Differentiation of real valued functions Theory

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June 29, 2021

Derivative of a real valued function

Theorem 1. ***Alternate definition of derivative of a function***

Let E be an open subset of $\mathbb{R}$.

Let $f : E \to \mathbb{R}$ be a function.

Let $c \in E$.

Since E is open, then c is an interior point of E, so there exists $\delta > 0$ such that $N(c;\delta) \subset E$.

Let $Q : (0,\delta) \to \mathbb{R}$ be a function defined by $Q(h) = \frac{f(c+h)-f(c)}{h}$.

If $\lim_{h\to 0} \frac{f(c+h)-f(c)}{h}$ exists, then f is differentiable at c and $f'(c) = \lim_{h\to 0} \frac{f(c+h)-f(c)}{h}$.

Proof. Suppose $\lim_{h\to 0} \frac{f(c+h)-f(c)}{h}$ exists.

Let $L = \lim_{h\to 0} \frac{f(c+h)-f(c)}{h}$.

Then $L \in \mathbb{R}$.

Since c is an interior point of E, then c is an accumulation point of E.

To prove f is differentiable at c, we prove $\lim_{x\to c} \frac{f(x)-f(c)}{x-c} = L$.

Let $\epsilon > 0$ be given.

Since $\lim_{h\to 0} \frac{f(c+h)-f(c)}{h} = L$, then there exists $\gamma > 0$ such that for all h, if $0 < |h| < \gamma$, then $|\frac{f(c+h)-f(c)}{h} - L| < \epsilon$.

Let $x \in E$ such that $0 < |x - c| < \gamma$.

Let $h = x - c$.

Then $0 < |h| < \gamma$, so $|\frac{f(c+h)-f(c)}{h} - L| < \epsilon$.

Since $c + h = x$, then $|\frac{f(x)-f(c)}{x-c} - L| < \epsilon$.

Thus, $\lim_{x\to c} \frac{f(x)-f(c)}{x-c} = L$.

Therefore, f is differentiable at c and $f'(c) = \lim_{x\to c} \frac{f(x)-f(c)}{x-c} = L = \lim_{h\to 0} \frac{f(c+h)-f(c)}{h}$. □

Proposition 2. ***the derivative of a constant is zero***

Let $k \in \mathbb{R}$ be fixed.

Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by $f(x) = k$.

Then $f'(x) = 0$ for all $x \in \mathbb{R}$.

Proof. We prove $f'(x) = 0$ for all $x \in \mathbb{R}$ using the definition of derivative.

Let $c \in \mathbb{R}$ be given.

We must prove $f'(c) = 0$.

Let $q : \mathbb{R} - \{c\} \to \mathbb{R}$ be a function defined by $q(x) = \frac{f(x)-f(c)}{x-c}$ for all $x \in \mathbb{R} - \{c\}$.

Since $c \in \mathbb{R}$ and every real number is an accumulation point of $\mathbb{R}$, then c is an accumulation point of $\mathbb{R}$, so c is an accumulation point of $\mathbb{R} - \{c\}$, the domain of q.

For $x \in \mathbb{R} - \{c\}$, we have $x \in \mathbb{R}$ and $x \neq c$, so $x - c \neq 0$.

Observe that

$$\begin{aligned}
\lim_{x\to c} q(x) &= \lim_{x\to c} \frac{f(x)-f(c)}{x-c} \\
&= \lim_{x\to c} \frac{k-k}{x-c} \\
&= \lim_{x\to c} \frac{0}{x-c} \\
&= \lim_{x\to c} 0 \\
&= 0.
\end{aligned}$$

Thus, $f'(c) = \lim_{x\to c} \frac{f(x)-f(c)}{x-c} = 0$, as desired. □

Proposition 3. ***the derivative of the identity function is 1***

Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by $f(x) = x$.

Then $f'(x) = 1$ for all $x \in \mathbb{R}$.

Proof. We prove $f'(x) = 1$ for all $x \in \mathbb{R}$ using the definition of derivative.

Let $c \in \mathbb{R}$ be given.

We must prove $f'(c) = 1$.

Let $q : \mathbb{R} - \{c\} \to \mathbb{R}$ be a function defined by $q(x) = \frac{f(x)-f(c)}{x-c}$ for all $x \in \mathbb{R} - \{c\}$.

Since $c \in \mathbb{R}$ and every real number is an accumulation point of $\mathbb{R}$, then c is an accumulation point of $\mathbb{R}$, so c is an accumulation point of $\mathbb{R} - \{c\}$, the domain of q.

For $x \in \mathbb{R} - \{c\}$, we have $x \in \mathbb{R}$ and $x \neq c$, so $x - c \neq 0$.

Observe that

$$\begin{aligned}
\lim_{x\to c} q(x) &= \lim_{x\to c} \frac{f(x)-f(c)}{x-c} \\
&= \lim_{x\to c} \frac{x-c}{x-c} \\
&= \lim_{x\to c} 1 \\
&= 1.
\end{aligned}$$

Thus, $f'(c) = \lim_{x\to c} \frac{f(x)-f(c)}{x-c} = 1$, as desired. □

Theorem 4. ***differentiability implies continuity***

Let f be a real valued function of a real variable x.

If f is differentiable at c, then f is continuous at c.

Proof. Suppose f is differentiable at c.

Then $c \in domf$ and c is an accumulation point of $domf$ and $\lim_{x\to c} \frac{f(x)-f(c)}{x-c} = f'(c)$ exists and $f'(c) \in \mathbb{R}$.

Let $q : domf - \{c\} \to \mathbb{R}$ be a function defined by $q(x) = \frac{f(x)-f(c)}{x-c}$.

For each $x \in domf - \{c\}$, we have $x \in domf$ and $x \neq c$, so $x - c \neq 0$.

Thus,

$$\begin{aligned} f(c) &= 0 \cdot f'(c) + f(c) \\ &= \lim_{x\to c}(x-c) \cdot \lim_{x\to c} \frac{f(x)-f(c)}{x-c} + \lim_{x\to c} f(c) \\ &= \lim_{x\to c}(x-c) \cdot \frac{f(x)-f(c)}{x-c} + \lim_{x\to c} f(c) \\ &= \lim_{x\to c}[f(x)-f(c)] + \lim_{x\to c} f(c) \\ &= \lim_{x\to c}[f(x)-f(c)+f(c)] \\ &= \lim_{x\to c} f(x). \end{aligned}$$

Since c is an accumulation point of $domf$ and $\lim_{x\to c} f(x) = f(c)$, then f is continuous at c. □

Corollary 5. ***Every differentiable function is continuous.***

Let f be a real valued function of a real variable x.

If f is differentiable, then f is continuous.

Proof. Suppose f is differentiable.

To prove f is continuous, let $x \in domf$.

We must prove f is continuous at x.

Since f is differentiable, then f is differentiable on $domf$.

Since $x \in domf$, then f is differentiable at x, so f is continuous at x. □

Algebraic properties of derivatives

Theorem 6. ***scalar multiple rule for derivatives***

Let f be a real valued function of a real variable x.

If f is differentiable at c, then for every $\lambda \in \mathbb{R}$, the function λf is differentiable at c and $(\lambda f)'(c) = \lambda f'(c)$.

Proof. Suppose f is differentiable at c.

Then $c \in domf$ and c is an accumulation point of $domf$ and $f'(c) \in \mathbb{R}$ and $f'(c) = \lim_{x\to c} \frac{f(x)-f(c)}{x-c}$.

Let $\lambda \in \mathbb{R}$ be given.

Since $dom(\lambda f) = domf$ and $c \in domf$ and c is an accumulation point of $domf$, then $c \in dom(\lambda f)$ and c is an accumulation point of $dom(\lambda f)$.

Observe that

$$\begin{aligned}
\lambda f'(c) &= \lambda \lim_{x \to c} \frac{f(x) - f(c)}{x - c} \\
&= \lim_{x \to c} \frac{\lambda[f(x) - f(c)]}{x - c} \\
&= \lim_{x \to c} \frac{\lambda f(x) - \lambda f(c)}{x - c} \\
&= \lim_{x \to c} \frac{(\lambda f)(x) - (\lambda f)(c)}{x - c} \\
&= (\lambda f)'(c).
\end{aligned}$$

Therefore, λf is differentiable at c and $(\lambda f)'(c) = \lambda f'(c)$. □

Theorem 7. ***derivative of a sum equals sum of a derivative***

Let f and g be real valued functions of a real variable x.

Let c be an accumulation point of $domf \cap domg$.

If f is differentiable at c and g is differentiable at c, then the function $f+g$ is differentiable at c and

$(f+g)'(c) = f'(c) + g'(c)$.

Proof. Suppose f is differentiable at c and g is differentiable at c.

Since f is differentiable at c, then $c \in domf$ and $f'(c) \in \mathbb{R}$ and $f'(c) = \lim_{x \to c} \frac{f(x)-f(c)}{x-c}$.

Since g is differentiable at c, then $c \in domg$ and $g'(c) \in \mathbb{R}$ and $g'(c) = \lim_{x \to c} \frac{g(x)-g(c)}{x-c}$.

Since $c \in domf$ and $c \in domg$, then $c \in domf \cap domg$.

Since $dom(f+g) = domf \cap domg$ and $c \in domf \cap domg$ and c is an accumulation point of $domf \cap domg$, then $c \in dom(f+g)$ and c is an accumulation point of $dom(f+g)$.

