desired. □

Proof. Let (x_n) be a sequence in S such that $\lim_{n\to\infty} x_n = c$.

Since (x_n) is in S, then $x_n \in S$ for all $n \in \mathbb{N}$, so $f(x_n) \geq g(x_n)$ for all $n \in \mathbb{N}$.

Suppose for the sake of contradiction $c \notin S$.

Then $c \in \mathbb{R}$ and $f(c) < g(c)$, so $g(c) - f(c) > 0$.

Let $h = g - f$.

Then h is a function defined by $h(x) = (g - f)(x) = g(x) - f(x)$ for all $x \in \mathbb{R}$.

Thus, $h(c) = g(c) - f(c) > 0$, so $h(c) > 0$.

Since g and f are continuous functions and $h = g - f$, then h is continuous, so h is continuous at c.

By the previous lemma, since h is continuous at c and $h(c) > 0$, then there exists $\epsilon > 0$ such that if $x \in N(c; \epsilon)$, then $h(x) > 0$.

Since $\lim_{n\to\infty} x_n = c$ and $\epsilon > 0$, then there exists $N \in \mathbb{N}$ such that if $n > N$, then $|x_n - c| < \epsilon$.

Let $n \in \mathbb{N}$ such that $n > N$.

Then $|x_n - c| < \epsilon$, so $x_n \in N(c; \epsilon)$.

Hence, $h(x_n) > 0$, so $g(x_n) - f(x_n) > 0$.

Thus, $g(x_n) > f(x_n)$, so $f(x_n) < g(x_n)$.

Since $n \in \mathbb{N}$, then $f(x_n) \geq g(x_n)$.

Therefore, we have $f(x_n) < g(x_n)$ and $f(x_n) \geq g(x_n)$, a contradiction.

Consequently, $c \in S$, as desired. □

Exercise 25. Let $f : E \to \mathbb{R}$ be a function continuous at $c \in E$.

Then for every $\epsilon > 0$, there exists $\delta > 0$ such that if $x, y \in E \cap N(c; \delta)$, then $|f(x) - f(y)| < \epsilon$.

Proof. Let $\epsilon > 0$ be given.

Then $\frac{\epsilon}{2} > 0$.

Since f is continuous at c, then there exists $\delta > 0$ such that for all $x \in E$, if $|x - c| < \delta$, then $|f(x) - f(c)| < \frac{\epsilon}{2}$.

Since $c \in E$ and $c \in N(c;\delta)$, then $c \in E \cap N(c;\delta)$, so $E \cap N(c;\delta) \neq \emptyset$.

Let $x, y \in E \cap N(c;\delta)$.

Then $x \in E \cap N(c;\delta)$ and $y \in E \cap N(c;\delta)$.

Hence, $x \in E$ and $x \in N(c;\delta)$ and $y \in E$ and $y \in N(c;\delta)$.

Since $x \in N(c;\delta)$ and $y \in N(c;\delta)$, then $|x - c| < \delta$ and $|y - c| < \delta$.

Since $x \in E$ and $|x - c| < \delta$, then $|f(x) - f(c)| < \frac{\epsilon}{2}$.

Since $y \in E$ and $|y - c| < \delta$, then $|f(y) - f(c)| < \frac{\epsilon}{2}$.

Observe that

$$\begin{aligned}
|f(x) - f(y)| &= |f(x) - f(c) + f(c) - f(y)| \\
&\leq |f(x) - f(c)| + |f(c) - f(y)| \\
&= |f(x) - f(c)| + |f(y) - f(c)| \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&= \epsilon.
\end{aligned}$$

Therefore, $|f(x) - f(y)| < \epsilon$, as desired. □

Algebraic properties of continuous functions

Exercise 26. The function $r : (0, \infty) \to \mathbb{R}$ defined by $r(x) = \sin(\frac{1}{x})$ is continuous on $(0, \infty)$.

Proof. Let $f : (0, \infty) \to \mathbb{R}$ be the function defined by $f(x) = \frac{1}{x}$.

Let $g : \mathbb{R} \to \mathbb{R}$ be the function defined by $g(x) = \sin(x)$.

Then $g \circ f$ is the composite function.

Since $dom(g \circ f) = \{x \in dom f : f(x) \in dom g\} = \{x \in (0, \infty) : \frac{1}{x} \in \mathbb{R}\} = \{x \in (0, \infty) : x \neq 0\} = (0, \infty) = dom r$, then $dom(g \circ f) = dom r$.

Let $x \in dom(g \circ f)$.

Then $(g \circ f)(x) = g(f(x)) = g(\frac{1}{x}) = \sin(\frac{1}{x}) = r(x)$, so $(g \circ f)(x) = r(x)$ for all $x \in dom(g \circ f)$.

Since $dom(g \circ f) = dom r$ and $(g \circ f)(x) = r(x)$ for all $x \in dom(g \circ f)$, then $g \circ f = r$.

Since f is continuous and g is continuous, then $g \circ f = r$ is continuous, so r is continuous on $(0, \infty)$. □

Exercise 27. The function $f : [-1, 1] \to \mathbb{R}$ defined by $f(x) = \sqrt{1 - x^2}$ is continuous at 1.

Solution. If $y = f(x) = \sqrt{1 - x^2}$, then $y^2 = 1 - x^2$, so $x^2 + y^2 = 1$.

Thus, we have the unit circle centered at the origin.

The graph of f is the top semicircle and the limit of f as x approaches 1 is 0 and $\lim_{x\to 1} f(x) = 0 = f(1)$. $\square$

Proof. Let $g : \mathbb{R} \to \mathbb{R}$ be the function defined by $g(x) = 1 - x^2$.

Let $h : [0, \infty) \to \mathbb{R}$ be the function defined by $h(x) = \sqrt{x}$.

Then $h \circ g$ is the composite function.

Since $dom(h \circ g) = \{x \in domg : g(x) \in domh\} = \{x \in \mathbb{R} : 1 - x^2 \in [0, \infty)\} = \{x \in \mathbb{R} : 1 - x^2 \geq 0\} = \{x \in \mathbb{R} : 1 \geq x^2\} = \{x \in \mathbb{R} : x^2 \leq 1\} = \{x \in \mathbb{R} : |x|^2 \leq 1\} = \{x \in \mathbb{R} : |x| \leq 1\} = [-1, 1] = domf$, then $dom(h \circ g) = domf$.

Let $x \in dom(h \circ g)$.

Then $(h \circ g)(x) = h(g(x)) = h(1 - x^2) = \sqrt{1 - x^2} = f(x)$, so $(h \circ g)(x) = f(x)$ for all $x \in dom(h \circ g)$.

Since $dom(h \circ g) = domf$ and $(h \circ g)(x) = f(x)$ for all $x \in dom(h \circ g)$, then $h \circ g = f$.

Since g is a polynomial function, then g is continuous.

Since the square root function is continuous, then h is continuous.

Since g is continuous and h is continuous, then $h \circ g = f$ is continuous, so f is continuous on $[-1, 1]$.

Since $1 \in [-1, 1]$, then f is continuous at 1. $\square$

Lemma 28. *Let* $x, y \in \mathbb{R}$.

Then $\max\{x, y\} = \frac{x+y}{2} + \frac{|x-y|}{2}$ *and* $\min\{x, y\} = \frac{x+y}{2} - \frac{|x-y|}{2}$.

Proof. Let $S = \{x, y\}$.

We must prove $\max S = \frac{x+y}{2} + \frac{|x-y|}{2}$ and $\min S = \frac{x+y}{2} - \frac{|x-y|}{2}$.

Since $x, y \in \mathbb{R}$, then either $x \geq y$ or $x < y$.

We consider these cases separately.

Case 1: Suppose $x \geq y$.

Then $x - y \geq 0$ and $\max S = x$ and $\min S = y$.

Observe that

$$\begin{aligned}
\max S &= x \\
&= \frac{2x}{2} \\
&= \frac{x + x}{2} \\
&= \frac{x + y + x - y}{2} \\
&= \frac{x + y}{2} + \frac{x - y}{2} \\
&= \frac{x + y}{2} + \frac{|x - y|}{2}.
\end{aligned}$$

Therefore, $\max S = \frac{x+y}{2} + \frac{|x-y|}{2}$, as desired.

Observe that

$$\begin{aligned}
\min S &= y \\
&= \frac{2y}{2} \\
&= \frac{y+y}{2} \\
&= \frac{x+y+y-x}{2} \\
&= \frac{x+y}{2} + \frac{y-x}{2} \\
&= \frac{x+y}{2} - \frac{x-y}{2} \\
&= \frac{x+y}{2} - \frac{|x-y|}{2}.
\end{aligned}$$

Therefore, $\min S = \frac{x+y}{2} - \frac{|x-y|}{2}$, as desired.

Case 2: Suppose $x < y$.

Then $x - y < 0$ and $\max S = y$ and $\min S = x$.

Observe that

$$\begin{aligned}
\max S &= y \\
&= \frac{2y}{2} \\
&= \frac{y+y}{2} \\
&= \frac{x+y+y-x}{2} \\
&= \frac{x+y}{2} + \frac{y-x}{2} \\
&= \frac{x+y}{2} + \frac{-(x-y)}{2} \\
&= \frac{x+y}{2} + \frac{|x-y|}{2}.
\end{aligned}$$

Therefore, $\max S = \frac{x+y}{2} + \frac{|x-y|}{2}$, as desired.

Observe that

$$
\begin{aligned}
\min S &= x \\
&= \frac{2x}{2} \\
&= \frac{x+x}{2} \\
&= \frac{x+y+x-y}{2} \\
&= \frac{x+y}{2} + \frac{x-y}{2} \\
&= \frac{x+y}{2} - \frac{-(x-y)}{2} \\
&= \frac{x+y}{2} - \frac{|x-y|}{2}.
\end{aligned}
$$

Therefore, $\min S = \frac{x+y}{2} - \frac{|x-y|}{2}$, as desired. □

Exercise 29. Let f and g be real valued functions continuous on $E \subset \mathbb{R}$.
Let $h : E \to \mathbb{R}$ be a function defined by $h(x) = \max\{f(x), g(x)\}$.
Then h is continuous.

Proof. Let $x \in E$.
Then $f(x) \in \mathbb{R}$ and $g(x) \in \mathbb{R}$.
Hence, by the previous lemma, $h(x) = \max\{f(x), g(x)\} = \frac{f(x)+g(x)}{2} + \frac{|f(x)-g(x)|}{2}$.
Thus, $h = \frac{f+g}{2} + \frac{|f-g|}{2}$.
Since f and g are continuous on E, then f and g are continuous, so the sum $f+g$ and difference $f-g$ are continuous.
Since $f+g$ is continuous, then the scalar multiple $\frac{f+g}{2}$ is continuous.
Since $f-g$ is continuous, then $|f-g|$ is continuous, so the scalar multiple $\frac{|f-g|}{2}$ is continuous.
Since $\frac{f+g}{2}$ is continuous and $\frac{|f-g|}{2}$ is continuous, then the sum $\frac{f+g}{2} + \frac{|f-g|}{2}$ is continuous.
Therefore, h is continuous, as desired. □

Exercise 30. Let $a, b, c \in \mathbb{R}$ such that $a < b < c$.
Let $f : [a, b] \to \mathbb{R}$ be a function continuous on $[a, b]$ and $g : [b, c] \to \mathbb{R}$ be a function continuous on $[b, c]$ such that $f(b) = g(b)$.
Let $h : [a, c] \to \mathbb{R}$ be a function defined by

$$
h(x) = \begin{cases} f(x) & \text{if } x \in [a, b] \\ g(x) & \text{if } x \in [b, c] \end{cases}
$$

Then h is continuous on $[a, c]$.

