Continuous functions Notes

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Sets of Numbers

$\mathbb{R} =$ set of all real numbers
$\mathbb{R}^+ = \{x \in \mathbb{R} : x > 0\} =$ set of all positive real numbers
$\mathbb{R}^* = \{x \in \mathbb{R} : x \neq 0\} = (-\infty, 0) \cup (0, \infty) =$ set of all nonzero real numbers
$\mathbb{R}_+ = \{x \in \mathbb{R} : x \geq 0\} = [0, \infty) = [0, \infty[=$ set of all nonnegative real numbers

Continuity

A function f is continuous at a point c iff $f(x)$ is arbitrarily close to $f(c)$ whenever x is sufficiently close to c.

Intuitively, this means f is continuous at c if there is no hole in the graph of f at c and there is no break in the graph of f in the immediate vicinity of c.

Definition 1. ϵ, δ **definition of continuity of a function at a point**

Let (E_1, d_1) and (E_2, d_2) be metric spaces.

Let $f : E_1 \to E_2$ be a function.

Then f **is continuous at** $c \in E_1$ iff for every real number $\epsilon > 0$, there exists a real number $\delta > 0$, such that $d_2(f(x), f(c)) < \epsilon$ whenever $x \in E_1$ and $d_1(x, c) < \delta$.

Therefore, f **is continuous at** $c \in E_1$ iff for every real number $\epsilon > 0$, there exists a real number $\delta > 0$ such that if $x \in E_1$ and $d_1(x, c) < \delta$, then $d_2(f(x), f(c)) < \epsilon$.

Therefore, f **is continuous at** $c \in E_1$ iff

$(\forall \epsilon > 0)(\exists \delta > 0)(\forall x \in E_1)(d_1(x, c) < \delta \to d_2(f(x), f(c)) < \epsilon)$.

Definition 2. ϵ, δ **definition of continuity of a real valued function at a point**

Let $E \subset \mathbb{R}$.

Let $f : E \to \mathbb{R}$ be a function.

Then f is **continuous at** $c \in E$ iff $(\forall \epsilon > 0)(\exists \delta > 0)(\forall x \in E)(|x - c| < \delta \to |f(x) - f(c)| < \epsilon)$.

If f is not continuous at $c \in E$, then we say f is **discontinuous at** c.

Observe that

$$(\forall \epsilon > 0)(\exists \delta > 0)(\forall x \in E)(|x - c| < \delta \to |f(x) - f(c)| < \epsilon) \Leftrightarrow$$
$$(\forall \epsilon > 0)(\exists \delta > 0)(\forall x \in E)(x \in N(c;\delta) \to f(x) \in N(f(c);\epsilon)) \Leftrightarrow$$
$$(\forall \epsilon > 0)(\exists \delta > 0)(\forall x \in E \cap N(c;\delta))(f(x) \in N(f(c);\epsilon)).$$

Therefore, f is continuous at $c \in E$ iff for every ϵ neighborhood of $f(c)$, there exists a δ neighborhood of c such that for each $x \in E$, if $x \in N(c;\delta)$, then $f(x) \in N(f(c);\epsilon)$.

Observe that

$$\neg(\forall \epsilon > 0)(\exists \delta > 0)(\forall x \in E)(|x - c| < \delta \to |f(x) - f(c)| < \epsilon) \Leftrightarrow$$
$$(\exists \epsilon > 0)(\forall \delta > 0)(\exists x \in E)(|x - c| < \delta \wedge |f(x) - f(c)| \geq \epsilon).$$

Therefore, f is discontinuous at $c \in E$ iff $(\exists \epsilon > 0)(\forall \delta > 0)(\exists x \in E)(|x-c| < \delta \wedge |f(x) - f(c)| \geq \epsilon)$.

Let $E \subset \mathbb{R}$.
Let $f : E \to \mathbb{R}$ be a function.
Suppose f is continuous at c.
Then $c \in E$, so $c \in domf$.
Therefore, if f is continuous at c, then $c \in domf$.
Hence, if $c \notin domf$, then f is not continuous at c.
Therefore, if $c \notin domf$, then f is discontinuous at c.