Observe that

$$\begin{aligned}
f'(c) + g'(c) &= \lim_{x \to c} \frac{f(x) - f(c)}{x - c} + \lim_{x \to c} \frac{g(x) - g(c)}{x - c} \\
&= \lim_{x \to c} [\frac{f(x) - f(c)}{x - c} + \frac{g(x) - g(c)}{x - c}] \\
&= \lim_{x \to c} \frac{f(x) - f(c) + g(x) - g(c)}{x - c} \\
&= \lim_{x \to c} \frac{f(x) + g(x) - f(c) - g(c)}{x - c} \\
&= \lim_{x \to c} \frac{[f(x) + g(x)] - [f(c) + g(c)]}{x - c} \\
&= \lim_{x \to c} \frac{(f+g)(x) - (f+g)(c)}{x - c} \\
&= (f+g)'(c).
\end{aligned}$$

Therefore, $f+g$ is differentiable at c and $(f+g)'(c) = f'(c) + g'(c)$. $\square$

Corollary 8. ***derivative of a difference equals difference of a derivative***

Let f and g be real valued functions of a real variable x.

Let c be an accumulation point of $dom f \cap dom g$.

If f is differentiable at c and g is differentiable at c, then the function $f-g$ is differentiable at c and

$(f-g)'(c) = f'(c) - g'(c)$.

Proof. Suppose f is differentiable at c and g is differentiable at c.

Since g is differentiable at c, then the function $-g$ is differentiable at c and $(-g)'(c) = (-1)g'(c)$.

Since c is an accumulation point of $dom f \cap dom g$ and $dom(-g) = dom g$, then c is an accumulation point of $dom f \cap dom(-g)$.

Since f is differentiable at c and $-g$ is differentiable at c, then the function $f + (-g)$ is differentiable at c and $(f+(-g))'(c) = f'(c) + (-g)'(c)$.

Observe that

$$\begin{aligned} f'(c) - g'(c) &= f'(c) + (-1)g'(c) \\ &= f'(c) + (-g)'(c) \\ &= (f + (-g))'(c) \\ &= (f-g)'(c). \end{aligned}$$

Therefore, $f-g$ is differentiable at c and $(f-g)'(c) = f'(c) - g'(c)$. $\square$

Theorem 9. ***product rule for derivatives***

Let f and g be real valued functions of a real variable x.

Let c be an accumulation point of $dom f \cap dom g$.

If f is differentiable at c and g is differentiable at c, then the function fg is differentiable at c and

$(fg)'(c) = f(c)g'(c) + g(c)f'(c)$.

Proof. Suppose f is differentiable at c and g is differentiable at c.

Since f is differentiable at c, then $c \in dom f$ and $f'(c) \in \mathbb{R}$ and $f'(c) = \lim_{x\to c} \frac{f(x)-f(c)}{x-c}$.

Since g is differentiable at c, then $c \in dom g$ and $g'(c) \in \mathbb{R}$ and $g'(c) = \lim_{x\to c} \frac{g(x)-g(c)}{x-c}$.

Since $c \in dom f$ and $c \in dom g$, then $c \in dom f \cap dom g$.

Since c is an accumulation point of $dom f \cap dom g$ and $dom f \cap dom g$ is a subset of $dom f$, then c is an accumulation point of $dom f$.

Since f is differentiable at c, then f is continuous at c.

Since $c \in dom f$ and c is an accumulation point of $dom f$ and f is continuous at c, then $\lim_{x\to c} f(x) = f(c)$.

Since $dom(fg) = dom f \cap dom g$ and $c \in dom f \cap dom g$ and c is an accumulation point of $dom f \cap dom g$, then $c \in dom(fg)$ and c is an accumulation point of $dom(fg)$.

Observe that

$$\begin{aligned}
f(c)g'(c)+g(c)f'(c) &= \lim_{x\to c} f(x)\cdot \lim_{x\to c}\frac{g(x)-g(c)}{x-c} + g(c)\cdot \lim_{x\to c}\frac{f(x)-f(c)}{x-c} \\
&= \lim_{x\to c}\frac{f(x)[g(x)-g(c)]}{x-c} + \lim_{x\to c}\frac{g(c)[f(x)-f(c)]}{x-c} \\
&= \lim_{x\to c}\frac{f(x)[g(x)-g(c)]}{x-c} + \frac{g(c)[f(x)-f(c)]}{x-c} \\
&= \lim_{x\to c}\frac{f(x)[g(x)-g(c)]+g(c)[f(x)-f(c)]}{x-c} \\
&= \lim_{x\to c}\frac{f(x)g(x)-f(x)g(c)+g(c)f(x)-g(c)f(c)}{x-c} \\
&= \lim_{x\to c}\frac{f(x)g(x)-f(x)g(c)+f(x)g(c)-f(c)g(c)}{x-c} \\
&= \lim_{x\to c}\frac{f(x)g(x)-f(c)g(c)}{x-c} \\
&= \lim_{x\to c}\frac{(fg)(x)-(fg)(c)}{x-c} \\
&= (fg)'(c).
\end{aligned}$$

Therefore, fg is differentiable at c and $(fg)'(c) = f(c)g'(c) + g(c)f'(c)$. ☐

Theorem 10. ***quotient rule for derivatives***

Let f and g be real valued functions of a real variable x.

Let c be an accumulation point of $dom f \cap dom g$.

If f is differentiable at c and g is differentiable at c and $g(c) \neq 0$, then the function $\frac{f}{g}$ is differentiable at c and

$(\frac{f}{g})'(c) = \frac{g(c)f'(c)-f(c)g'(c)}{(g(c))^2}$.

Proof. Suppose f is differentiable at c and g is differentiable at c.

Since f is differentiable at c, then $c \in dom f$ and $f'(c) \in \mathbb{R}$ and $f'(c) = \lim_{x\to c} \frac{f(x)-f(c)}{x-c}$.

Since g is differentiable at c, then $c \in dom g$ and $g'(c) \in \mathbb{R}$ and $g'(c) = \lim_{x\to c} \frac{g(x)-g(c)}{x-c}$.

Since c is an accumulation point of $dom f \cap dom g$ and $dom f \cap dom g$ is a subset of $dom g$, then c is an accumulation point of $dom g$.

Since g is differentiable at c, then g is continuous at c.

Since $c \in dom g$ and c is an accumulation point of $dom g$ and g is continuous at c, then $\lim_{x\to c} g(x) = g(c)$.

Since $\lim_{x\to c} g(x) = g(c)$ and $g(c) \neq 0$, then $\lim_{x\to c} \frac{1}{g(x)} = \frac{1}{\lim_{x\to c} g(x)} = \frac{1}{g(c)}$.

Since $c \in dom f$ and $c \in dom g$, then $c \in dom f \cap dom g$.

Since $c \in dom f \cap dom g$ and $g(c) \neq 0$, then c is in the domain of $\frac{f}{g}$.

Since c is an accumulation point of $dom f \cap dom g$, then c is an accumulation point of the domain of $\frac{f}{g}$.

Observe that

$$
\begin{aligned}
\frac{g(c)f'(c)-f(c)g'(c)}{(g(c))^2} &= \frac{1}{g(c)}\cdot f'(c)-\frac{f(c)}{g(c)}\cdot\frac{1}{g(c)}\cdot g'(c)\\
&= \lim_{x\to c}\frac{1}{g(x)}\cdot\lim_{x\to c}\frac{f(x)-f(c)}{x-c}-\frac{f(c)}{g(c)}\cdot\lim_{x\to c}\frac{1}{g(x)}\cdot\lim_{x\to c}\frac{g(x)-g(c)}{x-c}\\
&= \lim_{x\to c}\frac{f(x)-f(c)}{g(x)(x-c)}-\frac{f(c)}{g(c)}\cdot\lim_{x\to c}\frac{g(x)-g(c)}{g(x)(x-c)}\\
&= \lim_{x\to c}\frac{[f(x)-f(c)]g(c)}{g(x)g(c)(x-c)}-\lim_{x\to c}\frac{f(c)[g(x)-g(c)]}{g(x)g(c)(x-c)}\\
&= \lim_{x\to c}\frac{[f(x)-f(c)]g(c)-f(c)[g(x)-g(c)]}{g(x)g(c)(x-c)}\\
&= \lim_{x\to c}\frac{f(x)g(c)-f(c)g(c)-f(c)g(x)+f(c)g(c)}{g(x)g(c)(x-c)}\\
&= \lim_{x\to c}\frac{f(x)g(c)-f(c)g(x)}{g(x)g(c)(x-c)}\\
&= \lim_{x\to c}\frac{\frac{f(x)}{g(x)}-\frac{f(c)}{g(c)}}{x-c}\\
&= \lim_{x\to c}\frac{(\frac{f}{g})(x)-(\frac{f}{g})(c)}{x-c}\\
&= (\frac{f}{g})'(c).
\end{aligned}
$$

Therefore, $\frac{f}{g}$ is differentiable at c and $(\frac{f}{g})'(c)=\frac{g(c)f'(c)-f(c)g'(c)}{(g(c))^2}$. □

Corollary 11. ***power rule for derivatives***

Let $n\in\mathbb{Z}$ be fixed.

Let $f:\mathbb{R}^*\to\mathbb{R}$ be the function defined by $f(x)=x^n$.

Then $f'(x)=nx^{n-1}$.

Proof. We prove for $n=0$, if $f(x)=x^n$ for all nonzero $x\in\mathbb{R}$, then $f'(x)=nx^{n-1}$.

Let $n=0$.

Suppose $f(x)=x^n$ for all nonzero $x\in\mathbb{R}$.