Proof. To prove h is continuous on $[a, c]$, let $\alpha \in [a, c]$ be arbitrary.

We must prove h is continuous at α.

Since $\alpha \in [a, c]$ and $[a, c] = [a, b) \cup \{b\} \cup (b, c]$, then either $\alpha \in [a, b)$ or $\alpha = b$ or $\alpha \in (b, c]$.

We consider these cases separately.

Case 1: Suppose $\alpha \in [a, b)$.

Since $[a, b] \subset [a, c]$ and $f(x) = h(x)$ for all $x \in [a, b]$, then f is a restriction of h to $[a, b]$.

Since f is continuous on $[a, b]$, then f is continuous at x for all $x \in [a, b]$.

Hence, a restriction of h to $[a, b]$ is continuous at x for all $x \in [a, b]$, so h is continuous on $[a, b]$.

This is not correct!!! We must fix this!!!

Since $\alpha \in [a, b)$ and $[a, b) \subset [a, b]$, then $\alpha \in [a, b]$, so h is continuous at α.

Case 2: Suppose $\alpha \in (b, c]$.

Since $[b, c] \subset [a, c]$ and $g(x) = h(x)$ for all $x \in [b, c]$, then g is a restriction of h to $[b, c]$.

Since g is continuous on $[b, c]$, then g is continuous at x for all $x \in [b, c]$.

Hence, a restriction of h to $[b, c]$ is continuous at x for all $x \in [b, c]$, so h is continuous on $[b, c]$.

Since $\alpha \in (b, c]$ and $(b, c] \subset [b, c]$, then $\alpha \in [b, c]$, so h is continuous at α.

Case 3: Suppose $\alpha = b$.

We prove h is continuous at b.

Let $\epsilon > 0$ be given.

We must prove there exists $\delta > 0$ such that for all $x \in [a, c]$, if $|x - b| < \delta$, then $|h(x) - h(b)| < \epsilon$.

Since f is continuous on $[a, b]$ and $b \in [a, b]$, then f is continuous at b, so there exists $\delta_1 > 0$ such that for all $x \in [a, b]$, if $|x - b| < \delta_1$, then $|f(x) - f(b)| < \epsilon$.

Since g is continuous on $[b, c]$ and $b \in [b, c]$, then g is continuous at b, so there exists $\delta_2 > 0$ such that for all $x \in [b, c]$, if $|x - b| < \delta_2$, then $|g(x) - g(b)| < \epsilon$.

Let $\delta = \frac{\min\{\delta_1, \delta_2\}}{2}$.

Then $\delta > 0$ and $2\delta = \min\{\delta_1, \delta_2\}$, so $2\delta \le \delta_1$ and $2\delta \le \delta_2$.

Thus, $\delta \le \frac{\delta_1}{2}$ and $\delta \le \frac{\delta_2}{2}$.

Let $x \in [a, c]$ such that $|x - b| < \delta$.

Since $x \in [a, c]$ and $[a, c] = [a, b] \cup [b, c]$, then either $x \in [a, b]$ or $x \in [b, c]$.

We consider these cases separately.

Case 3.1: Suppose $x \in [a, b]$.

Since $|x - b| < \delta \le \frac{\delta_1}{2} < \delta_1$, then $|x - b| < \delta_1$.

Since $x \in [a, b]$ and $|x - b| < \delta_1$, then $|f(x) - f(b)| < \epsilon$.

Thus, $|h(x) - h(b)| = |f(x) - f(b)| < \epsilon$, so $|h(x) - h(b)| < \epsilon$.

Case 3.2: Suppose $x \in [b, c]$.

Since $|x - b| < \delta \le \frac{\delta_2}{2} < \delta_2$, then $|x - b| < \delta_2$.

Since $x \in [b, c]$ and $|x - b| < \delta_2$, then $|g(x) - g(b)| < \epsilon$.

Thus, $|h(x) - h(b)| = |g(x) - g(b)| < \epsilon$, so $|h(x) - h(b)| < \epsilon$.

Therefore, in all cases, $|h(x) - h(b)| < \epsilon$, so h is continuous at b. $\square$

Exercise 31. Let $E \subset \mathbb{R}$ and $c \in E$.

Let $f : E \to \mathbb{R}$ be a function.

If for every sequence (x_n) in E such that $\lim_{n\to\infty} x_n = c$, the sequence $(f(x_n))$ is convergent, then f is continuous at c.

Proof. We prove by contrapositive.

Suppose f is not continuous at c.

Then there exists $\epsilon_0 > 0$ such that for each $\delta > 0$ there corresponds $x \in E$ such that $|x - c| < \delta$ and $|f(x) - f(c)| \geq \epsilon_0$.

Let $\delta = \frac{1}{n}$ for each $n \in \mathbb{N}$.

Then for each $n \in \mathbb{N}$, there corresponds $x \in E$ such that $|x - c| < \frac{1}{n}$ and $|f(x) - f(c)| \geq \epsilon_0$.

Thus, there exists a function $g : \mathbb{N} \to E$ such that $g(n) \in E$ and $|g(n)-c| < \frac{1}{n}$ and $|f(g(n)) - f(c)| \geq \epsilon_0$ for each $n \in \mathbb{N}$, so there exists a sequence (x_n) such that $x_n \in E$ and $|x_n - c| < \frac{1}{n}$ and $|f(x_n) - f(c)| \geq \epsilon_0$ for each $n \in \mathbb{N}$.

Since $x_n \in E$ for each $n \in \mathbb{N}$, then (x_n) is a sequence of points in E.

We prove $\lim_{n\to\infty} x_n = c$.

Let $\epsilon > 0$ be given.

Then $\epsilon \neq 0$, so $\frac{1}{\epsilon} \in \mathbb{R}$.

Hence, by the Archimedean property of $\mathbb{R}$, there exists $N \in \mathbb{N}$ such that $N > \frac{1}{\epsilon}$.

Let $n \in \mathbb{N}$ such that $n > N$.

Then $n > N > \frac{1}{\epsilon}$, so $n > \frac{1}{\epsilon}$.

Hence, $\epsilon > \frac{1}{n}$, so $\frac{1}{n} < \epsilon$.

Since $n \in \mathbb{N}$ and $|x_n - c| < \frac{1}{n}$ for each $n \in \mathbb{N}$, then $|x_n - c| < \frac{1}{n}$.

Thus, $|x_n - c| < \frac{1}{n} < \epsilon$, so $|x_n - c| < \epsilon$.

Therefore, $\lim_{n\to\infty} x_n = c$, as desired.

We prove the sequence $(f(x_n))$ is divergent.

Thus, we must prove for every real L there exists $\epsilon_0 > 0$ such that for each $N \in \mathbb{N}$ there corresponds $n \in \mathbb{N}$ with $n > N$ and $|f(x_n) - L| \geq \epsilon_0$.

Let $L \in \mathbb{R}$ be arbitrary.

Let $N \in \mathbb{N}$.

Let $n = N + 1$.

Then $n \in \mathbb{N}$ and $n = N + 1 > N$ and

Use triangle inequality to figure out how we can ensure for any $L \in \mathbb{R}$ that if $|f(x_n) - f(c)| \geq \epsilon_0$ for each $n \in \mathbb{N}$, then $|f(x_n) - L| \geq \epsilon_0$ for each $n \in \mathbb{N}$.

□

Exercise 32. Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be functions.

Let $a, b \in \mathbb{R}$.

If $\lim_{x\to a} f = b$ and g is continuous at b, then $\lim_{x\to a} g \circ f = g(b)$.

Proof. Suppose $\lim_{x\to a} f = b$ and g is continuous at b.

Observe that a is an accumulation point of $\mathbb{R}$, the domain of $g \circ f$.

To prove $\lim_{x \to a} g \circ f = g(b)$, let $\epsilon > 0$ be given.

We must prove there exists $\delta > 0$ such that for all $x \in \mathbb{R}$, if $0 < |x - a| < \delta$, then $|(g \circ f)(x) - g(b)| < \epsilon$.

Since g is continuous at b and $\epsilon > 0$, then there exists $\delta_1 > 0$ such that for all $x \in \mathbb{R}$, if $|x - b| < \delta_1$, then $|g(x) - g(b)| < \epsilon$.

Since $\lim_{x \to a} f = b$ and $\delta_1 > 0$, then there exists $\delta_2 > 0$ such that for all $x \in \mathbb{R}$, if $0 < |x - a| < \delta_2$, then $|f(x) - b| < \delta_1$.

Let $\delta = \delta_2 > 0$.

Let $x \in \mathbb{R}$ such that $0 < |x - a| < \delta$.

Then $0 < |x - a| < \delta_2$, so $|f(x) - b| < \delta_1$.

Since $f(x) \in \mathbb{R}$ and $|f(x) - b| < \delta_1$, then $|g(f(x)) - g(b)| < \epsilon$, so $|(g \circ f)(x) - g(b)| < \epsilon$, as desired. ☐

Exercise 33. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function.

Let $k \in \mathbb{R}$.

The set $\{x \in \mathbb{R} : f(x) \neq k\}$ is open.

Proof. Let $S = \{x \in \mathbb{R} : f(x) \neq k\}$.

We must prove S is open.

Either S is empty or not.

We consider these cases separately.

Case 1: Suppose $S = \emptyset$.

Since the empty set is open, then S is open.

Case 2: Suppose $S \neq \emptyset$.

Then there is an element in S.

Let p be an arbitrary element of S.

Then $p \in S$, so $p \in \mathbb{R}$ and $f(p) \neq k$.

Since $f(p) \neq k$, then $f(p) - k \neq 0$, so $|f(p) - k| > 0$.

Since f is continuous on $\mathbb{R}$ and $p \in \mathbb{R}$, then f is continuous at p.

Thus, there exists $\delta > 0$ such that for all $x \in \mathbb{R}$, if $|x - p| < \delta$, then $|f(x) - f(p)| < |f(p) - k|$.

Let $x \in \mathbb{R}$ such that $|x - p| < \delta$.

Then $|f(x) - f(p)| < |f(p) - k|$.

Since $|f(x) - k| \in \mathbb{R}$, then $|f(x) - k| \geq 0$.

Suppose $|f(x) - k| = 0$.

Then $f(x) - k = 0$, so $f(x) = k$.

Thus, $|f(p) - k| = |k - f(p)| = |f(x) - f(p)| < |f(p) - k|$.

Hence, $|f(p) - k| < |f(p) - k|$, a contradiction.

Therefore, $|f(x) - k| \neq 0$.

Since $|f(x) - k| \geq 0$, then this implies $|f(x) - k| > 0$, so $f(x) - k \neq 0$.

Thus, $f(x) \neq k$.

Since $x \in \mathbb{R}$ and $f(x) \neq k$, then $x \in S$.

Therefore, there exists $\delta > 0$ such that for all $x \in \mathbb{R}$, if $|x - p| < \delta$, then $x \in S$.

Hence, there exists $\delta > 0$ such that for all $x \in \mathbb{R}$, if $x \in N(p;\delta)$, then $x \in S$, so there exists $\delta > 0$ such that $N(p;\delta) \subset S$.

Thus, p is an interior point of S.

Since p is arbitrary, then every point in S is an interior point of S, so S is open. $\square$

Exercise 34. Let $a, b \in \mathbb{R}$ with $a < b$.

Let $f : [a,b] \to \mathbb{R}$ and $g : [a,b] \to \mathbb{R}$ be continuous functions.

Let $S = \{x \in [a,b] : f(x) = g(x)\}$.

Then S is closed.

Proof. Let c be an arbitrary accumulation point of S.

Since $S \subset [a,b]$, then c is an accumulation point of $[a,b]$.