Proposition 3. ***characterization of continuity at a point***
Let $E \subset \mathbb{R}$.
Let $f : E \to \mathbb{R}$ be a function and $c \in E$. Then
1. If c is not an accumulation point of E, then f is continuous at c.
2. If c is an accumulation point of E, then f is continuous at c iff the limit of f at c exists and $\lim_{x\to c} f(x) = f(c)$.

Thus, if $c \in domf$ and c is an accumulation point of $domf$ and the limit of f at c exists and $\lim_{x\to c} f(x) = f(c)$, then f is continuous at c.

Since the existence of a limit of f at c implies c is an accumulation point of $domf$, then if $c \in domf$ and the limit of f at c exists and $\lim_{x\to c} f(x) = f(c)$, then f is continuous at c.

Therefore, f is continuous at c if
1. $c \in domf$.
2. limit of f at c exists.
3. $\lim_{x\to c} f(x) = f(c)$.

Theorem 4. ***sequential characterization of continuity***
Let $E \subset \mathbb{R}$.
Let $f : E \to \mathbb{R}$ be a function.
Let $c \in E$.
Then f is continuous at c iff for every sequence (x_n) of points in E such that $\lim_{n\to\infty} x_n = c$, $\lim_{n\to\infty} f(x_n) = f(c)$.

Therefore, f is discontinuous at c iff there exists a sequence (x_n) of points in E such that $\lim_{n\to\infty} x_n = c$ and $\lim_{n\to\infty} f(x_n) \neq f(c)$.

A function continuous on a set is continuous at each point of the set.

Definition 5. continuity of a function defined on a set
A function f is **continuous on a set** $E \subset dom f$ iff f is continuous at c for all $c \in E$.

Definition 6. continuous function
A function f is **continuous** iff f is continuous on $dom f$.

Therefore, a function f is continuous iff f is continuous at c for all $c \in dom f$.
Therefore, a function f is continuous iff $(\forall c \in dom f)(\forall \epsilon > 0)(\exists \delta > 0)(\forall x \in E)(|x - c| < \delta \to |f(x) - f(c)| < \epsilon)$.
Observe that δ depends on both $c \in dom f$ and ϵ.

Example 7. every constant function is continuous
Let $k \in \mathbb{R}$.
The function given by $f(x) = k$ is continuous.

Example 8. identity function is continuous
The function given by $f(x) = x$ is continuous.

Example 9. square function is continuous
The function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$ is continuous.

Example 10. The function $f : (0, \infty) \to \mathbb{R}$ defined by $f(x) = \frac{1}{x}$ is continuous.

Example 11. absolute value function is continuous
The function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = |x|$ is continuous.

Example 12. square root function is continuous
The function $f : [0, \infty) \to \mathbb{R}$ defined by $f(x) = \sqrt{x}$ is continuous.

Example 13. function with a removable discontinuity
Let $f : \mathbb{R} \to \mathbb{R}$ be a function defined by

$$f(x) = \begin{cases} x + 1 & \text{if } x \neq 1 \\ 5 & \text{if } x = 1 \end{cases}$$

Then f is discontinuous at 1.
If f is redefined so that $f(1) = 2$, then $\lim_{x\to 1} f(x) = 2 = f(1)$, so f would be continuous at 1.
Therefore, we can remove the discontinuity by redefining f at 1 so that f is continuous at 1, so 1 is a removable discontinuity.

Example 14. function with a jump discontinuity

Let $f : \mathbb{R} \to \mathbb{R}$ be a function defined by

$$f(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$

Then f is discontinuous at 0.

Example 15. unbounded function, infinite discontinuity

The function $f : (0, \infty) \to \mathbb{R}$ defined by $f(x) = \frac{1}{x}$ is discontinuous at 0 since $0 \notin (0, \infty)$, the domain of f.

Let $r \in \mathbb{R}$ be arbitrary.

Let $g : [0, \infty) \to \mathbb{R}$ be a function defined by

$$g(x) = \begin{cases} \frac{1}{x} & \text{if } x > 0 \\ r & \text{if } x = 0 \end{cases}$$

Then g is discontinuous at 0, so there is no way to extend f to make f continuous at 0.