Then $f(x)=x^n=x^0=1$ for all nonzero $x\in\mathbb{R}$, so $f'(x)=0$ for all nonzero $x\in\mathbb{R}$.

Let $x\in\mathbb{R}$ with $x\neq 0$.

Then $f'(x)=0=0\cdot x^{-1}=0x^{0-1}=nx^{n-1}$. □

Proof. We prove for all positive integers n, if $f(x)=x^n$ for all nonzero $x\in\mathbb{R}$, then $f'(x)=nx^{n-1}$ by induction on n.

Let $S=\{n\in\mathbb{Z}^+ :\ \text{if } f(x)=x^n \text{ for all nonzero } x\in\mathbb{R}, \text{ then } f'(x)=nx^{n-1}\}$.

Basis:

Suppose $f(x)=x^1=x$ for all nonzero $x\in\mathbb{R}$.

Then $f'(x) = 1$ for all nonzero $x \in \mathbb{R}$.
Let $x \in \mathbb{R}$ with $x \neq 0$.
Since $x \neq 0$, then $f'(x) = 1 = 1 \cdot 1 = 1 \cdot x^0 = 1 \cdot x^{1-1}$, so $1 \in S$.
Induction:
Suppose $k \in S$.
Then $k \in \mathbb{Z}^+$ and if $f(x) = x^k$ for all nonzero $x \in \mathbb{R}$, then $f'(x) = kx^{k-1}$.
Since $k \in \mathbb{Z}^+$, then $k + 1 \in \mathbb{Z}^+$.
Suppose $g(x) = x^{k+1}$ for all nonzero $x \in \mathbb{R}$.
Then $g(x) = x^k \cdot x$ for all nonzero $x \in \mathbb{R}$.
Let f and h be functions defined by $f(x) = x^k$ and $h(x) = x$ for all nonzero $x \in \mathbb{R}$.
Then $g = fh$ is the function defined by $g(x) = f(x) \cdot h(x)$ for all nonzero $x \in \mathbb{R}$.
Since $f(x) = x^k$ for all nonzero $x \in \mathbb{R}$, then $f'(x) = kx^{k-1}$.
Since $h(x) = x$ for all nonzero $x \in \mathbb{R}$, then $h'(x) = 1$ for all nonzero $x \in \mathbb{R}$.
Let $x \in \mathbb{R}$ with $x \neq 0$.
Since $dom f \cap dom h = \mathbb{R} \cap \mathbb{R} = \mathbb{R}$ and $x \in \mathbb{R}$ and every real number is an accumulation point of $\mathbb{R}$, then x is an accumulation point of $dom f \cap dom h$.
Since x is a nonzero real number, then $f(x) = x^k$ and $f'(x) = kx^{k-1}$ and $h(x) = x$ and $h'(x) = 1$.
Thus,

$$\begin{aligned}
g'(x) &= (fh)'(x) \\
&= f(x) \cdot h'(x) + h(x) \cdot f'(x) \\
&= x^k \cdot 1 + x \cdot kx^{k-1} \\
&= x^k + kx^k \\
&= (1+k)x^k \\
&= (k+1)x^k.
\end{aligned}$$

Hence, if $g(x) = x^{k+1}$ for all nonzero $x \in \mathbb{R}$, then $g'(x) = (k+1)x^k$.
Since $k + 1 \in \mathbb{Z}^+$, then this implies $k + 1 \in S$.
Hence, $k \in S$ implies $k + 1 \in S$ for all $k \in \mathbb{Z}^+$.
Since $1 \in S$ and $k \in S$ implies $k + 1 \in S$ for all $k \in \mathbb{Z}^+$, then by induction, $S = \mathbb{Z}^+$.
Therefore, for all $n \in \mathbb{Z}^+$, if $f(x) = x^n$ for all nonzero $x \in \mathbb{R}$, then $f'(x) = nx^{n-1}$. □

Proof. We prove for all negative integers n, if $f(x) = x^n$ for all nonzero $x \in \mathbb{R}$, then $f'(x) = nx^{n-1}$.
Let n be an arbitrary negative integer.
Then $n \in \mathbb{Z}$ and $n < 0$.
Thus, there exists $k \in \mathbb{Z}$ such that $k = -n > 0$, so $n = -k$ and k is a positive integer.
Suppose $f(x) = x^n$ for all nonzero $x \in \mathbb{R}$.

Let $x \in \mathbb{R}$ with $x \neq 0$.

Then $f(x) = x^n = x^{-k} = \frac{1}{x^k}$.

Let g and h be functions defined by $g(x) = 1$ and $h(x) = x^k$ for all nonzero $x \in \mathbb{R}$.

Then $f = \frac{g}{h}$ is the function defined by $f(x) = \frac{g(x)}{h(x)}$ for all nonzero $x \in \mathbb{R}$.

Since $g(x) = 1$ for all nonzero $x \in \mathbb{R}$, then $g'(x) = 0$ for all nonzero $x \in \mathbb{R}$.

Since k is a positive integer and $h(x) = x^k$ for all nonzero $x \in \mathbb{R}$, then $h'(x) = kx^{k-1}$.

Let $x \in \mathbb{R}^*$.

Then $x \in \mathbb{R}$ and $x \neq 0$.

Since $x \in \mathbb{R}^*$ and $\mathbb{R}^*$ is an open set, then x is an interior point of $\mathbb{R}^*$, so x is an accumulation point of $\mathbb{R}^*$.

Since $dom g \cap dom h = \mathbb{R}^* \cap \mathbb{R}^* = \mathbb{R}^*$, then this implies x is an accumulation point of $dom g \cap dom h$.

Since x is a nonzero real number, then $g(x) = 1$ and $g'(x) = 0$ and $h(x) = x^k$ and $h'(x) = kx^{k-1}$.

Since $x \neq 0$ and $k > 0$, then $x^k \neq 0$, so $h(x) \neq 0$.

Thus,

$$\begin{aligned} f'(x) &= (\frac{g}{h})'(x) \\ &= \frac{h(x) \cdot g'(x) - g(x) \cdot h'(x)}{[h(x)]^2} \\ &= \frac{x^k \cdot 0 - 1 \cdot kx^{k-1}}{(x^k)^2} \\ &= \frac{-kx^{k-1}}{x^{2k}} \\ &= -kx^{-k-1} \\ &= nx^{n-1}. \end{aligned}$$

Therefore, if $f(x) = x^n$ for all nonzero $x \in \mathbb{R}$, then $f'(x) = nx^{n-1}$. ☐

Proposition 12. ***derivatives of trig functions***

1. $\frac{d}{dx}(\tan x) = \sec^2 x$.

2. $\frac{d}{dx}(\cot x) = -\csc^2 x$.

3. $\frac{d}{dx}(\sec x) = \sec x \tan x$.

4. $\frac{d}{dx}(\csc x) = -\csc x \cot x$.

Proof. We prove 1.

Observe that

$$
\begin{aligned}
\frac{d}{dx}(\tan x) &= \frac{d}{dx}(\frac{\sin x}{\cos x}) \\
&= \frac{(\cos x)(\cos x) - (\sin x)(-\sin x)}{\cos^2 x} \\
&= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} \\
&= \frac{1}{\cos^2 x} \\
&= \sec^2 x.
\end{aligned}
$$

□

Proof. We prove 2.

Observe that

$$
\begin{aligned}
\frac{d}{dx}(\cot x) &= \frac{d}{dx}(\frac{\cos x}{\sin x}) \\
&= \frac{(\sin x)(-\sin x) - (\cos x)(\cos x)}{\sin^2 x} \\
&= \frac{-\sin^2 x - \cos^2 x}{\sin^2 x} \\
&= \frac{-(\sin^2 x + \cos^2 x)}{\sin^2 x} \\
&= \frac{-1}{\sin^2 x} \\
&= -\csc^2 x.
\end{aligned}
$$

□

Proof. We prove 3.

Observe that

$$
\begin{aligned}
\frac{d}{dx}(\sec x) &= \frac{d}{dx}(\frac{1}{\cos x}) \\
&= \frac{(\cos x)(0) - (1)(-\sin x)}{\cos^2 x} \\
&= \frac{\sin x}{\cos^2 x} \\
&= \frac{1 \cdot \sin x}{\cos x \cdot \cos x} \\
&= \frac{1}{\cos x} \cdot \frac{\sin x}{\cos x} \\
&= \sec x \tan x.
\end{aligned}
$$

□

Proof. We prove 4.

Observe that

$$\begin{aligned}
\frac{d}{dx}(\csc x) &= \frac{d}{dx}(\frac{1}{\sin x}) \\
&= \frac{(\sin x)(0) - (1)(\cos x)}{\sin^2 x} \\
&= \frac{-\cos x}{\sin^2 x} \\
&= \frac{-1 \cdot \cos x}{\sin x \cdot \sin x} \\
&= \frac{-1}{\sin x} \cdot \frac{\cos x}{\sin x} \\
&= -\csc x \cot x.
\end{aligned}$$

□

Theorem 13. ***chain rule for derivatives***

Let f and g be real valued functions such that $rngf \subset domg$.

If f is differentiable at c and g is differentiable at $f(c)$, then the function $g \circ f$ is differentiable at c and

$(g \circ f)'(c) = g'(f(c)) \cdot f'(c)$.

Proof. Suppose f is differentiable at c and g is differentiable at $f(c)$.

Since f is differentiable at c, then $c \in domf$ and c is an accumulation point of $domf$ and $f'(c) = \lim_{x \to c} \frac{f(x)-f(c)}{x-c}$ and f is continuous at c.