Since the interval $[a,b]$ is closed, then $c \in [a,b]$.

Since f and g are continuous on $[a,b]$, then f and g are continuous at c.

Suppose $f(c) \neq g(c)$.

Then $f(c) - g(c) \neq 0$, so $|f(c) - g(c)| > 0$.

Hence, $\frac{|f(c)-g(c)|}{2} > 0$.

Since f is continuous at c, then there exists $\delta_1 > 0$ such that for all $x \in [a,b]$, if $|x - c| < \delta_1$, then $|f(x) - f(c)| < \frac{|f(c)-g(c)|}{2}$.

Since g is continuous at c, then there exists $\delta_2 > 0$ such that for all $x \in [a,b]$, if $|x - c| < \delta_2$, then $|g(x) - g(c)| < \frac{|f(c)-g(c)|}{2}$.

Let $\delta = \min\{\delta_1, \delta_2\}$.

Then $\delta \leq \delta_1$ and $\delta \leq \delta_2$ and $\delta > 0$.

Since c is an accumulation point of S and $\delta > 0$, then there exists $x \in S$ such that $x \in N'(c;\delta)$.

Since $x \in S$, then $x \in [a,b]$ and $f(x) = g(x)$.

Since $x \in N'(c;\delta)$ and $N'(c;\delta) \subset N(c;\delta)$, then $x \in N(c;\delta)$, so $|x - c| < \delta$.

Since $|x - c| < \delta$ and $\delta \leq \delta_1$, then $|x - c| < \delta_1$.

Since $x \in [a,b]$ and $|x - c| < \delta_1$, then $|f(x) - f(c)| < \frac{|f(c)-g(c)|}{2}$.

Since $|x - c| < \delta$ and $\delta \leq \delta_2$, then $|x - c| < \delta_2$.

Since $x \in [a,b]$ and $|x - c| < \delta_2$, then $|g(x) - g(c)| < \frac{|f(c)-g(c)|}{2}$.

Observe that

$$\begin{aligned}
|f(c) - g(c)| &= |f(c) - f(x) + f(x) - g(c)| \\
&= |f(c) - f(x) + g(x) - g(c)| \\
&\leq |f(c) - f(x)| + |g(x) - g(c)| \\
&= |f(x) - f(c)| + |g(x) - g(c)| \\
&< \frac{|f(c) - g(c)|}{2} + \frac{|f(c) - g(c)|}{2} \\
&= |f(c) - g(c)|.
\end{aligned}$$

Hence, $|f(c) - g(c)| < |f(c) - g(c)|$, a contradiction.

Thus, $f(c) = g(c)$.
Since $c \in [a, b]$ and $f(c) = g(c)$, then $c \in S$.
Therefore, S is closed. $\square$

Exercise 35. Let I be a closed set.
Let $f : I \to \mathbb{R}$ be a continuous function.
Let $S = \{x \in I : f(x) = k\}$.
Then S is closed.

Proof. Either $S = \emptyset$ or $S \neq \emptyset$.
We consider these cases separately.
Case 1: Suppose $S = \emptyset$.
Since the empty set is closed, then S is closed.
Case 2: Suppose $S \neq \emptyset$.
Let p be an arbitrary accumulation point of S.
To prove S is closed, we must prove $p \in S$.
Since p is an accumulation point of S and $S \subset I$, then p is an accumulation point of I.
Since I is closed, then $p \in I$.
Since f is continuous on I, then f is continuous at p.
Since p is an accumulation point of S, then for every $\delta > 0$ there exists $x \in S$ such that $x \in N'(p; \delta)$.
Let $\delta = \frac{1}{n}$ for each $n \in \mathbb{N}$.
Then for each $n \in \mathbb{N}$, there exists $x \in S$ such that $x \in N'(p; \frac{1}{n})$, so there exists a function $f : \mathbb{N} \to S$ such that $f(n) \in S$ and $f(n) \in N'(p; \frac{1}{n})$ for each $n \in \mathbb{N}$.
Hence, there exists a sequence (x_n) such that $x_n \in S$ and $x_n \in N'(p; \frac{1}{n})$ for each $n \in \mathbb{N}$.
Let $n \in \mathbb{N}$.
Then $x_n \in S$, so $x_n \in I$ and $f(x_n) = k$.
Hence, $x_n \in I$ and $f(x_n) = k$ for all $n \in \mathbb{N}$.
Since $x_n \in I$ for all $n \in \mathbb{N}$, then (x_n) is a sequence in I.
Since $f(x_n) = k$ for all $n \in \mathbb{N}$, then $(f(x_n))$ is the constant sequence k, so $\lim_{n\to\infty} f(x_n) = k$.
We prove the sequence (x_n) converges to p.
Let $\epsilon > 0$ be given.
Then $\frac{1}{\epsilon} > 0$, so by the Archimedean property of $\mathbb{R}$, there exists $N \in \mathbb{N}$ such that $N > \frac{1}{\epsilon}$.
Let $n \in \mathbb{N}$ such that $n > N$.
Then $n > N > \frac{1}{\epsilon}$, so $n > \frac{1}{\epsilon}$.
Hence, $\epsilon > \frac{1}{n}$.
Since $n \in \mathbb{N}$, then $x_n \in N'(p; \frac{1}{n})$, so $x_n \in N(p; \frac{1}{n})$.
Thus, $|x_n - p| < \frac{1}{n} < \epsilon$, so $|x_n - p| < \epsilon$.
Therefore, $\lim_{n\to\infty} x_n = p$.

Since f is continuous at p and (x_n) is a sequence of points in I and $\lim_{n\to\infty} x_n = p$, then by the sequential characterization of continuity, $k = \lim_{n\to\infty} f(x_n) = f(p)$.

Therefore, $f(p) = k$.

Since $p \in I$ and $f(p) = k$, then $p \in S$, so S is closed.

Thus, in all cases, S is closed, as desired. □

Exercise 36. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that $f(x) = 0$ for all $x \in \mathbb{Q}$.

Then $f(x) = 0$ for all $x \in \mathbb{R}$.

Proof. We prove $f(c) = 0$ for all $c \in \mathbb{R}$ by contradiction.

Suppose there exists $c \in \mathbb{R}$ such that $f(c) \neq 0$.

Since $f(c) \neq 0$, then $|f(c)| > 0$.

Since f is continuous on $\mathbb{R}$ and $c \in \mathbb{R}$, then f is continuous at c.

Since $|f(c)| > 0$, then there exists $\delta > 0$ such that for all $x \in \mathbb{R}$, if $|x-c| < \delta$, then $|f(x) - f(c)| < |f(c)|$.

Since $c < c + \delta$ and $\mathbb{Q}$ is dense in $\mathbb{R}$, then there exists $q \in \mathbb{Q}$ such that $c < q < c + \delta$.

Since $q \in \mathbb{Q}$ and $\mathbb{Q} \subset \mathbb{R}$, then $q \in \mathbb{R}$.

Since $c < q < c + \delta$, then $0 < q - c < \delta$, so $|q - c| < \delta$.

Since $q \in \mathbb{R}$ and $|q - c| < \delta$, then $|f(q) - f(c)| < |f(c)|$.

Since $q \in \mathbb{Q}$, then $f(q) = 0$.

Therefore, $|0 - f(c)| < |f(c)|$, so $|-f(c)| < |f(c)|$.

Thus, $|f(c)| < |f(c)|$, a contradiction.

Therefore, $f(c) = 0$ for all $c \in \mathbb{R}$, as desired □

Exercise 37. Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be continuous functions such that $f(x) = g(x)$ for all $x \in \mathbb{Q}$.

Then $f(x) = g(x)$ for all $x \in \mathbb{R}$. (Hence, function $f = g$).

Proof. We prove $f(c) = g(c)$ for all $c \in \mathbb{R}$ by contradiction.

Suppose there exists $c \in \mathbb{R}$ such that $f(c) \neq g(c)$.

Then $|f(c) - g(c)| > 0$, so $\frac{|f(c)-g(c)|}{2} > 0$.

Since f is continuous on $\mathbb{R}$ and $c \in \mathbb{R}$, then f is continuous at c.

Since g is continuous on $\mathbb{R}$ and $c \in \mathbb{R}$, then g is continuous at c.

Since $\frac{|f(c)-g(c)|}{2} > 0$ and f is continuous at c, then there exists $\delta_1 > 0$ such that for all $x \in \mathbb{R}$, if $|x - c| < \delta_1$, then $|f(x) - f(c)| < \frac{|f(c)-g(c)|}{2}$.

Since $\frac{|f(c)-g(c)|}{2} > 0$ and g is continuous at c, then there exists $\delta_2 > 0$ such that for all $x \in \mathbb{R}$, if $|x - c| < \delta_2$, then $|g(x) - g(c)| < \frac{|f(c)-g(c)|}{2}$.

Let $\delta = \min\{\delta_1, \delta_2\}$.

Then $\delta \leq \delta_1$ and $\delta \leq \delta_2$ and $\delta > 0$.

Since $c < c + \delta$ and $\mathbb{Q}$ is dense in $\mathbb{R}$, then there exists $q \in \mathbb{Q}$ such that $c < q < c + \delta$.

Hence, $0 < q - c < \delta$, so $|q - c| < \delta$.

Since $|q - c| < \delta \leq \delta_1$, then $|q - c| < \delta_1$, so $|f(q) - f(c)| < \frac{|f(c)-g(c)|}{2}$.

Since $|q - c| < \delta \leq \delta_2$, then $|q - c| < \delta_2$, so $|g(q) - g(c)| < \frac{|f(c)-g(c)|}{2}$.
Since $q \in \mathbb{Q}$, then $f(q) = g(q)$, so $|f(q) - g(c)| < \frac{|f(c)-g(c)|}{2}$.
Observe that

$$\begin{aligned}
|f(c) - g(c)| &= |f(c) - f(q) + f(q) - g(c)| \\
&\leq |f(c) - f(q)| + |f(q) - g(c)| \\
&= |f(q) - f(c)| + |f(q) - g(c)| \\
&< \frac{|f(c) - g(c)|}{2} + \frac{|f(c) - g(c)|}{2} \\
&= |f(c) - g(c)|.
\end{aligned}$$

Hence, $|f(c) - g(c)| < |f(c) - g(c)|$, a contradiction.
Therefore, $f(c) = g(c)$ for all $c \in \mathbb{R}$, as desired. □

Definition 38. additive map
An additive map preserves the operation of addition.
Let f be a real valued function.
Let $dom f$ be an additive group.
Then f is said to be **additive** iff $f(x+y) = f(x) + f(y)$ for all $x, y \in dom f$.

Lemma 39. *Let $f : \mathbb{R} \to \mathbb{R}$ be a function.*
If there exist real numbers k and L such that $\lim_{x\to 0} f(x) = L$ and $k \neq 0$, then $\lim_{x\to 0} f(kx) = L$.

Proof. Suppose there exist real numbers k and L such that $\lim_{x\to 0} f(x) = L$ and $k \neq 0$.
To prove $\lim_{x\to 0} f(kx) = L$, let $\epsilon > 0$ be given.
We must prove there exists $\delta > 0$ such that for all $x \in \mathbb{R}$, if $0 < |x| < \delta$, then $|f(kx) - L| < \epsilon$.
Since $\lim_{x\to 0} f(x) = L$, then there exists $\delta_1 > 0$ such that for all $x \in \mathbb{R}$, if $0 < |x| < \delta_1$, then $|f(x) - L| < \epsilon$.
Let $\delta = \frac{\delta_1}{|k|}$.
Since $k \neq 0$, then $|k| > 0$.
Since $|k| > 0$ and $\delta_1 > 0$, then $\delta > 0$.
Let $x \in \mathbb{R}$ such that $0 < |x| < \delta$.
Then $0 < |x| < \frac{\delta_1}{|k|}$, so $0 < |k||x| < \delta_1$.
Hence, $0 < |kx| < \delta_1$.
Since $kx \in \mathbb{R}$ and $0 < |kx| < \delta_1$, then $|f(kx) - L| < \epsilon$, as desired. □

Exercise 40. Let $f : \mathbb{R} \to \mathbb{R}$ be an additive function.
If the limit of f at 0 exists, then $\lim_{x\to 0} f(x) = 0$.