Therefore, 0 is a non removable discontinuity of f.

Example 16. oscillating function

The function $f : \mathbb{R}^* \to \mathbb{R}$ defined by $f(x) = \sin(\frac{1}{x})$ is discontinuous at 0 since $0 \notin \mathbb{R}^*$, the domain of f.

Let $r \in \mathbb{R}$ be arbitrary.

Let $g : \mathbb{R} \to \mathbb{R}$ be a function defined by

$$g(x) = \begin{cases} \sin(\frac{1}{x}) & \text{if } x \neq 0 \\ r & \text{if } x = 0 \end{cases}$$

Since 0 is an accumulation point of $\mathbb{R}$, but $\lim_{x \to 0} \sin(\frac{1}{x})$ does not exist in $\mathbb{R}$, then g is discontinuous at 0.

Hence, there is no way to define f at 0 to make f continuous at 0.

Therefore, 0 is a non removable discontinuity of f.

Example 17. Dirichlet function is discontinuous everywhere

Let $f : \mathbb{R} \to \mathbb{R}$ be a function defined by

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$$

Then f is not continuous at any point in its domain.

Example 18. Thomae's (popcorn) function
Let $f : (0, 1) \to \mathbb{R}$ be a function defined by

$$f(x) = \begin{cases} \frac{1}{m} & \text{if } k, m \in \mathbb{N} \text{ and } \gcd(k, m) = 1 \text{ and } x = \frac{k}{m} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$$

Then f is

Proposition 19. ***restriction of a continuous function is continuous***
Let f be a real valued function of a real variable.
Let g be a restriction of f to a nonempty set $E \subset dom f$.
If f is continuous, then the restriction g is continuous.

Algebraic properties of continuous functions

Theorem 20. *Let $\lambda \in \mathbb{R}$.*
Let f be a real valued function.
Let $c \in dom f$.
If f is continuous at c, then λf is continuous at c.

Corollary 21. ***scalar multiple of a continuous function is continuous***
Let $\lambda \in \mathbb{R}$.
Let f be a real valued function.
If f is continuous, then λf is continuous.

Theorem 22. *Let f and g be real valued functions.*
Let $c \in dom f \cap dom g$.
If f is continuous at c and g is continuous at c, then $f + g$ is continuous at c.

Corollary 23. ***sum of continuous functions is continuous***
Let f and g be real valued functions.
If f is continuous and g is continuous, then $f + g$ is continuous.

Corollary 24. *Let f and g be real valued functions.*
Let $c \in dom f \cap dom g$.
If f is continuous at c and g is continuous at c, then $f - g$ is continuous at c.

Corollary 25. ***difference of continuous functions is continuous***
Let f and g be real valued functions.
If f is continuous and g is continuous, then $f - g$ is continuous.

Theorem 26. *Let f and g be real valued functions.*
Let $c \in dom f \cap dom g$.
If f is continuous at c and g is continuous at c, then fg is continuous at c.

Corollary 27. ***product of continuous functions is continuous***

Let f and g be real valued functions.

If f is continuous and g is continuous, then fg is continuous.

Theorem 28. *Let f and g be real valued functions.*

Let $c \in dom f \cap dom g$.

If f is continuous at c and g is continuous at c and $g(c) \neq 0$, then $\frac{f}{g}$ is continuous at c.

Corollary 29. ***quotient of continuous functions is continuous wherever defined***

Let f and g be real valued functions.

If f is continuous and g is continuous, then $\frac{f}{g}$ is continuous for all $x \in dom f \cap dom g$ such that $g(x) \neq 0$.

Theorem 30. ***polynomial functions are continuous***

Every polynomial function is continuous.

Theorem 31. ***rational functions are continuous wherever defined***

Let r be a rational function defined by $r(x) = \frac{p(x)}{q(x)}$ such that p and q are polynomial functions.

Then r is continuous for all $x \in \mathbb{R}$ such that $q(x) \neq 0$.

Theorem 32. *Let f and g be real valued functions of a real variable.*

If f is continuous at c and g is continuous at $f(c)$, then $g \circ f$ is continuous at c.