Since g is differentiable at $f(c)$, then $f(c) \in domg$ and $f(c)$ is an accumulation point of $domg$ and $g'(f(c))$ exists.

Let $\delta > 0$ be given.

Since c is an accumulation point of $domf$, then there exists $x \in domf$ such that $x \in N'(c; \delta)$.

Let $x \in domf$ such that $x \in N'(c; \delta)$.

Since $x \in domf$, then $f(x) \in rngf$.

Since $rngf \subset domg$, then $f(x) \in domg$.

Since $x \in N'(c; \delta)$, then $x \in N(c; \delta)$ and $x \neq c$.

Since $f(x) \in domg$ and $f(c) \in domg$, then either $f(x) = f(c)$ or $f(x) \neq f(c)$.

We consider these cases separately.

Case 1: Suppose $f(x) \neq f(c)$.

Then $f(x) - f(c) \neq 0$.

Let $h : domg \to \mathbb{R}$ be a function defined by $h(y) = \frac{g(y)-g(f(c))}{y-f(c)}$ if $y \neq f(c)$ and $h(f(c)) = g'(f(c))$.

Since $f(x) \in domg$ and $f(x) \neq f(c)$, then $h(f(x)) = \frac{g(f(x))-g(f(c))}{f(x)-f(c)}$.

If $y \in domg$ and $y \neq f(c)$, then

$$\begin{aligned} h(f(c)) &= g'(f(c)) \\ &= \lim_{y \to f(c)} \frac{g(y) - g(f(c))}{y - f(c)} \\ &= \lim_{y \to f(c)} h(y). \end{aligned}$$

Let $y = f(x)$.

Then $y \in domg$ and $y \neq f(c)$ and $h(y) = \frac{g(y)-g(f(c))}{y-f(c)}$.

Since $y \in domg$ and $y \neq f(c)$, then we conclude $h(f(c)) = \lim_{y \to f(c)} h(y)$.

Since $f(c)$ is an accumulation point of $domg$ and $domh = domg$, then $f(c)$ is an accumulation point of $domh$.

Since $f(c)$ is an accumulation point of $domh$ and $\lim_{y \to f(c)} h(y) = h(f(c))$, then by the characterization of continuity at a point, h is continuous at $f(c)$.

Since f is continuous at c and h is continuous at $f(c)$, then $h \circ f$ is continuous at c.

Observe that $dom(h \circ f) = \{x \in domf : f(x) \in domh\} = \{x \in domf : f(x) \in domg\}$.

Since $x \in domf$ and $f(x) \in domg$, then $x \in dom(h \circ f)$.

Thus, there exists $x \in dom(h \circ f)$ such that $x \in N'(c;\delta)$ for every $\delta > 0$.

Therefore, c is an accumulation point of $dom(h \circ f)$.

Since c is an accumulation point of $dom(h \circ f)$ and $h \circ f$ is continuous at c, then by the characterization of continuity at a point, $\lim_{x \to c}(h \circ f)(x) = (h \circ f)(c) = h(f(c)) = g'(f(c))$, so $\lim_{x \to c}(h \circ f)(x) = g'(f(c))$.

Since $x \neq c$ and $f(x) - f(c) \neq 0$, then

$$\begin{aligned} g'(f(c)) \cdot f'(c) &= \lim_{x \to c}(h \circ f)(x) \cdot \lim_{x \to c} \frac{f(x) - f(c)}{x - c} \\ &= \lim_{x \to c}(h \circ f)(x) \cdot \frac{f(x) - f(c)}{x - c} \\ &= \lim_{x \to c}(h(f(x)) \cdot \frac{f(x) - f(c)}{x - c} \\ &= \lim_{x \to c} \frac{g(f(x)) - g(f(c))}{f(x) - f(c)} \cdot \frac{f(x) - f(c)}{x - c} \\ &= \lim_{x \to c} \frac{g(f(x)) - g(f(c))}{x - c} \\ &= \lim_{x \to c} \frac{(g \circ f)(x) - (g \circ f)(c)}{x - c} \\ &= (g \circ f)'(c). \end{aligned}$$

Therefore, if $f(x) \neq f(c)$, then $(g \circ f)'(c) = g'(f(c)) \cdot f'(c)$.

Case 2: Suppose $f(x) = f(c)$.

Since f is differentiable at c and $x \neq c$, then $f'(c) = \lim_{x \to c} \frac{f(x)-f(c)}{x-c} = \lim_{x \to c} \frac{f(c)-f(c)}{x-c} = \lim_{x \to c} \frac{0}{x-c} = 0$, so $f'(c) = 0$.

Observe that

$$\begin{aligned} g'(f(c)) \cdot f'(c) &= g'(f(c)) \cdot 0 \\ &= 0 \\ &= \lim_{x \to c} \frac{0}{x-c} \\ &= \lim_{x \to c} \frac{g(f(c)) - g(f(c))}{x-c} \\ &= \lim_{x \to c} \frac{g(f(x)) - g(f(c))}{x-c} \\ &= \lim_{x \to c} \frac{(g \circ f)(x) - (g \circ f)(c)}{x-c} \\ &= (g \circ f)'(c). \end{aligned}$$

Therefore, if $f(x) = f(c)$, then $(g \circ f)'(c) = g'(f(c)) \cdot f'(c)$. □

Mean Value Theorem

Lemma 14. *Let $E \subset \mathbb{R}$.*
Let $f : E \to \mathbb{R}$ be a function.
Let $c \in E$.
1. If $f(c)$ is a relative maximum and f is differentiable at c, then $f'(c) = 0$.
2. If $f(c)$ is a relative minimum and f is differentiable at c, then $f'(c) = 0$.

Proof. We prove 1.
Suppose $f(c)$ is a relative maximum and f is differentiable at c.
Since $f(c)$ is a relative maximum, then there exists $\delta_1 > 0$ such that $N(c; \delta_1) \subset E$ and $f(c) \geq f(x)$ for all $x \in N(c; \delta_1)$.
Since f is differentiable at c, then $c \in E$ and there is a real number $f'(c)$ such that $f'(c) = \lim_{x \to c} \frac{f(x)-f(c)}{x-c}$.
Since $f'(c) \in \mathbb{R}$, then either $f'(c) > 0$ or $f'(c) = 0$ or $f'(c) < 0$.

Suppose $f'(c) > 0$.
Since $f'(c) = \lim_{x \to c} \frac{f(x)-f(c)}{x-c}$, then there exists $\delta_2 > 0$ such that for all $x \in E - \{c\}$, if $0 < |x - c| < \delta_2$, then $|\frac{f(x)-f(c)}{x-c} - f'(c)| < f'(c)$.
Let $\delta = \min\{\delta_1, \delta_2\}$.
Then $\delta \leq \delta_1$ and $\delta \leq \delta_2$ and $\delta > 0$.
Since $\delta > 0$, then $\frac{\delta}{2} > 0$.
Let $x = c + \frac{\delta}{2}$.
Since $d(x, c) = |x - c| = |\frac{\delta}{2}| = \frac{\delta}{2} < \delta \leq \delta_1$, then $d(x, c) < \delta_1$, so $x \in N(c; \delta_1)$.
Hence, $f(c) \geq f(x)$.
Since $x \in N(c; \delta_1)$ and $N(c; \delta_1) \subset E$, then $x \in E$.
Since $x - c = \frac{\delta}{2} > 0$, then $x - c > 0$, so $x - c \neq 0$.
Hence, $x \neq c$.
Since $x \in E$ and $x \neq c$, then $x \in E - \{c\}$.

Since $0 < |x - c| = |\frac{\delta}{2}| = \frac{\delta}{2} < \delta \leq \delta_2$, then $0 < |x - c| < \delta_2$.
Since $x \in E - \{c\}$ and $0 < |x - c| < \delta_2$, then $|\frac{f(x)-f(c)}{x-c} - f'(c)| < f'(c)$.
Thus, $-f'(c) < \frac{f(x)-f(c)}{x-c} - f'(c) < f'(c)$, so $-f'(c) < \frac{f(x)-f(c)}{x-c} - f'(c)$.
Hence, $0 < \frac{f(x)-f(c)}{x-c}$.
Since $x - c > 0$, then $0 < f(x) - f(c)$, so $f(c) < f(x)$.
Thus, we have $f(c) \geq f(x)$ and $f(c) < f(x)$, a contradiction.
Therefore, $f'(c)$ cannot be positive.

Suppose $f'(c) < 0$.
Then $-f'(c) > 0$.
Since $f'(c) = \lim_{x \to c} \frac{f(x)-f(c)}{x-c}$, then there exists $\delta_2 > 0$ such that for all $x \in E - \{c\}$, if $0 < |x - c| < \delta_2$, then $|\frac{f(x)-f(c)}{x-c} - f'(c)| < -f'(c)$.
Let $\delta = \min\{\delta_1, \delta_2\}$.
Then $\delta \leq \delta_1$ and $\delta \leq \delta_2$ and $\delta > 0$.
Since $\delta > 0$, then $\frac{\delta}{2} > 0$.
Let $x = c - \frac{\delta}{2}$.
Since $d(x, c) = |x - c| = |\frac{-\delta}{2}| = |\frac{\delta}{2}| = \frac{\delta}{2} < \delta \leq \delta_1$, then $d(x, c) < \delta_1$, so $x \in N(c; \delta_1)$.
Hence, $f(c) \geq f(x)$.
Since $x \in N(c; \delta_1)$ and $N(c; \delta_1) \subset E$, then $x \in E$.
Since $x - c = \frac{-\delta}{2} < 0$, then $x - c < 0$, so $x - c \neq 0$.
Hence, $x \neq c$.
Since $x \in E$ and $x \neq c$, then $x \in E - \{c\}$.
Since $0 < |x - c| = |\frac{-\delta}{2}| = |\frac{\delta}{2}| = \frac{\delta}{2} < \delta \leq \delta_2$, then $0 < |x - c| < \delta_2$.
Since $x \in E - \{c\}$ and $0 < |x - c| < \delta_2$, then $|\frac{f(x)-f(c)}{x-c} - f'(c)| < -f'(c)$.
Thus, $f'(c) < \frac{f(x)-f(c)}{x-c} - f'(c) < -f'(c)$, so $\frac{f(x)-f(c)}{x-c} - f'(c) < -f'(c)$.
Hence, $\frac{f(x)-f(c)}{x-c} < 0$.
Since $x - c < 0$, then $f(x) - f(c) > 0$, so $f(x) > f(c)$.
Thus, we have $f(c) \geq f(x)$ and $f(c) < f(x)$, a contradiction.
Therefore, $f'(c)$ cannot be negative.