Proof. Suppose the limit of f at 0 exists.
Then there exists a real number L such that $\lim_{x\to 0} f(x) = L$.
To prove $\lim_{x\to 0} f(x) = 0$, we must prove $L = 0$.
Let $x \in \mathbb{R}$.

Then $f(2x) = f(x + x) = f(x) + f(x) = 2f(x)$, so $f(2x) = 2f(x)$ for all $x \in \mathbb{R}$.

By the previous lemma, if $\lim_{x\to 0} f(x) = L$ and $k \neq 0$, then $\lim_{x\to 0} f(kx) = L$.

Thus, if $\lim_{x\to 0} f(x) = L$, then $\lim_{x\to 0} f(2x) = L$.

Since $\lim_{x\to 0} f(x) = L$, then $\lim_{x\to 0} f(2x) = L$, so $\lim_{x\to 0} f(x) = \lim_{x\to 0} f(2x)$.

Hence,

$$\begin{aligned} L &= \lim_{x\to 0} f(x) \\ &= \lim_{x\to 0} f(2x) \\ &= \lim_{x\to 0} 2f(x) \\ &= 2\lim_{x\to 0} f(x) \\ &= 2L. \end{aligned}$$

Thus, $L = 2L$, so $2L = L$.

Subtracting L from both sides, we obtain $L = 0$, as desired. □

Lemma 41. *Let $f : \mathbb{R} \to \mathbb{R}$ be an additive function.*
Then $f(a - b) = f(a) - f(b)$ for all $a, b \in \mathbb{R}$ and $f(0) = 0$.

Proof. We prove $f(a - b) = f(a) - f(b)$ for all $a, b \in \mathbb{R}$.

Let $a, b \in \mathbb{R}$.

Then $f(a) = f(a - b + b) = f(a - b) + f(b)$, so $f(a) = f(a - b) + f(b)$.

Hence, $f(a) - f(b) = f(a - b)$. □

Proof. We prove $f(0) = 0$.

Since $f(0) = f(0 + 0) = f(0) + f(0)$, then $f(0) = f(0) + f(0)$.

Therefore, $f(0) = f(0) - f(0) = 0$. □

Lemma 42. *Let $f : \mathbb{R} \to \mathbb{R}$ be an additive function.*
If $\lim_{x\to 0} f(x) = 0$, then $\lim_{x\to a} f(x) = f(a)$ for all $a \in \mathbb{R}$.

Proof. Suppose $\lim_{x\to 0} f(x) = 0$.

Let $a \in \mathbb{R}$ be arbitrary.

Observe that a is an accumulation point of $\mathbb{R}$, the domain of f.

To prove $\lim_{x\to a} f(x) = f(a)$, let $\epsilon > 0$ be given.

We must prove there exists $\delta > 0$ such that for all $x \in \mathbb{R}$, if $0 < |x - a| < \delta$, then $|f(x) - f(a)| < \epsilon$.

Since $\lim_{x\to 0} f(x) = 0$, then there exists $\delta > 0$ such that for all $x \in \mathbb{R}$, if $0 < |x| < \delta$, then $|f(x)| < \epsilon$.

Let $x \in \mathbb{R}$ such that $0 < |x - a| < \delta$.

Since $x - a \in \mathbb{R}$ and $0 < |x - a| < \delta$, then $|f(x - a)| < \epsilon$.

Since f is additive, then $|f(x) - f(a)| = |f(x - a)| < \epsilon$, as desired. □

Lemma 43. *Let $f : \mathbb{R} \to \mathbb{R}$ be an additive function.*
If f is continuous at $x_0 \in \mathbb{R}$, then f is continuous at 0.

Proof. Suppose f is continuous at $x_0 \in \mathbb{R}$.
To prove f is continuous at 0, let $\epsilon > 0$ be given.
We must prove there exists $\delta > 0$ such that for all $x \in \mathbb{R}$, if $|x| < \delta$, then $|f(x) - f(0)| < \epsilon$.

Since f is continuous at x_0, then there exists $\delta > 0$ such that for all $x \in \mathbb{R}$, if $|x - x_0| < \delta$, then $|f(x) - f(x_0)| < \epsilon$.
Let $x \in \mathbb{R}$ such that $|x| < \delta$.
Let $y = x + x_0$.
Then $y \in \mathbb{R}$ and $x = y - x_0$, so $|y - x_0| < \delta$.
Since $y \in \mathbb{R}$ and $|y - x_0| < \delta$, then $|f(y) - f(x_0)| < \epsilon$.
Since f is additive, then $|f(x) - f(0)| = |f(x - 0)| = |f(x)| = |f(y - x_0)| = |f(y) - f(x_0)| < \epsilon$, so $|f(x) - f(0)| < \epsilon$, as desired. □

Exercise 44. Let $f : \mathbb{R} \to \mathbb{R}$ be an additive function.
If f is continuous at $x_0 \in \mathbb{R}$, then f is continuous.

Proof. Suppose f is continuous at $x_0 \in \mathbb{R}$.
To prove f is continuous, let $c \in \mathbb{R}$ be given.
We must prove f is continuous at c.
Since f is additive and f is continuous at x_0, then by a previous lemma, f is continuous at 0.
Since 0 is an accumulation point of $\mathbb{R}$ and f is continuous at 0, then by the characterization of continuity, $\lim_{x\to 0} f(x) = f(0)$.

Since f is additive, then $f(0) = 0$.
Thus, $\lim_{x\to 0} f(x) = f(0) = 0$.
Since f is additive and $\lim_{x\to 0} f(x) = 0$, then by a previous lemma, $\lim_{x\to a} f(x) = f(a)$ for all $a \in \mathbb{R}$, so $\lim_{x\to c} f(x) = f(c)$,
Since c is an accumulation point of $\mathbb{R}$ and $\lim_{x\to c} f(x) = f(c)$, then by the characterization of continuity, f is continuous at c, as desired. □

Lemma 45. *Let $f : \mathbb{R} \to \mathbb{R}$ be an additive function.*
Then $f(nr) = nf(r)$ for all $n \in \mathbb{Z}, r \in \mathbb{R}$.

Proof. Let $r \in \mathbb{R}$ be given.
To prove $f(nr) = nf(r)$ for all $n \in \mathbb{Z}$, we must prove $f(nr) = nf(r)$ for all $n \in \mathbb{Z}^+$ and $f(-nr) = -nf(r)$ for all $n \in \mathbb{Z}^+$.

We prove $f(nr) = nf(r)$ for all $n \in \mathbb{Z}^+$ by induction on n.
Let $S = \{n \in \mathbb{Z}^+ : f(nr) = nf(r)\}$.
Basis:
Since $1 \in \mathbb{Z}^+$ and $f(1 \cdot r) = f(r) = 1 \cdot f(r)$, then $1 \in S$.
Induction:
Let $k \in S$.
Then $k \in \mathbb{Z}^+$ and $f(kr) = kf(r)$.
Since $k \in \mathbb{Z}^+$, then $k + 1 \in \mathbb{Z}^+$.

Observe that

$$\begin{aligned} f((k+1)r) &= f(kr+r) \\ &= f(kr)+f(r) \\ &= kf(r)+f(r) \\ &= (k+1)f(r). \end{aligned}$$

Since $k+1 \in \mathbb{Z}^+$ and $f((k+1)r) = (k+1)f(r)$, then $k+1 \in S$.
Hence, $k \in S$ implies $k+1 \in S$, so by PMI, $S = \mathbb{Z}^+$.
Therefore, $f(nr) = nf(r)$ for all $n \in \mathbb{Z}^+$. ☐

Proof. We prove $f(-nr) = -nf(r)$ for all $n \in \mathbb{Z}^+$ by induction on n.
Let $S = \{n \in \mathbb{Z}^+ : f(-nr) = -nf(r)\}$.
Basis:
Since f is additive, then $f(-r) = f(0-r) = f(0)-f(r) = 0-f(r) = -f(r)$.
Since $1 \in \mathbb{Z}^+$ and $f(-r) = -f(r)$, then $1 \in S$.
Induction:
Let $k \in S$.
Then $k \in \mathbb{Z}^+$ and $f(-kr) = -kf(r)$.
Since $k \in \mathbb{Z}^+$, then $k+1 \in \mathbb{Z}^+$.
Observe that

$$\begin{aligned} f(-(k+1)r) &= f(-kr-r) \\ &= f(-kr)-f(r) \\ &= -kf(r)-f(r) \\ &= -(k+1)f(r). \end{aligned}$$

Since $k+1 \in \mathbb{Z}^+$ and $f(-(k+1)r) = -(k+1)f(r)$, then $k+1 \in S$.
Hence, $k \in S$ implies $k+1 \in S$, so by PMI, $S = \mathbb{Z}^+$.
Therefore, $f(-nr) = -nf(r)$ for all $n \in \mathbb{Z}^+$. ☐

Lemma 46. *Let $f : \mathbb{R} \to \mathbb{R}$ be an additive function.*
Then $f(\frac{1}{n}) = \frac{f(1)}{n}$ for all nonzero integers n.

Proof. Let n be an arbitrary nonzero integer.
Then $n \in \mathbb{Z}$ and $n \neq 0$.
Since $n \in \mathbb{Z}$ and $\mathbb{Z} \subset \mathbb{R}$, then $n \in \mathbb{R}$.
Since $n \in \mathbb{R}$ and $n \neq 0$, then $\frac{1}{n} \in \mathbb{R}$.
Since f is additive and $n \in \mathbb{Z}$ and $\frac{1}{n} \in \mathbb{R}$, then $f(1) = f(n \cdot \frac{1}{n}) = n \cdot f(\frac{1}{n})$.
Since $f(1) = n \cdot f(\frac{1}{n})$ and $n \neq 0$, then $\frac{f(1)}{n} = f(\frac{1}{n})$. ☐

Exercise 47. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous additive function.
If $f(1) = c$, then $f(x) = cx$ for all $x \in \mathbb{R}$.

Proof. Suppose $f(1) = c$.

We first prove $f(q) = cq$ for all $q \in \mathbb{Q}$.

Let $q \in \mathbb{Q}$.

Then there exist integers a, b with $b \neq 0$ such that $q = \frac{a}{b}$.

Since a and b are integers, then $a \in \mathbb{Z}$ and $b \in \mathbb{Z}$.

Since $b \in \mathbb{Z}$ and $b \neq 0$, then b is a nonzero integer.

Since f is additive and $a \in \mathbb{Z}$ and b is a nonzero integer, then

$$\begin{aligned}
f(q) &= f(\frac{a}{b}) \\
&= f(a \cdot \frac{1}{b}) \\
&= a \cdot f(\frac{1}{b}) \\
&= a \cdot \frac{f(1)}{b} \\
&= a \cdot \frac{c}{b} \\
&= \frac{ac}{b} \\
&= \frac{ca}{b} \\
&= c \cdot \frac{a}{b} \\
&= cq.
\end{aligned}$$

Hence, $f(q) = cq$ for all $q \in \mathbb{Q}$.

Let $g : \mathbb{R} \to \mathbb{R}$ be a function defined by $g(x) = cx$ for all $x \in \mathbb{R}$.