Corollary 33. ***composition of continuous functions is continuous***

Let f and g be real valued functions of a real variable.

If f is continuous and g is continuous, then $g \circ f$ is continuous.

Proposition 34. ***If f is a continuous function, then so is $|f|$.***

Let $f : E \to \mathbb{R}$ be a function.

Let $|f| : E \to \mathbb{R}$ be a function defined by $|f|(x) = |f(x)|$.

If f is continuous, then $|f|$ is continuous.

Proposition 35. ***If f is a continuous function, then so is $\sqrt{f}$.***

Let $f : E \to \mathbb{R}$ be a function such that $f(x) \geq 0$ for all $x \in E$.

Let $\sqrt{f}$ be a function defined by $\sqrt{f}(x) = \sqrt{f(x)}$ for all $x \in E$ such that $f(x) \geq 0$.

If f is continuous, then $\sqrt{f}$ is continuous.

Transcendental functions like $e^x, \ln x, \sin x, \cos x$ are continuous wherever defined.

Definition 36. Continuous from left and right

Let $a, r \in \mathbb{R}$.

Let a be fixed.

Let $f : \mathbb{R} \to \mathbb{R}$ be a function defined on $[a, a + r)$ for some $r > 0$.

Then f is **right continuous at a** iff $\lim_{x \to a^+} f(x) = f(a)$.

Let $f : \mathbb{R} \to \mathbb{R}$ be a function defined on $(a - r, a]$ for some $r > 0$.
Then f is **left continuous at a** iff $\lim_{x\to a^-} f(x) = f(a)$.

Continuous functions on compact sets

Lemma 37. ***Bolzano-Weierstrass property of compact sets***
Let $E \subset \mathbb{R}$ be a closed bounded set.
Then every sequence in E has a subsequence (y_n) in E such that $\lim_{n\to\infty} y_n \in E$.

Theorem 38. ***Boundedness Theorem***
Every real valued function continuous on a closed bounded set is bounded.

Let $f : [a, b] \to \mathbb{R}$ be a continuous function.
Since $[a, b]$ is closed and bounded, then by the boundedness theorem, f is bounded.
Therefore, there exists $B \in \mathbb{R}$ such that $|f(x)| \leq B$ for all $x \in [a, b]$.

Theorem 39. ***Extreme Value Theorem***
Every real valued function continuous on a nonempty closed bounded set attains a maximum and minimum on the set.

Let $a, b \in \mathbb{R}$ with $a < b$.
Then by the density of $\mathbb{R}$, the interval $[a, b]$ is not empty.
Let f be a real valued function continuous on the closed interval $[a, b]$.
Since the closed bounded interval $[a, b]$ is compact, then by EVT, f attains a maximum and minimum on $[a, b]$.
Since f has a maximum on $[a, b]$, then there exists $M \in [a, b]$ such that $f(x) \leq f(M)$ for all $x \in [a, b]$.
Since f has a minimum on $[a, b]$, then there exists $m \in [a, b]$ such that $f(m) \leq f(x)$ for all $x \in [a, b]$.
Therefore, there exist $m, M \in [a, b]$ such that $f(m) \leq f(x) \leq f(M)$ for all $x \in [a, b]$.
Therefore, f is bounded.

Lemma 40. *Let f be a continuous real valued function.*
Let x_0, c, d be real numbers such that $x_0 \in dom f$ and $c < f(x_0) < d$.
Then there exists a positive real number δ such that $c < f(x) < d$ for all $x \in N(x_0; \delta) \cap dom f$.

Theorem 41. ***Intermediate Value Theorem***
Let $a, b \in \mathbb{R}$.
Let f be a real valued function continuous on the closed interval $[a, b]$.
For every real number k such that $f(a) < k < f(b)$, there exists $c \in (a, b)$ such that $f(c) = k$.

Therefore, a continuous function $f : [a, b] \to \mathbb{R}$ has all values between $f(a)$ and $f(b)$.

Therefore, the graph of a function continuous on a closed and bounded interval $[a, b]$ must pass through every horizontal line $y = y_0$, where $f(a) < y_0 < f(b)$. It doesn't jump over any such line, so there are no 'breaks' or 'missing points'.