Since $f'(c)$ is neither positive nor negative, then $f'(c)$ must be zero, so $f'(c) = 0$. □

Proof. We prove 2.
Suppose $f(c)$ is a relative minimum and f is differentiable at c.
Since $f(c)$ is a relative minimum, then there exists $\delta_1 > 0$ such that $N(c; \delta_1) \subset E$ and $f(c) \leq f(x)$ for all $x \in N(c; \delta_1)$.
Since f is differentiable at c, then $c \in E$ and there is a real number $f'(c)$ such that $f'(c) = \lim_{x \to c} \frac{f(x)-f(c)}{x-c}$.
Since $f'(c) \in \mathbb{R}$, then either $f'(c) > 0$ or $f'(c) = 0$ or $f'(c) < 0$.

Suppose $f'(c) > 0$.

Since $f'(c) = \lim_{x \to c} \frac{f(x)-f(c)}{x-c}$, then there exists $\delta_2 > 0$ such that for all $x \in E - \{c\}$, if $0 < |x - c| < \delta_2$, then $|\frac{f(x)-f(c)}{x-c} - f'(c)| < f'(c)$.

Let $\delta = \min\{\delta_1, \delta_2\}$.

Then $\delta \leq \delta_1$ and $\delta \leq \delta_2$ and $\delta > 0$.

Since $\delta > 0$, then $\frac{\delta}{2} > 0$.

Let $x = c - \frac{\delta}{2}$.

Since $d(x, c) = |x - c| = |\frac{-\delta}{2}| = |\frac{\delta}{2}| = \frac{\delta}{2} < \delta \leq \delta_1$, then $d(x, c) < \delta_1$, so $x \in N(c; \delta_1)$.

Hence, $f(c) \leq f(x)$.

Since $x \in N(c; \delta_1)$ and $N(c; \delta_1) \subset E$, then $x \in E$.

Since $x - c = \frac{-\delta}{2} < 0$, then $x - c < 0$, so $x - c \neq 0$.

Hence, $x \neq c$.

Since $x \in E$ and $x \neq c$, then $x \in E - \{c\}$.

Since $0 < |x - c| = |\frac{-\delta}{2}| = |\frac{\delta}{2}| = \frac{\delta}{2} < \delta \leq \delta_2$, then $0 < |x - c| < \delta_2$.

Since $x \in E - \{c\}$ and $0 < |x - c| < \delta_2$, then $|\frac{f(x)-f(c)}{x-c} - f'(c)| < f'(c)$.

Thus, $-f'(c) < \frac{f(x)-f(c)}{x-c} - f'(c) < f'(c)$, so $-f'(c) < \frac{f(x)-f(c)}{x-c} - f'(c)$.

Hence, $0 < \frac{f(x)-f(c)}{x-c}$.

Since $x - c < 0$, then $0 > f(x) - f(c)$, so $f(c) > f(x)$.

Thus, we have $f(c) \leq f(x)$ and $f(c) > f(x)$, a contradiction.

Therefore, $f'(c)$ cannot be positive.

Suppose $f'(c) < 0$.

Then $-f'(c) > 0$.

Since $f'(c) = \lim_{x \to c} \frac{f(x)-f(c)}{x-c}$, then there exists $\delta_2 > 0$ such that for all $x \in E - \{c\}$, if $0 < |x - c| < \delta_2$, then $|\frac{f(x)-f(c)}{x-c} - f'(c)| < -f'(c)$.

Let $\delta = \min\{\delta_1, \delta_2\}$.

Then $\delta \leq \delta_1$ and $\delta \leq \delta_2$ and $\delta > 0$.

Since $\delta > 0$, then $\frac{\delta}{2} > 0$.

Let $x = c + \frac{\delta}{2}$.

Since $d(x, c) = |x - c| = |\frac{\delta}{2}| = \frac{\delta}{2} < \delta \leq \delta_1$, then $d(x, c) < \delta_1$, so $x \in N(c; \delta_1)$.

Hence, $f(c) \leq f(x)$.

Since $x \in N(c; \delta_1)$ and $N(c; \delta_1) \subset E$, then $x \in E$.

Since $x - c = \frac{\delta}{2} > 0$, then $x - c > 0$, so $x - c \neq 0$.

Hence, $x \neq c$.

Since $x \in E$ and $x \neq c$, then $x \in E - \{c\}$.

Since $0 < |x - c| = |\frac{\delta}{2}| = \frac{\delta}{2} < \delta \leq \delta_2$, then $0 < |x - c| < \delta_2$.

Since $x \in E - \{c\}$ and $0 < |x - c| < \delta_2$, then $|\frac{f(x)-f(c)}{x-c} - f'(c)| < -f'(c)$.

Thus, $f'(c) < \frac{f(x)-f(c)}{x-c} - f'(c) < -f'(c)$, so $\frac{f(x)-f(c)}{x-c} - f'(c) < -f'(c)$.

Hence, $\frac{f(x)-f(c)}{x-c} < 0$.

Since $x - c > 0$, then $f(x) - f(c) < 0$, so $f(x) < f(c)$.

Thus, we have $f(c) \leq f(x)$ and $f(c) > f(x)$, a contradiction.

Therefore, $f'(c)$ cannot be negative.

Since $f'(c)$ is neither positive nor negative, then $f'(c)$ must be zero, so $f'(c) = 0$. $\square$

Theorem 15. ***Rolle's Theorem***

Let $a, b \in \mathbb{R}$ with $a < b$.

Let f be a real valued function continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b).

If $f(a) = f(b)$, then there exists $c \in (a, b)$ such that $f'(c) = 0$.

Proof. Suppose $f(a) = f(b)$.

We must prove there exists $c \in (a, b)$ such that $f'(c) = 0$.

Let $k = f(a) = f(b)$.

Either $f(x) = k$ for all $x \in [a, b]$ or not.

We consider these cases separately.

Case 1: Suppose $f(x) = k$ for all $x \in [a, b]$.

Since $a < b$, then by the density of $\mathbb{R}$, there exists a real number c such that $a < c < b$.

Hence, $c \in (a, b)$.

Since $(a, b) \subset [a, b]$, then $c \in [a, b]$.

Since every point in $[a, b]$ is an accumulation point of $[a, b]$, then c is an accumulation point of $[a, b]$.

Since $[a, b] \subset \mathbb{R}$ and c is an accumulation point of $[a, b]$, then c is an accumulation point of $[a, b] - \{c\}$.

Let $q : [a, b] - \{c\} \to \mathbb{R}$ be a function defined by $q(x) = \frac{f(x)-f(c)}{x-c}$ for all $x \in [a, b] - \{c\}$.

Let $x \in [a, b] - \{c\}$.

Then $x \in [a, b]$ and $x \neq c$.

Since $x \neq c$, then $x - c \neq 0$.

Since $x \in [a, b]$ and $c \in [a, b]$ and $f(x) = k$ for all $x \in [a, b]$, then $f(x) = k = f(c)$.

Thus, $f(x) - f(c) = 0$.

Since $0 = \lim_{x\to c} \frac{0}{x-c} = \lim_{x\to c} \frac{f(x)-f(c)}{x-c} = \lim_{x\to c} q(x) = f'(c)$, then $f'(c) = 0$.

Therefore, there exists $c \in (a, b)$ such that $f'(c) = 0$, as desired.

Case 2: Suppose it is not the case that $f(x) = k$ for all $x \in [a, b]$.

Then there exists $x \in [a, b]$ such that $f(x) \neq k$.

Since $f(x) \neq k$, then either $f(x) > k$ or $f(x) < k$.

We consider these cases separately.

Case 2a: Suppose $f(x) > k$.

Since f is a real valued function continuous on the nonempty closed bounded interval $[a, b]$, then by EVT, f has a maximum on $[a, b]$.

Let $f(c)$ be a maximum of f on $[a,b]$.
Then there exists $c \in [a,b]$ such that $f(c) \geq f(x)$ for all $x \in [a,b]$.
Since $c \in [a,b]$, then $a \leq c \leq b$, so $a \leq c$ and $c \leq b$.
Since $x \in [a,b]$ and $f(c)$ is a maximum of f on $[a,b]$, then $f(c) \geq f(x)$.
Since $f(c) \geq f(x)$ and $f(x) > k$, then $f(c) > k$.
Since $c = a$ implies $f(c) = f(a)$, then $f(c) \neq f(a)$ implies $c \neq a$.
Since $f(c) > k = f(a)$, then $f(c) > f(a)$, so $f(c) \neq f(a)$.
Hence, $c \neq a$.
Since $c \geq a$ and $c \neq a$, then $c > a$.
Since $c = b$ implies $f(c) = f(b)$, then $f(c) \neq f(b)$ implies $c \neq b$.
Since $f(c) > k = f(b)$, then $f(c) > f(b)$, so $f(c) \neq f(b)$.
Hence, $c \neq b$.
Since $c \leq b$ and $c \neq b$, then $c < b$.
Hence, $a < c$ and $c < b$, so $a < c < b$.
Thus, $c \in (a,b)$.