Since g is a polynomial function, then g is continuous.

Let $x \in \mathbb{Q}$.

Since $\mathbb{Q} \subset \mathbb{R}$, then $x \in \mathbb{R}$, so $g(x) = cx = f(x)$.

Hence, $f(x) = g(x)$ for all $x \in \mathbb{Q}$.

Since f and g are continuous and $f(x) = g(x)$ for all $x \in \mathbb{Q}$, then by a previous exercise, $f(x) = g(x)$ for all $x \in \mathbb{R}$.

Therefore, $f(x) = cx$ for all $x \in \mathbb{R}$, as desired. □

Exercise 48. Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that $f(x + y) = f(x)f(y)$ for all $x, y \in \mathbb{R}$.

Then

1. If $f(c) = 0$ for some $c \in \mathbb{R}$, then $f(x) = 0$ for all $x \in \mathbb{R}$.
2. If the limit of f at 0 exists, then the limit of f at c exists for all $c \in \mathbb{R}$.

Proof. Let $x \in \mathbb{R}$.

Then $f(x) \in \mathbb{R}$ and $f(x) = f(\frac{x}{2} + \frac{x}{2}) = f(\frac{x}{2})f(\frac{x}{2}) = [f(\frac{x}{2})]^2 \geq 0$, so $f(x) \geq 0$ for all $x \in \mathbb{R}$.

Either there is $c \in \mathbb{R}$ such that $f(c) = 0$ or there is no $c \in \mathbb{R}$ such that $f(c) = 0$.

We consider these cases separately.

Case 1: Suppose there is $c \in \mathbb{R}$ such that $f(c) = 0$.

Let $x \in \mathbb{R}$.

Then $f(x) = f(0 + x) = f(c - c + x) = f(c + x - c) = f(c)f(x - c) = 0 \cdot f(x - c) = 0$, so $f(x) = 0$.

Therefore, $f(x) = 0$ for all $x \in \mathbb{R}$.

Case 2: Suppose there is no $c \in \mathbb{R}$ such that $f(c) = 0$.

Since $f(0) = f(0 + 0) = f(0)f(0)$, then $f(0) = f(0)f(0)$, so $0 = f(0)f(0) - f(0) = f(0)[f(0) - 1]$.

Thus, $0 = f(0)[f(0) - 1]$, so either $f(0) = 0$ or $f(0) - 1 = 0$.

Since there is no $c \in \mathbb{R}$ such that $f(c) = 0$, then $f(0) \neq 0$.

Therefore, $f(0) - 1 = 0$, so $f(0) = 1$.

Suppose the limit of f at 0 exists.

Then there is a real number L such that $\lim_{x \to 0} f(x) = L$.

We must prove $L = 1$.

Let $c \in \mathbb{R}$.

To prove the limit of f at c exists, we must prove there exists $M \in \mathbb{R}$ such that $\lim_{x \to c} f(x) = M$. $\square$

Continuous functions on compact sets

Exercise 49. Let $a, b \in \mathbb{R}$ with $a < b$.

Let $f : [a, b] \to \mathbb{R}$ be continuous.

If f has no zeroes on $[a, b]$, then either $f(x) > 0$ for all $x \in [a, b]$ or $f(x) < 0$ for all $x \in [a, b]$.

Proof. Suppose f has no zeroes on $[a, b]$.

To prove either $f(x) > 0$ for all $x \in [a, b]$ or $f(x) < 0$ for all $x \in [a, b]$, we prove by contradiction.

Suppose it is not the case that either $f(x) > 0$ for all $x \in [a, b]$ or $f(x) < 0$ for all $x \in [a, b]$.

Then it is not the case that $f(x) > 0$ for all $x \in [a, b]$ and it is not the case that $f(x) < 0$ for all $x \in [a, b]$.

Hence, there exists $x \in [a, b]$ such that $f(x) \leq 0$ and there exists $y \in [a, b]$ such that $f(y) \geq 0$.

Since $x \in [a, b]$ and f has no zeroes on $[a, b]$, then $f(x) \neq 0$.

Since $f(x) \leq 0$ and $f(x) \neq 0$, then $f(x) < 0$.

Since $y \in [a, b]$ and f has no zeroes on $[a, b]$, then $f(y) \neq 0$.

Since $f(y) \geq 0$ and $f(y) \neq 0$, then $f(y) > 0$.

Since $x \in [a, b]$ and $y \in [a, b]$, then $[x, y] \subset [a, b]$.

Since f is continuous on $[a, b]$ and $[x, y] \subset [a, b]$, then f is continuous on $[x, y]$.

Since $f(x) < 0 < f(y)$, then by IVT, there exists $c \in (x, y)$ such that $f(c) = 0$.

Since $(x, y) \subset [x, y] \subset [a, b]$, then $(x, y) \subset [a, b]$.

Since $c \in (x, y)$ and $(x, y) \subset [a, b]$, then $c \in [a, b]$.

Thus, there exists $c \in [a, b]$ such that $f(c) = 0$.

But, this contradicts the assumption that f has no zeroes on $[a, b]$.

Therefore, either $f(x) > 0$ for all $x \in [a, b]$ or $f(x) < 0$ for all $x \in [a, b]$. $\square$

Exercise 50. Let $a, b \in \mathbb{R}$.

Let $f : [a, b] \to \mathbb{R}$ be a continuous function.

If there exists $k \in \mathbb{R}$ such that $f(a) \geq k \geq f(b)$, then there exists $c \in [a, b]$ such that $f(c) = k$.

Proof. Suppose there exists $k \in \mathbb{R}$ such that $f(a) \geq k \geq f(b)$.

Since $f(a) \geq k \geq f(b)$, then $f(a) \geq k$ and $k \geq f(b)$, so either $f(a) = k$ or $f(b) = k$ or $f(a) > k > f(b)$.

We consider these cases separately.

Case 1: Suppose $f(a) = k$.

Observe that $a \in [a, b]$ and $f(a) = k$.

Case 2: Suppose $f(b) = k$.

Observe that $b \in [a, b]$ and $f(b) = k$.

Case 3: Suppose $f(a) > k > f(b)$.

Since f is continuous on the interval $[a, b]$ and $f(a) > k > f(b)$, then by IVT, there exists $c \in (a, b)$ such that $f(c) = k$. $\square$

Exercise 51. Let E be a nonempty closed bounded set.

Let $f : E \to \mathbb{R}$ be a continuous function such that $f(x) > 0$ for all $x \in E$.

Then there exists $k > 0$ such that $f(x) \geq k$ for all $x \in E$.

Proof. Let $\frac{1}{f} : E \to \mathbb{R}$ be a function defined by $\frac{1}{f}(x) = \frac{1}{f(x)}$ for all $x \in E$.

Since $E \neq \emptyset$, then there is some element in E.

Let $c \in E$ be given.

Then $f(c) > 0$, so $f(c) \neq 0$.

Since f is continuous on E and $c \in E$, then f is continuous at c.

Since the constant function 1 is continuous at c and f is continuous at c and $f(c) \neq 0$, then $\frac{1}{f}$ is continuous at c.

Since c is arbitrary, then $\frac{1}{f}$ is continuous on E.

Since E is a closed bounded set, then by the boundedness theorem, the function $\frac{1}{f}$ is bounded.

Hence, there exists $M \in \mathbb{R}$ such that $|\frac{1}{f(x)}| \leq M$ for all $x \in E$.

Let $x \in E$.

Then $f(x) > 0$ and $|\frac{1}{f(x)}| \leq M$.

Thus, $0 < \frac{1}{f(x)} = \frac{1}{|f(x)|} = |\frac{1}{f(x)}| \leq M$, so $0 < M$ and $\frac{1}{f(x)} \leq M$.

Hence, $0 < \frac{1}{M} \leq f(x)$, so $f(x) \geq \frac{1}{M} > 0$.

Let $k = \frac{1}{M}$.

Then $k > 0$ and $f(x) \geq k$.

Therefore, there exists $k > 0$ such that $f(x) \geq k$ for all $x \in E$, as desired. $\square$

Proof. Since f is continuous on E and E is a nonempty closed bounded set, then by EVT, f has a minimum on E.

Hence, there exists $m \in E$ such that $f(m) \leq f(x)$ for all $x \in E$.

Since $m \in E$ and $f(x) > 0$ for all $x \in E$, then $f(m) > 0$.

Let $k = f(m)$.

Then $k > 0$ and $k \leq f(x)$ for all $x \in E$, so there exists $k > 0$ such that $f(x) \geq k$ for all $x \in E$, as desired. $\square$

Exercise 52. Let $E \subset \mathbb{R}$ be a closed, bounded set.

Let f and g be real valued functions continuous on E.

Let $S = \{x \in E : f(x) = g(x)\}$.

If (x_n) is a sequence in S and $\lim_{n\to\infty} x_n = c$, then $c \in S$.

Proof. Suppose (x_n) is a sequence in S and $\lim_{n\to\infty} x_n = c$.

Since (x_n) is a sequence in S, then $x_n \in S$ for all $n \in \mathbb{N}$, so $x_n \in E$ and $f(x_n) = g(x_n)$ for all $n \in \mathbb{N}$.

Thus, $x_n \in E$ for all $n \in \mathbb{N}$ and $f(x_n) = g(x_n)$ for all $n \in \mathbb{N}$.

Since $x_n \in E$ for all $n \in \mathbb{N}$, then (x_n) is a sequence in E.

Since E is a closed and bounded set and (x_n) is a sequence in E, then by the Bolzano-Weierstrass property of compact sets, there exists a subsequence (y_n) in E such that $\lim_{n\to\infty} y_n \in E$.

Since (x_n) is a convergent sequence in $\mathbb{R}$ and (y_n) is a subsequence of (x_n), then $\lim_{n\to\infty} y_n = \lim_{n\to\infty} x_n = c$.

Hence, $c \in E$.

Let $h = f - g$.

Then $h : E \to \mathbb{R}$ is the function defined by $h(x) = f(x) - g(x)$.

Since f is continuous on E and g is continuous on E, then the difference $f - g = h$ is continuous on E.

Since $c \in E$, then h is continuous at c.

Since $f(x_n) = g(x_n)$ for all $n \in \mathbb{N}$, then $h(x_n) = f(x_n) - g(x_n) = 0$ for all $n \in \mathbb{N}$, so $(h(x_n))$ is the 0 constant sequence.

Hence, $\lim_{n\to\infty} h(x_n) = 0$.

Since h is continuous at c and (x_n) is a sequence of points in E such that $\lim_{n\to\infty} x_n = c$, then by the sequential characterization of continuity, we have $\lim_{n\to\infty} h(x_n) = h(c)$.

Thus, $f(c) - g(c) = h(c) = \lim_{n\to\infty} h(x_n) = 0$, so $f(c) - g(c) = 0$.

Hence, $f(c) = g(c)$.

Since $c \in E$ and $f(c) = g(c)$, then $c \in S$, as desired. $\square$

Exercise 53. Let E be a closed bounded infinite set.

Let $f : E \to \mathbb{R}$ be a continuous function.

If for every $x \in E$, there exists $y \in E$ such that $|f(y)| \leq \frac{1}{2}|f(x)|$, then there exists $c \in E$ such that $f(c) = 0$.

Proof. Since E is infinite, then $E \neq \emptyset$, so let $x_0 \in E$.

Then there exists $x_1 \in E$ such that $|f(x_1)| \leq \frac{1}{2}|f(x_0)|$.

Suppose there exists $k \in \mathbb{N}$ such that $x_k \in E$ and $|f(x_k)| \leq \frac{|f(x_0)|}{2^k}$.

Since $k \in \mathbb{N}$, then $k+1 \in \mathbb{N}$.