Lemma 42. *Let I be an interval and $a \in I$ and $b \in I$ and $a < b$.*
Then $[a, b] \subset I$.

Theorem 43. ***intervals are preserved by continuous functions***
Let f be a real valued function continuous on an interval I.
Then $f(I)$ is an interval.

Therefore, the image of an interval under a continuous function is an interval.

Uniform continuity is a condition that guarantees continuity of a function f in which δ depends on ϵ only and not on a particular point $c \in E$.

Uniform continuity

Definition 44. uniform continuity
Let $E \subset \mathbb{R}$.
Let $f : E \to \mathbb{R}$ be a function.
Then f is **uniformly continuous on** E iff $(\forall \epsilon > 0)(\exists \delta > 0)(\forall x \in E)(\forall y \in E)(|x - y| < \delta \to |f(x) - f(y)| < \epsilon)$.

The definition of uniform continuity implies that δ depends on ϵ and not on a specific point in E.

Thus, δ chosen is independent of a specific point c in the domain of a function f, so the same δ works for any point $c \in dom f$.

Let $E \subset \mathbb{R}$.
Let $f : E \to \mathbb{R}$ be a function.
Then f is not uniformly continuous on E iff $(\exists \epsilon > 0)(\forall \delta > 0)(\exists x \in E)(\exists y \in E)(|x - y| < \delta \land |f(x) - f(y)| \geq \epsilon)$.

Example 45. Let $a > 0$.
Let $f : (0, a) \to \mathbb{R}$ be the function given by $f(x) = x^2$.
Then f is uniformly continuous on the interval $(0, a)$.

Example 46. Let $f : \mathbb{R} \to \mathbb{R}$ be the function given by $f(x) = x^2$.
Then f is not uniformly continuous on $\mathbb{R}$.

Example 47. Let $f : (1, \infty) \to \mathbb{R}$ be a function defined by $f(x) = \frac{1}{x}$.
Then f is uniformly continuous on the interval $(1, \infty)$.

Example 48. Let $f : (0,1) \to \mathbb{R}$ be the function given by $f(x) = \frac{1}{x}$.
Then f is not uniformly continuous on the interval $(0,1)$.

Proposition 49. ***uniform continuity implies continuity***
Let $E \subset \mathbb{R}$.
Let $f : E \to \mathbb{R}$ be a function.
If f is uniformly continuous on E, then f is continuous on E.

Example 50. continuity does not necessarily imply uniform continuity
Let $f : (0,1) \to \mathbb{R}$ be the function given by $f(x) = \frac{1}{x}$.
Then f is not uniformly continuous on $(0,1)$.
Since the reciprocal function is continuous on $(0,\infty)$ and f is a restriction of the reciprocal function to $(0,1) \subset (0,\infty)$, then f is continuous on $(0,1)$.
Since f is continuous on $(0,1)$, but f is not uniformly continuous on $(0,1)$, then f is continuous on $(0,1)$ does not imply f is uniformly continuous on $(0,1)$.

Lemma 51. *Let $E \subset \mathbb{R}$.*
Let $f : E \to \mathbb{R}$ be a function.
If f is not uniformly continuous on E, then there exist $\epsilon_1 > 0$ and sequences (x_n) and (y_n) in E such that $\lim_{n\to\infty}(x_n - y_n) = 0$ and $|f(x_n) - f(y_n)| \geq \epsilon_1$ for all $n \in \mathbb{N}$.

Theorem 52. ***Heine-Cantor Uniform Continuity Theorem***
Let $E \subset \mathbb{R}$.
Let $f : E \to \mathbb{R}$ be a function.
If f is continuous on E and E is compact, then f is uniformly continuous on E.

Therefore, if f is a continuous function on a compact set E, then f is uniformly continuous on E.
Thus, a function continuous on a compact set is uniformly continuous.

Let $f : [a,b] \to \mathbb{R}$ be a continuous function.
Since the closed bounded interval $[a,b]$ is compact, then f is uniformly continuous on $[a,b]$.
Therefore, every real valued function continuous on a closed bounded interval is uniformly continuous.