Since the open interval (a,b) is an open set, then c is an interior point of (a,b), so there exists $\delta > 0$ such that $N(c;\delta) \subset (a,b)$.
Since $N(c;\delta) \subset (a,b)$ and $(a,b) \subset [a,b]$, then $N(c;\delta) \subset [a,b]$.
Thus, there exists $\delta > 0$ such that $N(c;\delta) \subset [a,b]$.

Let $p \in N(c;\delta)$.
Since $N(c;\delta) \subset [a,b]$, then $p \in [a,b]$.
Since $f(c)$ is a maximum of f on $[a,b]$, then $f(c) \geq f(p)$.
Thus, $f(c) \geq f(p)$ for all $p \in N(c;\delta)$.

Since there exists $\delta > 0$ such that $N(c;\delta) \subset [a,b]$ and $f(c) \geq f(p)$ for all $p \in N(c;\delta)$, then $f(c)$ is a relative maximum of f on $[a,b]$.
Since f is differentiable on (a,b) and $c \in (a,b)$, then f is differentiable at c.
Since $f(c)$ is a relative maximum and f is differentiable at c, then by the previous lemma, $f'(c) = 0$.
Therefore, there exists $c \in (a,b)$ such that $f'(c) = 0$, as desired.
Case 2b: Suppose $f(x) < k$.
Since f is a real valued function continuous on the nonempty closed bounded interval $[a,b]$, then by EVT, f has a minimum on $[a,b]$.

Let $f(c)$ be a minimum of f on $[a,b]$.
Then there exists $c \in [a,b]$ such that $f(c) \leq f(x)$ for all $x \in [a,b]$.
Since $c \in [a,b]$, then $a \leq c \leq b$, so $a \leq c$ and $c \leq b$.
Since $x \in [a,b]$ and $f(c)$ is a minimum of f on $[a,b]$, then $f(c) \leq f(x)$.
Since $f(c) \leq f(x)$ and $f(x) < k$, then $f(c) < k$.
Since $c = a$ implies $f(c) = f(a)$, then $f(c) \neq f(a)$ implies $c \neq a$.
Since $f(c) < k = f(a)$, then $f(c) < f(a)$, so $f(c) \neq f(a)$.
Hence, $c \neq a$.
Since $c \geq a$ and $c \neq a$, then $c > a$.

Since $c = b$ implies $f(c) = f(b)$, then $f(c) \neq f(b)$ implies $c \neq b$.
Since $f(c) < k = f(b)$, then $f(c) < f(b)$, so $f(c) \neq f(b)$.
Hence, $c \neq b$.
Since $c \leq b$ and $c \neq b$, then $c < b$.
Hence, $a < c$ and $c < b$, so $a < c < b$.
Thus, $c \in (a, b)$.

Since the open interval (a, b) is an open set, then c is an interior point of (a, b), so there exists $\delta > 0$ such that $N(c; \delta) \subset (a, b)$.
Since $N(c; \delta) \subset (a, b)$ and $(a, b) \subset [a, b]$, then $N(c; \delta) \subset [a, b]$.
Thus, there exists $\delta > 0$ such that $N(c; \delta) \subset [a, b]$.

Let $p \in N(c; \delta)$.
Since $N(c; \delta) \subset [a, b]$, then $p \in [a, b]$.
Since $f(c)$ is a minimum of f on $[a, b]$, then $f(c) \leq f(p)$.
Thus, $f(c) \leq f(p)$ for all $p \in N(c; \delta)$.

Since there exists $\delta > 0$ such that $N(c; \delta) \subset [a, b]$ and $f(c) \leq f(p)$ for all $p \in N(c; \delta)$, then $f(c)$ is a relative minimum of f on $[a, b]$.
Since f is differentiable on (a, b) and $c \in (a, b)$, then f is differentiable at c.
Since $f(c)$ is a relative minimum and f is differentiable at c, then by the previous lemma, $f'(c) = 0$.
Therefore, there exists $c \in (a, b)$ such that $f'(c) = 0$, as desired. □

Theorem 16. ***Mean Value Theorem***
Let $a, b \in \mathbb{R}$ with $a < b$.
Let f be a real valued function continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b).
Then there exists $c \in (a, b)$ such that $f'(c) = \frac{f(b)-f(a)}{b-a}$.

Proof. We must prove there exists $c \in (a, b)$ such that $f'(c) = \frac{f(b)-f(a)}{b-a}$.
Since $a < b$, then $b - a > 0$, so $\frac{f(b)-f(a)}{b-a}$ is a real number.
Let L be a real valued function defined by $L(x) = \frac{f(b)-f(a)}{b-a}(x - a) + f(a)$ for all $x \in [a, b]$.
Let g be defined by $g = f - L$.
Then g is a real valued function defined by $g(x) = f(x) - L(x)$ for all $x \in [a, b]$.
Thus, $g(x) = f(x) - \frac{f(b)-f(a)}{b-a}(x - a) - f(a)$ for all $x \in [a, b]$.
Since f is continuous on $[a, b]$, then f is continuous.
Since L is a linear function, then L is continuous.
Since the difference of continuous functions is continuous, then g is continuous, so g is continuous on $[a, b]$.

Since L is a linear function, then L is differentiable, so L is differentiable on $[a,b]$.

Since $(a,b) \subset [a,b]$, then L is differentiable on (a,b).

Since f is differentiable on (a,b) and L is differentiable on (a,b), then the difference $g = f - L$ is differentiable on (a,b).

Let $x \in (a,b)$.

Since $L(x) = \frac{f(b)-f(a)}{b-a}(x-a) + f(a)$, then $L'(x) = \frac{f(b)-f(a)}{b-a}$.

Since $g(x) = f(x) - L(x)$, then $g'(x) = f'(x) - L'(x) = f'(x) - \frac{f(b)-f(a)}{b-a}$, so $g'(x) = f'(x) - \frac{f(b)-f(a)}{b-a}$.

Hence, g is differentiable at x.

Thus, g is differentiable at x for all $x \in (a,b)$, so g is differentiable on (a,b).

Observe that

$$\begin{aligned} g(a) &= f(a) - \frac{f(b)-f(a)}{b-a}(a-a) - f(a) \\ &= 0 \\ &= f(b) - f(b) + f(a) - f(a) \\ &= f(b) - (f(b) - f(a)) - f(a) \\ &= f(b) - \frac{f(b)-f(a)}{b-a}(b-a) - f(a) \\ &= g(b). \end{aligned}$$

Since g is continuous on $[a,b]$ and g is differentiable on (a,b) and $g(a) = g(b)$, then by Rolle's theorem, there exists $c \in (a,b)$ such that $g'(c) = 0$.

Since $0 = g'(c) = f'(c) - L'(c) = f'(c) - \frac{f(b)-f(a)}{b-a}$, then $\frac{f(b)-f(a)}{b-a} = f'(c)$.

Therefore, there exists $c \in (a,b)$ such that $f'(c) = \frac{f(b)-f(a)}{b-a}$, as desired. $\square$

Corollary 17. *Let f be a real valued function differentiable on an interval I. Then*

1. *If $f'(x) = 0$ for all $x \in I$, then f is constant on I.*
2. *If $f'(x) > 0$ for all $x \in I$, then f is strictly increasing on I.*
3. *If $f'(x) < 0$ for all $x \in I$, then f is strictly decreasing on I.*

Proof. We prove 1.

Suppose $f'(x) = 0$ for all $x \in I$.

To prove f is constant on I, let $a, b \in I$ such that $a \neq b$.

We must prove $f(a) = f(b)$.

Since $a \neq b$, then either $a < b$ or $a > b$.

Without loss of generality, assume $a < b$.

Since $a \in I$ and $b \in I$ and $a < b$, then $[a,b] \subset I$.

Since f is differentiable on I and $[a,b] \subset I$, then f is differentiable on $[a,b]$, so f is continuous on $[a,b]$.

Since $(a,b) \subset [a,b]$ and $[a,b] \subset I$, then $(a,b) \subset I$.

Since f is differentiable on I and $(a,b) \subset I$, then f is differentiable on (a,b).

Since f is continuous on $[a, b]$ and differentiable on (a, b), then by MVT, there exists $c \in (a, b)$ such that $f'(c) = \frac{f(b)-f(a)}{b-a}$.

Since $c \in (a, b)$ and $(a, b) \subset I$, then $c \in I$, so $f'(c) = 0$.

Hence, $0 = f'(c) = \frac{f(b)-f(a)}{b-a}$, so $0 = \frac{f(b)-f(a)}{b-a}$.

Since $b \neq a$, then $b - a \neq 0$, so $0 = f(b) - f(a)$.

Therefore, $f(a) = f(b)$, as desired. □

Proof. We prove 2.

Suppose $f'(x) > 0$ for all $x \in I$.

To prove f is strictly increasing on I, let $a, b \in I$ such that $a < b$.

We must prove $f(a) < f(b)$.

Since $a \in I$ and $b \in I$ and $a < b$, then $[a, b] \subset I$.