Since $x_k \in E$, then there exists $x_{k+1} \in E$ such that $|f(x_{k+1})| \leq \frac{1}{2}|f(x_k)|$.

Thus, $|f(x_{k+1})| \leq \frac{1}{2}|f(x_k)| \leq \frac{|f(x_0)|}{2^{k+1}}$, so $|f(x_{k+1})| \leq \frac{|f(x_0)|}{2^{k+1}}$.

Hence, by PMI, $x_n \in E$ for all $n \in \mathbb{N}$ and $|f(x_n)| \leq \frac{|f(x_0)|}{2^n}$ for all $n \in \mathbb{N}$.

Since $x_n \in E$ for all $n \in \mathbb{N}$, then (x_n) is a sequence in E.

Since $0 \leq |f(x_n)| \leq \frac{|f(x_0)|}{2^n}$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} 0 = 0 = \lim_{n\to\infty} \frac{|f(x_0)|}{2^n}$, then by the squeeze rule, $\lim_{n\to\infty} |f(x_n)| = 0$.

Since $-|f(x_n)| \leq f(x_n) \leq |f(x_n)|$ for all $n \in \mathbb{N}$ and $0 = -0 = -\lim_{n\to\infty} |f(x_n)| = \lim_{n\to\infty} -|f(x_n)|$, then by the squeeze rule, $\lim_{n\to\infty} f(x_n) = 0$, so the sequence $(f(x_n))$ is convergent.

Since (x_n) is a sequence in E and E is a closed bounded set, then by the Bolzano-Weierstrass property of compact sets, there exists a subsequence (y_n) in E such that $\lim_{n\to\infty} y_n \in E$.

Let $c = \lim_{n\to\infty} y_n$.

Then $c \in E$.

Since f is continuous on E, then f is continuous at c.

Since (y_n) is a sequence in E and $\lim_{n\to\infty} y_n = c$, then by the sequential characterization of continuity, we have $\lim_{n\to\infty} f(y_n) = f(c)$.

Since (y_n) is a subsequence of (x_n), then there exists a strictly increasing function $g : \mathbb{N} \to \mathbb{N}$ such that $y_n = x_{g(n)}$ for all $n \in \mathbb{N}$.

Let $n \in \mathbb{N}$.

Then $y_n = x_{g(n)}$, so $f(y_n) = f(x_{g(n)})$ for all $n \in \mathbb{N}$.

Since $g : \mathbb{N} \to \mathbb{N}$ is a strictly increasing function such that $f(y_n) = f(x_{g(n)})$ for all $n \in \mathbb{N}$, then $(f(y_n))$ is a subsequence of $(f(x_n))$.

Since $(f(x_n))$ is convergent and $(f(y_n))$ is a subsequence of $(f(x_n))$, then $f(c) = \lim_{n\to\infty} f(y_n) = \lim_{n\to\infty} f(x_n) = 0$, so $f(c) = 0$.

Therefore, there exists $c \in E$ such that $f(c) = 0$, as desired. $\square$

Exercise 54. Let $f : [0,1] \to \mathbb{R}$ be a continuous function such that $f(0) = f(1)$.

Then there exists $c \in [0, \frac{1}{2}]$ such that $f(c) = f(c + \frac{1}{2})$.

Proof. Either $f(\frac{1}{2}) = f(0)$ or $f(\frac{1}{2}) \neq f(0)$.

We consider these cases separately.

Case 1: Suppose $f(\frac{1}{2}) = f(0)$.

Let $c = 0$.

Since $0 \in [0, \frac{1}{2}]$, then $c \in \frac{1}{2}$.

Observe that $f(c) = f(0) = f(\frac{1}{2}) = f(0 + \frac{1}{2}) = f(c + \frac{1}{2})$.

Case 2: Suppose $f(\frac{1}{2}) \neq f(0)$.

Then $f(\frac{1}{2}) - f(0) \neq 0$.

Let $k = f(\frac{1}{2}) - f(0)$.

Then $k \neq 0$.

Let $g : [0, \frac{1}{2}] \to \mathbb{R}$ be a function defined by $g(x) = f(x + \frac{1}{2})$.

To prove g is continuous, let $c \in [0, \frac{1}{2}]$ be given.

Then $0 \leq c \leq \frac{1}{2}$, so $\frac{1}{2} \leq c + \frac{1}{2} \leq 1$.

Let $a = c + \frac{1}{2}$.

Then $\frac{1}{2} \leq a \leq 1$, so $a \in [\frac{1}{2}, 1]$.

Since $[\frac{1}{2}, 1] \subset [0, 1]$, then $a \in [0, 1]$.

Since f is continuous on $[0, 1]$, then f is continuous at a.

Let $\epsilon > 0$ be given.

Then there exists $\delta > 0$ such that for all $x \in [0, 1]$ if $|x - a| < \delta$, then $|f(x) - f(a)| < \epsilon$.

Let $x \in [0, \frac{1}{2}]$ such that $|x - c| < \delta$.

Since $x \in [0, \frac{1}{2}]$, then $0 \leq x \leq \frac{1}{2}$, so $\frac{1}{2} \leq x + \frac{1}{2} \leq 1$.

Hence, $x + \frac{1}{2} \in [\frac{1}{2}, 1]$.

Since $[\frac{1}{2}, 1] \subset [0, 1]$, then $x + \frac{1}{2} \in [0, 1]$.

Since $|x - c| < \delta$ and $-c = \frac{1}{2} - a$, then $|x + \frac{1}{2} - a| < \delta$.

Since $x + \frac{1}{2} \in [0, 1]$ and $|(x + \frac{1}{2}) - a| < \delta$, then we conclude $|f(x + \frac{1}{2}) - f(a)| < \epsilon$.

Thus, $|g(x) - g(c)| = |f(x + \frac{1}{2}) - f(c + \frac{1}{2})| = |f(x + \frac{1}{2}) - f(a)| < \epsilon$.

Therefore, g is continuous at c, so g is continuous on $[0, \frac{1}{2}]$.

Let $h = g - f$ be defined on $[0, \frac{1}{2}]$.

Then $h : [0, \frac{1}{2}] \to \mathbb{R}$ is the function defined by $h(x) = f(x + \frac{1}{2}) - f(x)$.

Since g is continuous on $[0, \frac{1}{2}]$ and f is continuous on $[0, 1]$, then $g - f = h$ is continuous on the intersection $[0, \frac{1}{2}] \cap [0, 1] = [0, \frac{1}{2}]$.

Observe that $h(0) = f(0 + \frac{1}{2}) - f(0) = f(\frac{1}{2}) - f(0) = k$ and $h(\frac{1}{2}) = f(\frac{1}{2} + \frac{1}{2}) - f(\frac{1}{2}) = f(1) - f(\frac{1}{2}) = f(0) - f(\frac{1}{2}) = -k$.

Since $k \neq 0$, then either $k > 0$ or $k < 0$.

If $k > 0$, then $h(\frac{1}{2}) = -k < 0 < k = h(0)$, so $h(\frac{1}{2}) < 0 < h(0)$.

If $k < 0$, then $h(0) = k < 0 < -k = h(\frac{1}{2})$, so $h(0) < 0 < h(\frac{1}{2})$.

Thus, in either case, 0 is between $h(0)$ and $h(\frac{1}{2})$.

Since h is continuous on $[0, \frac{1}{2}]$ and the interval $[0, \frac{1}{2}]$ is closed and bounded and 0 is between $h(0)$ and $h(\frac{1}{2})$, then by IVT, there exists $c \in (0, \frac{1}{2})$ such that $h(c) = 0$.

Thus, $0 = h(c) = f(c + \frac{1}{2}) - f(c)$, so $f(c) = f(c + \frac{1}{2})$.

Since $c \in (0, \frac{1}{2})$ and $(0, \frac{1}{2}) \subset [0, \frac{1}{2}]$, then $c \in [0, \frac{1}{2}]$.

Therefore, there exists $c \in [0, \frac{1}{2}]$ such that $f(c) = f(c + \frac{1}{2})$, as desired. □

Proposition 55. ***Fixed Point Theorem***

Let $a, b \in \mathbb{R}$ with $a < b$.

Let $f : [a, b] \to [a, b]$ be a continuous function.

Then there exists $x \in [a, b]$ such that $f(x) = x$.

Proof. Since $a \in [a, b]$ and $[a, b] = rngf$, then $a \in rngf$, so there exists $x_1 \in [a, b]$ such that $f(x_1) = a$.

Since $b \in [a, b]$ and $[a, b] = rngf$, then $b \in rngf$, so there exists $x_2 \in [a, b]$ such that $f(x_2) = b$.

Suppose $x_1 = x_2$.

Then $a = f(x_1) = f(x_2) = b$, so $a = b$.

But, this contradicts the assumption $a < b$.

Hence, $x_1 \neq x_2$, so either $x_1 < x_2$ or $x_1 > x_2$.

Without loss of generality, assume $x_1 < x_2$.

Since $x_1 \in [a, b]$, then $a \leq x_1 \leq b$, so $a \leq x_1$.

Since $x_2 \in [a, b]$, then $a \leq x_2 \leq b$, so $x_2 \leq b$.

Thus, $a \leq x_1$ and $x_2 \leq b$.

Hence, either $x_1 = a$ or $x_2 = b$ or both $a < x_1$ and $x_2 < b$.

We consider these cases separately.

Case 1: Suppose $x_1 = a$.

Then $f(a) = f(x_1) = a$.

Therefore, we have $a \in [a, b]$ and $f(a) = a$.

Case 2: Suppose $x_2 = b$.

Then $f(b) = f(x_2) = b$.

Therefore, we have $b \in [a, b]$ and $f(b) = b$.

Case 3: Suppose $a < x_1$ and $x_2 < b$.

Then $a - x_1 < 0$ and $0 < b - x_2$.

Let $g : [a, b] \to \mathbb{R}$ be a function defined by $g(x) = f(x) - x$.

Since $x_1 \in [a, b]$, then $g(x_1) = f(x_1) - x_1 = a - x_1 < 0$.

Since $x_2 \in [a, b]$, then $g(x_2) = f(x_2) - x_2 = b - x_2 > 0$.

Since f is continuous and the line $y = x$ is continuous, then the difference g is continuous, so g is continuous on $[a, b]$.

Since g is continuous on the closed interval $[a, b]$ and $g(x_1) < 0 < g(x_2)$, then by IVT, there exists $c \in (x_1, x_2)$ such that $g(c) = 0$.

Since $0 = g(c) = f(c) - c$, then $f(c) = c$.

Since $c \in (x_1, x_2)$, then $x_1 < c < x_2$, so $x_1 < c$ and $c < x_2$.

Since $a < x_1$ and $x_1 < c$ and $c < x_2$ and $x_2 < b$, then $a < c < b$, so $c \in (a, b)$.

Since $(a, b) \subset [a, b]$, then $c \in [a, b]$.

Therefore, there exists $c \in [a, b]$ such that $f(c) = c$. □

Exercise 56. Let $f : [-1, 1] \to \mathbb{R}$ be a function defined by $f(x) = x^3 - 3x^2 + 17$.

Then f is not one to one on the interval $[-1, 1]$.

Proof. Since f is a polynomial function, then f is continuous, so f is continuous on $[-1, 1]$.

Since $[-1, 0] \subset [-1, 1]$, then f is continuous on $[-1, 0]$.

Since $f(-1) = 13 < 15 < 17 = f(0)$, then by IVT, there is $c \in (-1, 0)$ such that $f(c) = 15$.

Since $f(c) = 15 = f(1)$, then $f(c) = f(1)$.

Since $c \in (-1, 0)$, then $-1 < c < 0$, so $c < 0$.

Since $c < 0 < 1$, then $c < 1$, so $c \neq 1$.

Since $c \in (-1, 0)$ and $(-1, 0) \subset [-1, 1]$, then $c \in [-1, 1]$.

Thus, there is $c \in [-1, 1]$ such that $c \neq 1$ and $f(c) = f(1)$.

Therefore, f is not one to one. □

Uniform continuity

Exercise 57. Let $f : (0, 6) \to \mathbb{R}$ be a function defined by $f(x) = x^2 + 2x - 5$.
Then f is uniformly continuous on the interval $(0, 6)$.

Proof. To prove f is uniformly continuous on $(0, 6)$, let $\epsilon > 0$ be given.
Let $\delta = \frac{\epsilon}{14}$.
Then $\delta > 0$.
Let $x, y \in (0, 6)$ such that $|x - y| < \delta$.
Then $0 < x < 6$ and $0 < y < 6$, so $0 < x + y < 12$.
Hence, $0 < 2 < x + y + 2 < 14$, so $0 < x + y + 2 < 14$.
Thus, $|x + y + 2| < 14$.
Observe that

$$\begin{aligned} |f(x) - f(y)| &= |(x^2 + 2x - 5) - (y^2 + 2y - 5)| \\ &= |x^2 - y^2 + 2x - 2y| \\ &= |(x - y)(x + y) + 2(x - y)| \\ &= |(x - y)(x + y + 2)| \\ &= |x - y||x + y + 2| \\ &< 14\delta \\ &= \epsilon. \end{aligned}$$

Therefore, $|f(x) - f(y)| < \epsilon$, as desired. $\square$

Exercise 58. Let $f : [2.5, 3] \to \mathbb{R}$ be a function defined by $f(x) = \frac{3}{x-2}$.
Then f is uniformly continuous on the interval $[2.5, 3]$.

Proof. To prove f is uniformly continuous on $[2.5, 3]$, let $\epsilon > 0$ be given.
Let $\delta = \frac{\epsilon}{12}$.
Then $\delta > 0$.
Let $x, y \in [2.5, 3]$ such that $|x - y| < \delta$.
Then $2.5 \leq x \leq 3$ and $2.5 \leq y \leq 3$, so $\frac{1}{2} \leq x - 2 \leq 1$ and $\frac{1}{2} \leq y - 2 \leq 1$.
Hence, $\frac{1}{4} \leq (x - 2)(y - 2) \leq 1$, so $0 < \frac{1}{4} \leq (x - 2)(y - 2)$.
Thus, $0 < \frac{1}{(x-2)(y-2)} \leq 4$.

Observe that

$$\begin{aligned}
|f(x) - f(y)| &= |\frac{3}{x-2} - \frac{3}{y-2}| \\
&= |\frac{3(y-2) - 3(x-2)}{(x-2)(y-2)}| \\
&= |\frac{3y - 3x}{(x-2)(y-2)}| \\
&= |\frac{3x - 3y}{(x-2)(y-2)}| \\
&= 3|\frac{x-y}{(x-2)(y-2)}| \\
&= 3|x-y||\frac{1}{(x-2)(y-2)}| \\
&= 3|x-y|(\frac{1}{(x-2)(y-2)}) \\
&< 3\delta \cdot 4 \\
&= 12\delta \\
&= \epsilon.
\end{aligned}$$

Therefore, $|f(x) - f(y)| < \epsilon$, as desired. □

Exercise 59. Let $f : [3.4, 5] \to \mathbb{R}$ be a function defined by $f(x) = \frac{2}{x-3}$.
Then f is uniformly continuous on the interval $[3.4, 5]$.

Proof. To prove f is uniformly continuous on $[3.4, 5]$, let $\epsilon > 0$ be given.
Let $\delta = 0.08\epsilon$.
Then $\delta > 0$.
Let $x, y \in [3.4, 5]$ such that $|x - y| < \delta$.
Then $3.4 \leq x \leq 5$ and $3.4 \leq y \leq 5$, so $0.4 \leq x - 3 \leq 2$ and $0.4 \leq y - 3 \leq 2$.
Hence, $0.16 \leq (x-3)(y-3) \leq 4$, so $0.16 \leq (x-3)(y-3)$.
Thus, $0 < \frac{1}{(x-3)(y-3)} \leq 6.25$.
Since $x \geq 3.4 > 3$, then $x > 3$, so $x - 3 > 0$.
Since $y \geq 3.4 > 3$, then $y > 3$, so $y - 3 > 0$.

Observe that

$$\begin{aligned}
|f(x) - f(y)| &= |\frac{2}{x-3} - \frac{2}{y-3}| \\
&= |\frac{2(y-3) - 2(x-3)}{(x-3)(y-3)}| \\
&= |\frac{2y - 2x}{(x-3)(y-3)}| \\
&= |\frac{2x - 2y}{(x-3)(y-3)}| \\
&= 2|\frac{x-y}{(x-3)(y-3)}| \\
&= 2|x-y||\frac{1}{(x-3)(y-3)}| \\
&= 2|x-y|\frac{1}{(x-3)(y-3)} \\
&< 2\delta \cdot 6.25 \\
&= 12.5\delta \\
&= \epsilon.
\end{aligned}$$

Therefore, $|f(x) - f(y)| < \epsilon$, as desired. □

Exercise 60. Let $f : (2,7) \to \mathbb{R}$ be a function defined by $f(x) = x^3 - x + 1$.
Then f is uniformly continuous on the interval $(2,7)$.

Proof. To prove f is uniformly continuous on $(2,7)$, let $\epsilon > 0$ be given.

Let $\delta = \frac{\epsilon}{148}$.

Then $\delta > 0$.

Let $x, y \in (2,7)$ such that $|x - y| < \delta$.

Then $2 < x < 7$ and $2 < y < 7$, so $4 < x^2 < 49$ and $4 < y^2 < 49$ and $4 < xy < 49$.

Thus, $|x^2 + xy + y^2 - 1| \leq |x^2| + |xy| + |y^2| + |-1| = x^2 + xy + y^2 + 1 < 49 + 49 + 49 + 1 = 148$.

Observe that

$$\begin{aligned}
|f(x) - f(y)| &= |(x^3 - x + 1) - (y^3 - y + 1)| \\
&= |x^3 - y^3 - x + y| \\
&= |(x-y)(x^2 + xy + y^2) - (x - y)| \\
&= |(x-y)(x^2 + xy + y^2 - 1)| \\
&= |x-y||x^2 + xy + y^2 - 1| \\
&< \delta \cdot 148 \\
&= \epsilon.
\end{aligned}$$

Therefore, $|f(x) - f(y)| < \epsilon$, as desired. □

Exercise 61. Let $a > 0$.
Let $f : [a, \infty) \to \mathbb{R}$ be a function defined by $f(x) = \frac{1}{x}$.
Then f is uniformly continuous on the interval $[a, \infty)$.

Proof. To prove f is uniformly continuous on $[a, \infty)$, let $\epsilon > 0$ be given.
Let $\delta = \epsilon a^2$.
Since $\epsilon > 0$ and $a^2 > 0$, then $\delta > 0$.
Let $x, y \in [a, \infty)$ such that $|x - y| < \delta$.
Then $x \geq a$ and $y \geq a$.
Since $x \geq a > 0$, then $\frac{1}{a} \geq \frac{1}{x} > 0$.
Since $y \geq a > 0$, then $\frac{1}{a} \geq \frac{1}{y} > 0$.
Thus, $\frac{1}{a^2} \geq \frac{1}{xy}$.
Observe that

$$\begin{aligned} |f(x) - f(y)| &= |\frac{1}{x} - \frac{1}{y}| \\ &= |\frac{y - x}{xy}| \\ &= |\frac{x - y}{xy}| \\ &= \frac{1}{xy}|x - y| \\ &< \frac{\delta}{a^2} \\ &= \epsilon. \end{aligned}$$

Therefore, $|f(x) - f(y)| < \epsilon$, as desired. □

Exercise 62. Let $f : [1, \infty) \to \mathbb{R}$ be a function defined by $f(x) = \frac{1}{x^2}$.
Then f is uniformly continuous on the interval $[1, \infty)$.

Proof. To prove f is uniformly continuous on $[1, \infty)$, let $\epsilon > 0$ be given.
Let $\delta = \frac{\epsilon}{2}$.
Then $\delta > 0$.
Let $x, y \in [1, \infty)$ such that $|x - y| < \delta$.
Then $x \geq 1$ and $y \geq 1$, so $xy \geq 1$.
Hence, $1 \geq \frac{1}{xy} > 0$, so $0 < \frac{1}{xy} \leq 1$.
Observe that

$$\begin{aligned} |\frac{1}{x} - \frac{1}{y}| &= |\frac{y - x}{xy}| \\ &= |\frac{x - y}{xy}| \\ &= \frac{1}{xy}|x - y| \\ &< \delta. \end{aligned}$$

Thus, $|\frac{1}{x} - \frac{1}{y}| < \delta$.
Since $x \geq 1$, then $1 \geq \frac{1}{x} > 0$.
Since $y \geq 1$, then $1 \geq \frac{1}{y} > 0$.
Thus, $2 \geq \frac{1}{x} + \frac{1}{y} > 0$, so $0 < \frac{1}{x} + \frac{1}{y} \leq 2$.
Observe that

$$\begin{aligned}
|f(x) - f(y)| &= |\frac{1}{x^2} - \frac{1}{y^2}| \\
&= |(\frac{1}{x} - \frac{1}{y})(\frac{1}{x} + \frac{1}{y})| \\
&= |\frac{1}{x} - \frac{1}{y}||\frac{1}{x} + \frac{1}{y}| \\
&= |\frac{1}{x} - \frac{1}{y}|(\frac{1}{x} + \frac{1}{y}) \\
&< 2\delta \\
&= \epsilon.
\end{aligned}$$

Therefore, $|f(x) - f(y)| < \epsilon$, as desired. □

Exercise 63. Let m and b be fixed real numbers.
Let I be an interval.
Then the linear function $f(x) = mx + b$ is uniformly continuous on I.

Proof. To prove f is uniformly continuous on I, let $\epsilon > 0$ be given.
We must prove there exists $\delta > 0$ such that for all $x, y \in I$, if $|x - y| < \delta$, then $|f(x) - f(y)| < \epsilon$.
Either $m = 0$ or $m \neq 0$.
We consider these cases separately.
Case 1: Suppose $m = 0$.
Then $f(x) = 0x + b = b$ for all $x \in \mathbb{R}$.
Let $\delta = \epsilon$.
Then $\delta > 0$.
Let $x, y \in I$.
Then $|f(x) - f(y)| = |b - b| = 0 < \epsilon$.
Hence, the implication if $|x - y| < \delta$, then $|f(x) - f(y)| < \epsilon$ is trivially true.
Case 2: Suppose $m \neq 0$.
Let $\delta = \frac{\epsilon}{|m|}$.
Since $m \neq 0$, then $|m| > 0$.
Since $\epsilon > 0$ and $|m| > 0$, then $\delta > 0$.
Let $x, y \in I$ such that $|x - y| < \delta$.

Then

$$
\begin{aligned}
|f(x) - f(y)| &= |(mx+b) - (my+b)| \\
&= |mx + b - my - b| \\
&= |mx - my| \\
&= |m(x-y)| \\
&= |m||x-y| \\
&< |m|\delta \\
&= |m| \cdot \frac{\epsilon}{|m|} \\
&= \epsilon.
\end{aligned}
$$

Therefore, $|f(x) - f(y)| < \epsilon$, as desired. ☐

Exercise 64. Let $f : (0, \infty) \to \mathbb{R}$ be the function given by $f(x) = x^2$.
Then f is not uniformly continuous on $(0, \infty)$.