Since f is differentiable on I and $[a, b] \subset I$, then f is differentiable on $[a, b]$, so f is continuous on $[a, b]$.

Since f is differentiable on $[a, b]$ and $(a, b) \subset [a, b]$, then f is differentiable on (a, b).

Since f is continuous on $[a, b]$ and differentiable on (a, b), then by MVT, there exists $c \in (a, b)$ such that $f'(c) = \frac{f(b)-f(a)}{b-a}$.

Since $c \in (a, b)$ and $(a, b) \subset [a, b] \subset I$, then $c \in I$.

Hence, $f'(c) > 0$.

Thus, $0 < f'(c) = \frac{f(b)-f(a)}{b-a}$, so $0 < \frac{f(b)-f(a)}{b-a}$.

Since $a < b$, then $b - a > 0$, so $0 < f(b) - f(a)$.

Therefore, $f(a) < f(b)$, as desired. □

Proof. We prove 3.

Suppose $f'(x) < 0$ for all $x \in I$.

To prove f is strictly decreasing on I, let $a, b \in I$ such that $a < b$.

We must prove $f(a) > f(b)$.

Since $a \in I$ and $b \in I$ and $a < b$, then $[a, b] \subset I$.

Since f is differentiable on I and $[a, b] \subset I$, then f is differentiable on $[a, b]$, so f is continuous on $[a, b]$.

Since f is differentiable on $[a, b]$ and $(a, b) \subset [a, b]$, then f is differentiable on (a, b).

Since f is continuous on $[a, b]$ and differentiable on (a, b), then by MVT, there exists $c \in (a, b)$ such that $f'(c) = \frac{f(b)-f(a)}{b-a}$.

Since $c \in (a, b)$ and $(a, b) \subset [a, b] \subset I$, then $c \in I$.

Hence, $f'(c) < 0$.

Thus, $0 > f'(c) = \frac{f(b)-f(a)}{b-a}$, so $0 > \frac{f(b)-f(a)}{b-a}$.

Since $a < b$, then $b - a > 0$, so $0 > f(b) - f(a)$.

Therefore, $f(a) > f(b)$, as desired. □

Corollary 18. ***functions with the same derivative on an interval differ by a constant***

Let f and g be real valued functions differentiable on an interval I.

If $f'(x) = g'(x)$ for all $x \in I$, then there exists $C \in \mathbb{R}$ such that $f - g = C$.

Proof. Suppose $f'(x) = g'(x)$ for all $x \in I$.
Let $h = f - g$.
Then $h : I \to \mathbb{R}$ is the function defined by $h(x) = f(x) - g(x)$.
Let $x \in I$.
Then $h(x) = f(x) - g(x)$, so $h'(x) = f'(x) - g'(x) = f'(x) - f'(x) = 0$.
Hence, $h'(x) = 0$ for all $x \in I$, so h is differentiable on I.
Since h is differentiable on I and $h'(x) = 0$ for all $x \in I$, then h is constant on I.
Thus, there exists $C \in \mathbb{R}$ such that $h(x) = C$ for all $x \in I$.
Hence, $C = h(x) = f(x) - g(x)$, so $C = f(x) - g(x)$ for all $x \in I$.
Therefore, $C = f - g$. □

Theorem 19. ***First Derivative Test for relative extrema of a function continuous on an open interval***
Let $E \subset \mathbb{R}$.
Let $f : E \to \mathbb{R}$ be a function.
Let $c \in E$.
Let there exist $\delta > 0$ such that $(c - \delta, c + \delta) \subset E$.
Suppose f is continuous on $(c - \delta, c + \delta)$ and differentiable on $(c - \delta, c)$ and $(c, c + \delta)$.
1. If $f'(x) < 0$ for all $x \in (c - \delta, c)$ and $f'(x) > 0$ for all $x \in (c, c + \delta)$, then $f(c)$ is a relative minimum.
2. If $f'(x) > 0$ for all $x \in (c - \delta, c)$ and $f'(x) < 0$ for all $x \in (c, c + \delta)$, then $f(c)$ is a relative maximum.

Proof. We prove 1.
Suppose $f'(x) < 0$ for all $x \in (c - \delta, c)$ and $f'(x) > 0$ for all $x \in (c, c + \delta)$.
Since $N(c; \delta) = (c - \delta, c + \delta)$ and $(c - \delta, c + \delta) \subset E$, then $N(c; \delta) \subset E$.
Thus, there exists $\delta > 0$ such that $N(c; \delta) \subset E$.
To prove $f(c)$ is a relative minimum, let $x \in N(c; \delta)$.
We must prove $f(c) \leq f(x)$.
Since $x, c \in \mathbb{R}$, then either $x < c$ or $x = c$ or $x > c$.
We consider these cases separately.
Case 1: Suppose $x = c$.
Then $f(c) = f(x)$.
Case 2: Suppose $x < c$.
Since $x \in N(c; \delta)$, then $|x - c| < \delta$.
Since $x < c$, then $x - c < 0$, so $|x - c| = -(x - c) = c - x < \delta$.
Hence, $c - \delta < x$.
Since $c - \delta < x < c < c + \delta$, then $[x, c] \subset (c - \delta, c + \delta)$ and $(x, c) \subset (c - \delta, c)$.
Since f is continuous on $(c - \delta, c + \delta)$ and $[x, c] \subset (c - \delta, c + \delta)$, then f is continuous on $[x, c]$.
Since f is differentiable on $(c - \delta, c)$ and $(x, c) \subset (c - \delta, c)$, then f is differentiable on (x, c).
Thus, by MVT, there exists $p \in (x, c)$ such that $f'(p) = \frac{f(c) - f(x)}{c - x}$.
Since $p \in (x, c)$ and $(x, c) \subset (c - \delta, c)$, then $p \in (c - \delta, c)$, so $f'(p) < 0$.

Hence, $\frac{f(c)-f(x)}{c-x} < 0$.
Since $x < c$, then $c - x > 0$, so $f(c) - f(x) < 0$.
Thus, $f(c) < f(x)$.
Case 3: Suppose $x > c$.
Since $x \in N(c;\delta)$, then $|x - c| < \delta$.
Since $x > c$, then $x - c > 0$, so $|x - c| = x - c < \delta$.
Hence, $x < c + \delta$.
Since $c - \delta < c < x < c + \delta$, then $[c, x] \subset (c - \delta, c + \delta)$ and $(c, x) \subset (c, c + \delta)$.
Since f is continuous on $(c - \delta, c + \delta)$ and $[c, x] \subset (c - \delta, c + \delta)$, then f is continuous on $[c, x]$.
Since f is differentiable on $(c, c + \delta)$ and $(c, x) \subset (c, c + \delta)$, then f is differentiable on (c, x).
Thus, by MVT, there exists $q \in (c, x)$ such that $f'(q) = \frac{f(x)-f(c)}{x-c}$.
Since $q \in (c, x)$ and $(c, x) \subset (c, c + \delta)$, then $q \in (c, c + \delta)$, so $f'(q) > 0$.
Hence, $\frac{f(x)-f(c)}{x-c} > 0$.
Since $x > c$, then $x - c > 0$, so $f(x) - f(c) > 0$.
Thus, $f(x) > f(c)$, so $f(c) < f(x)$.
Therefore, in all cases, $f(c) \leq f(x)$, as desired. $\square$

Proof. We prove 2.
Suppose $f'(x) > 0$ for all $x \in (c - \delta, c)$ and $f'(x) < 0$ for all $x \in (c, c + \delta)$.
Since $N(c;\delta) = (c - \delta, c + \delta)$ and $(c - \delta, c + \delta) \subset E$, then $N(c;\delta) \subset E$.
Thus, there exists $\delta > 0$ such that $N(c;\delta) \subset E$.
To prove $f(c)$ is a relative maximum, let $x \in N(c;\delta)$.
We must prove $f(c) \geq f(x)$.
Since $x, c \in \mathbb{R}$, then either $x < c$ or $x = c$ or $x > c$.
We consider these cases separately.
Case 1: Suppose $x = c$.
Then $f(c) = f(x)$.
Case 2: Suppose $x < c$.
Since $x \in N(c;\delta)$, then $|x - c| < \delta$.
Since $x < c$, then $x - c < 0$, so $|x - c| = -(x - c) = c - x < \delta$.
Hence, $c - \delta < x$.
Since $c - \delta < x < c < c + \delta$, then $[x, c] \subset (c - \delta, c + \delta)$ and $(x, c) \subset (c - \delta, c)$.
Since f is continuous on $(c - \delta, c + \delta)$ and $[x, c] \subset (c - \delta, c + \delta)$, then f is continuous on $[x, c]$.
Since f is differentiable on $(c - \delta, c)$ and $(x, c) \subset (c - \delta, c)$, then f is differentiable on (x, c).
Thus, by MVT, there exists $p \in (x, c)$ such that $f'(p) = \frac{f(c)-f(x)}{c-x}$.
Since $p \in (x, c)$ and $(x, c) \subset (c - \delta, c)$, then $p \in (c - \delta, c)$, so $f'(p) > 0$.
Hence, $\frac{f(c)-f(x)}{c-x} > 0$.
Since $x < c$, then $c - x > 0$, so $f(c) - f(x) > 0$.
Thus, $f(c) > f(x)$.
Case 3: Suppose $x > c$.
Since $x \in N(c;\delta)$, then $|x - c| < \delta$.

Since $x > c$, then $x - c > 0$, so $|x - c| = x - c < \delta$.

Hence, $x < c + \delta$.