Proof. To prove f is not uniformly continuous on $(0, \infty)$, we prove $(\exists \epsilon > 0)(\forall \delta > 0)(\exists x, y \in (0, \infty))(|x - y| < \delta \wedge |f(x) - f(y)| \geq \epsilon)$.
Let $\epsilon = 1$.
Let $\delta > 0$ be given.
Let $\alpha = \min\{2, \delta\}$.
Then $\alpha \leq 2$ and $\alpha \leq \delta$ and $\alpha > 0$.
Let $x = \frac{1}{\alpha} - \frac{\alpha}{5}$.
Let $y = x + \frac{\alpha}{2}$.
Since $0 < \alpha \leq 2$, then $0 < \alpha^2 \leq 4 < 5$, so $0 < \alpha^2 < 5$.
Hence, $\frac{\alpha}{5} < \frac{1}{\alpha}$, so $\frac{1}{\alpha} - \frac{\alpha}{5} > 0$.
Thus, $x > 0$, so $x \in (0, \infty)$.
Since $x > 0$ and $\alpha > 0$, then $y > 0$, so $y \in (0, \infty)$.
Since $|x - y| = |y - x| = |\frac{\alpha}{2}| = \frac{\alpha}{2} < \alpha \leq \delta$, then $|x - y| < \delta$.
Since $4 < 5$ and $\alpha > 0$, then $4\alpha < 5\alpha$, so $\frac{\alpha}{5} < \frac{\alpha}{4}$.
Hence, $\frac{-\alpha}{5} > \frac{-\alpha}{4}$, so $\frac{1}{\alpha} - \frac{\alpha}{5} > \frac{1}{\alpha} - \frac{\alpha}{4}$.
Thus, $x > \frac{1}{\alpha} - \frac{\alpha}{4}$, so $x + \frac{\alpha}{4} > \frac{1}{\alpha}$.
Observe that

$$\begin{aligned}
|f(x)-f(y)| &= |x^2-y^2| \\
&= |(x-y)(x+y)| \\
&= |x-y||x+y| \\
&= |x-y|(x+y) \\
&= \frac{\alpha}{2}[x+(x+\frac{\alpha}{2})] \\
&= \frac{\alpha}{2}(2x+\frac{\alpha}{2}) \\
&= \alpha(x+\frac{\alpha}{4}) \\
&> \alpha\cdot\frac{1}{\alpha} \\
&= 1.
\end{aligned}$$

Therefore, $|f(x)-f(y)| > 1 = \epsilon$, so f is not uniformly continuous on the interval $(0,\infty)$. □

Exercise 65. Let $f:(0,\infty)\to\mathbb{R}$ be the function given by $f(x)=\frac{1}{x^2}$.

Then f is not uniformly continuous on $(0,\infty)$.

Proof. To prove f is not uniformly continuous on $(0,\infty)$, we prove $(\exists\epsilon>0)(\forall\delta>0)(\exists x,y\in(0,\infty))(|x-y|<\delta\wedge|f(x)-f(y)|\geq\epsilon)$.

Let $\epsilon=1$.

Let $\delta>0$ be given.

Let $\alpha=\min\{1,\delta\}$.

Then $\alpha\leq 1$ and $\alpha\leq\delta$ and $\alpha>0$.

Let $x=\alpha$.

Let $y=\frac{\alpha}{2}$.

Then $x>0$ and $y>0$, so $x\in(0,\infty)$ and $y\in(0,\infty)$.

Since $|x-y|=|\alpha-\frac{\alpha}{2}|=\frac{\alpha}{2}<\alpha\leq\delta$, then $|x-y|<\delta$.

Since $0<\alpha\leq 1$, then $0<\alpha^2\leq 1<3$, so $0<\alpha^2<3$.

Hence, $1<\frac{3}{\alpha^2}$.

Observe that

$$\begin{aligned}
|f(x)-f(y)| &= |f(\alpha)-f(\frac{\alpha}{2})| \\
&= |\frac{1}{\alpha^2}-\frac{4}{\alpha^2}| \\
&= \frac{3}{\alpha^2} \\
&> 1.
\end{aligned}$$

Therefore, $|f(x)-f(y)| > 1 = \epsilon$, so f is not uniformly continuous on the interval $(0,\infty)$. □

Proposition 66. ***the sum of uniformly continuous functions is uniformly continuous***

Let f and g be real valued functions defined on a set E.

If f is uniformly continuous on E and g is uniformly continuous on E, then $f+g$ is uniformly continuous on E.

Proof. To prove the function $f+g$ is uniformly continuous on E, let $\epsilon > 0$ be given.

We must prove there exists $\delta > 0$ such that for all $x, y \in E$, if $|x-y| < \delta$, then $|(f+g)(x) - (f+g)(y)| < \epsilon$.

Since $\epsilon > 0$, then $\frac{\epsilon}{2} > 0$.

Since f is uniformly continuous on E and $\frac{\epsilon}{2} > 0$, then there exists $\delta_1 > 0$ such that for all $x, y \in E$, if $|x-y| < \delta_1$, then $|f(x) - f(y)| < \frac{\epsilon}{2}$.

Since g is uniformly continuous on E and $\frac{\epsilon}{2} > 0$, then there exists $\delta_2 > 0$ such that for all $x, y \in E$, if $|x-y| < \delta_2$, then $|g(x) - g(y)| < \frac{\epsilon}{2}$.

Let $\delta = \min\{\delta_1, \delta_2\}$.

Then $\delta \leq \delta_1$ and $\delta \leq \delta_2$ and $\delta > 0$.

Let $x, y \in E$ such that $|x-y| < \delta$.

Since $|x-y| < \delta \leq \delta_1$, then $|x-y| < \delta_1$, so $|f(x) - f(y)| < \frac{\epsilon}{2}$.

Since $|x-y| < \delta \leq \delta_2$, then $|x-y| < \delta_2$, so $|g(x) - g(y)| < \frac{\epsilon}{2}$.

Therefore,

$$\begin{aligned}
|(f+g)(x) - (f+g)(y)| &= |f(x) + g(x) - (f(y) + g(y))| \\
&= |f(x) + g(x) - f(y) - g(y)| \\
&= |f(x) - f(y) + g(x) - g(y)| \\
&\leq |f(x) - f(y)| + |g(x) - g(y)| \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&= \epsilon.
\end{aligned}$$

∎

Exercise 67. Let f and g be real valued functions defined on a set E.

If f is uniformly continuous on E and g is uniformly continuous on E, show that fg is not necessarily uniformly continuous on E.

Solution. Let $f : (0, \infty) \to \mathbb{R}$ be the identity function $f(x) = x$.

Let g be a function such that $g = f$.

Then $g(x) = f(x) = x$ for all $x \in (0, \infty)$.

Since f is a linear function defined on the interval $(0, \infty)$, then f is uniformly continuous on $(0, \infty)$.

Since $g = f$, then g is uniformly continuous on $(0, \infty)$.

However, the function fg given by $(fg)(x) = f(x)g(x) = xx = x^2$ is not uniformly continuous on $(0, \infty)$. ∎

Proposition 68. ***the product of uniformly continuous bounded functions is uniformly continuous***

Let f and g be bounded real valued functions defined on a set E.

If f is uniformly continuous on E and g is uniformly continuous on E, then fg is uniformly continuous on E.

Proof. To prove the function fg is uniformly continuous on E, let $\epsilon > 0$ be given.

We must prove there exists $\delta > 0$ such that for all $x, y \in E$, if $|x - y| < \delta$, then $|(fg)(x) - (fg)(y)| < \epsilon$.

Since f is bounded in $\mathbb{R}$, then there exists a real number $K > 0$ such that $|f(x)| < K$ for all $x \in E$.

Since g is bounded in $\mathbb{R}$, then there exists a real number $M > 0$ such that $|g(x)| < M$ for all $x \in E$.

Since $\epsilon > 0$ and $M > 0$, then $\frac{\epsilon}{2M} > 0$.

Since f is uniformly continuous on E and $\frac{\epsilon}{2M} > 0$, then there exists $\delta_1 > 0$ such that for all $x, y \in E$, if $|x - y| < \delta_1$, then $|f(x) - f(y)| < \frac{\epsilon}{2M}$.

Since $\epsilon > 0$ and $K > 0$, then $\frac{\epsilon}{2K} > 0$.

Since g is uniformly continuous on E and $\frac{\epsilon}{2K} > 0$, then there exists $\delta_2 > 0$ such that for all $x, y \in E$, if $|x - y| < \delta_2$, then $|g(x) - g(y)| < \frac{\epsilon}{2K}$.

Let $\delta = \min\{\delta_1, \delta_2\}$.

Then $\delta \leq \delta_1$ and $\delta \leq \delta_2$ and $\delta > 0$.

Let $x, y \in E$ such that $|x - y| < \delta$.

Since $|x - y| < \delta \leq \delta_1$, then $|x - y| < \delta_1$, so $|f(x) - f(y)| < \frac{\epsilon}{2M}$.

Since $|x - y| < \delta \leq \delta_2$, then $|x - y| < \delta_2$, so $|g(x) - g(y)| < \frac{\epsilon}{2K}$.

Since $x \in E$, then $|f(x)| < K$.

Since $y \in E$, then $|g(y)| < M$.

Therefore,

$$
\begin{aligned}
|(fg)(x) - (fg)(y)| &= |f(x)g(x) - f(y)g(y)| \\
&= |f(x)(g(x) - g(y)) + f(x)g(y) - f(y)g(y)| \\
&= |f(x)(g(x) - g(y)) + g(y)(f(x) - f(y))| \\
&\leq |f(x)(g(x) - g(y))| + |g(y)(f(x) - f(y))| \\
&= |f(x)||g(x) - g(y)| + |g(y)||f(x) - f(y)| \\
&< K \cdot \frac{\epsilon}{2K} + M \cdot \frac{\epsilon}{2M} \\
&= \epsilon.
\end{aligned}
$$

□

Proposition 69. ***composition of uniformly continuous functions is uniformly continuous***

Let f and g be real valued functions of a real variable.

If f is uniformly continuous and g is uniformly continuous, then $g \circ f$ is uniformly continuous.

Proof. Suppose f is uniformly continuous and g is uniformly continuous.

To prove the function $g \circ f$ is uniformly continuous, let $\epsilon > 0$ be given.

Since g is uniformly continuous, then there exists $\delta_1 > 0$ such that for all $x, y \in dom g$, if $|x - y| < \delta_1$, then $|g(x) - g(y)| < \epsilon$.

Since f is uniformly continuous and $\delta_1 > 0$, then there exists $\delta > 0$ such that for all $x, y \in dom f$, if $|x - y| < \delta$, then $|f(x) - f(y)| < \delta_1$.

Let $x, y \in dom(g \circ f)$ such that $|x - y| < \delta$.

Since $x \in dom(g \circ f)$ and $dom(g \circ f) = \{x \in dom f : f(x) \in dom g\}$, then $x \in dom f$ and $f(x) \in dom g$.

Since $y \in dom(g \circ f)$ and $dom(g \circ f) = \{x \in dom f : f(x) \in dom g\}$, then $y \in dom f$ and $f(y) \in dom g$.

Since $x \in dom f$ and $y \in dom f$ and $|x - y| < \delta$, then $|f(x) - f(y)| < \delta_1$.

Since $f(x) \in dom g$ and $f(y) \in dom g$ and $|f(x) - f(y)| < \delta_1$, then $|g(f(x) - g(f(y))| < \epsilon$.

Therefore, $|(g \circ f)(x) - (g \circ f)(y)| < \epsilon$, as desired. □