Since $c - \delta < c < x < c + \delta$, then $[c, x] \subset (c - \delta, c + \delta)$ and $(c, x) \subset (c, c + \delta)$.

Since f is continuous on $(c - \delta, c + \delta)$ and $[c, x] \subset (c - \delta, c + \delta)$, then f is continuous on $[c, x]$.

Since f is differentiable on $(c, c + \delta)$ and $(c, x) \subset (c, c + \delta)$, then f is differentiable on (c, x).

Thus, by MVT, there exists $q \in (c, x)$ such that $f'(q) = \frac{f(x)-f(c)}{x-c}$.

Since $q \in (c, x)$ and $(c, x) \subset (c, c + \delta)$, then $q \in (c, c + \delta)$, so $f'(q) < 0$.

Hence, $\frac{f(x)-f(c)}{x-c} < 0$.

Since $x > c$, then $x - c > 0$, so $f(x) - f(c) < 0$.

Thus, $f(x) < f(c)$, so $f(c) > f(x)$.

Therefore, in all cases, $f(c) \geq f(x)$, as desired. □

Lemma 20. *Let $E \subset \mathbb{R}$.*

Let $f : E \to \mathbb{R}$ be a function.

Let c be a point.

1. If the limit of f at c exists and is positive, then there exists $\delta > 0$ such that $f(x) > 0$ for all $x \in N'(c; \delta) \cap E$.

2. If the limit of f at c exists and is negative, then there exists $\delta > 0$ such that $f(x) < 0$ for all $x \in N'(c; \delta) \cap E$.

Proof. We prove 1.

Suppose the limit of f at c exists and is positive.

Then there exists $L \in \mathbb{R}$ with $L > 0$ such that $L = \lim_{x \to c} f(x)$.

Since $L > 0$, then there exists $\delta > 0$ such that for all $x \in E$, if $0 < |x - c| < \delta$, then $|f(x) - L| < L$.

Let $x \in N'(c; \delta) \cap E$.

Then $x \in N'(c; \delta)$ and $x \in E$.

Since $x \in N'(c; \delta)$, then $x \in N(c; \delta)$ and $x \neq c$.

Since $x \in N(c; \delta)$, then $d(x, c) = |x - c| < \delta$.

Since $x \neq c$, then $x - c \neq 0$, so $|x - c| > 0$.

Thus, $0 < |x - c|$ and $|x - c| < \delta$, so $0 < |x - c| < \delta$.

Since $x \in E$ and $0 < |x - c| < \delta$, then $|f(x) - L| < L$.

Hence, $-L < f(x) - L < L$, so $-L < f(x) - L$.

Thus, $0 < f(x)$, so $f(x) > 0$.

Consequently, $f(x) > 0$ for all $x \in N'(c; \delta) \cap E$.

Therefore, there exists $\delta > 0$ such that $f(x) > 0$ for all $x \in N'(c; \delta) \cap E$, as desired. □

Proof. We prove 2.

Suppose the limit of f at c exists and is negative.

Then there exists $L \in \mathbb{R}$ with $L < 0$ such that $L = \lim_{x \to c} f(x)$.

Since $L < 0$, then $-L > 0$, so there exists $\delta > 0$ such that for all $x \in E$, if $0 < |x - c| < \delta$, then $|f(x) - L| < -L$.

Let $x \in N'(c;\delta) \cap E$.
Then $x \in N'(c;\delta)$ and $x \in E$.
Since $x \in N'(c;\delta)$, then $x \in N(c;\delta)$ and $x \neq c$.
Since $x \in N(c;\delta)$, then $d(x,c) = |x-c| < \delta$.
Since $x \neq c$, then $x - c \neq 0$, so $|x-c| > 0$.
Thus, $0 < |x-c|$ and $|x-c| < \delta$, so $0 < |x-c| < \delta$.
Since $x \in E$ and $0 < |x-c| < \delta$, then $|f(x) - L| < -L$.
Hence, $L < f(x) - L < -L$, so $f(x) - L < -L$.
Thus, $f(x) < 0$.
Consequently, $f(x) < 0$ for all $x \in N'(c;\delta) \cap E$.
Therefore, there exists $\delta > 0$ such that $f(x) < 0$ for all $x \in N'(c;\delta) \cap E$, as desired. ∎

Lemma 21. *Let $E \subset \mathbb{R}$.*
Let $f : E \to \mathbb{R}$ be a function.
Let f be differentiable at $c \in E$.
1. If $f'(c) > 0$, then there exists $\delta > 0$ such that $f(c) > f(x)$ for all $x \in (c-\delta, c) \cap E$.
2. If $f'(c) < 0$, then there exists $\delta > 0$ such that $f(c) > f(x)$ for all $x \in (c, c+\delta) \cap E$.

Proof. We prove 1.
Suppose $f'(c) > 0$.
Then $f'(c) = \lim_{x \to c} \frac{f(x)-f(c)}{x-c} > 0$.
Since $E - \{c\} \subset E$ and $E \subset \mathbb{R}$, then $E - \{c\} \subset \mathbb{R}$.
Let $q : E - \{c\} \to \mathbb{R}$ be a function defined by $q(x) = \frac{f(x)-f(c)}{x-c}$.
Then $f'(c) = \lim_{x \to c} q(x) > 0$, so the limit of q at c exists and is positive.
Hence, by the previous lemma, there exists $\delta > 0$ such that $q(x) > 0$ for all $x \in N'(c;\delta) \cap E$.
Thus, there exists $\delta > 0$ such that $\frac{f(x)-f(c)}{x-c} > 0$ for all $x \in N'(c;\delta) \cap E$.

Let $x \in (c-\delta, c) \cap E$.
Then $x \in (c-\delta, c)$ and $x \in E$.
Since $(c-\delta, c) \subset (c-\delta, c) \cup (c, c+\delta) = N'(c;\delta)$, then $(c-\delta, c) \subset N'(c,\delta)$.
Since $x \in (c-\delta, c)$ and $(c-\delta, c) \subset N'(c,\delta)$, then $x \in N'(c,\delta)$.
Since $x \in N'(c,\delta)$ and $x \in E$, then $x \in N'(c,\delta) \cap E$, so $\frac{f(x)-f(c)}{x-c} > 0$.
Since $x \in (c-\delta, c)$, then $c - \delta < x < c$, so $x < c$.
Thus, $x - c < 0$.
Since $\frac{f(x)-f(c)}{x-c} > 0$, then $f(x) - f(c) < 0$, so $f(x) < f(c)$.
Hence, $f(c) > f(x)$, so $f(c) > f(x)$ for all $x \in (c-\delta, c) \cap E$.
Therefore, there exists $\delta > 0$ such that $f(c) > f(x)$ for all $x \in (c-\delta, c) \cap E$, as desired. ∎

Proof. We prove 2.
Suppose $f'(c) < 0$.
Then $f'(c) = \lim_{x \to c} \frac{f(x)-f(c)}{x-c} < 0$.

Since $E - \{c\} \subset E$ and $E \subset \mathbb{R}$, then $E - \{c\} \subset \mathbb{R}$.

Let $q : E - \{c\} \to \mathbb{R}$ be a function defined by $q(x) = \frac{f(x)-f(c)}{x-c}$.

Then $f'(c) = \lim_{x \to c} q(x) < 0$, so the limit of q at c exists and is negative.

Hence, by the previous lemma, there exists $\delta > 0$ such that $q(x) < 0$ for all $x \in N'(c;\delta) \cap E$.

Thus, there exists $\delta > 0$ such that $\frac{f(x)-f(c)}{x-c} < 0$ for all $x \in N'(c;\delta) \cap E$.

Let $x \in (c, c+\delta) \cap E$.

Then $x \in (c, c+\delta)$ and $x \in E$.

Since $(c, c+\delta) \subset (c-\delta, c) \cup (c, c+\delta) = N'(c;\delta)$, then $(c, c+\delta) \subset N'(c, \delta)$.

Since $x \in (c, c+\delta)$ and $(c, c+\delta) \subset N'(c, \delta)$, then $x \in N'(c, \delta)$.

Since $x \in N'(c, \delta)$ and $x \in E$, then $x \in N'(c, \delta) \cap E$, so $\frac{f(x)-f(c)}{x-c} < 0$.

Since $x \in (c, c+\delta)$, then $c < x < c+\delta$, so $c < x$.

Thus, $x - c > 0$.

Since $\frac{f(x)-f(c)}{x-c} < 0$, then $f(x) - f(c) < 0$, so $f(x) < f(c)$.

Hence, $f(c) > f(x)$, so $f(c) > f(x)$ for all $x \in (c, c+\delta) \cap E$.

Therefore, there exists $\delta > 0$ such that $f(c) > f(x)$ for all $x \in (c, c+\delta) \cap E$, as desired. □

Theorem 22. ***Intermediate Value property of Derivatives***

Let $a, b \in \mathbb{R}$ with $a < b$.

Let f be a real valued function differentiable on the closed interval $[a, b]$.

For every real number k such that $f'(a) < k < f'(b)$, there exists $c \in (a, b)$ such that $f'(c) = k$.

Proof. Let k be an arbitrary real number such that $f'(a) < k < f'(b)$.

Then $f'(a) < k$ and $k < f'(b)$, so $f'(a) - k < 0$ and $0 < f'(b) - k$.

Let $g : [a, b] \to \mathbb{R}$ be a function defined by $g(x) = f(x) - kx$.

Since f is differentiable on $[a, b]$, then $g'(x) = f'(x) - k$ for all $x \in [a, b]$, so g is differentiable on $[a, b]$.

Hence, g is differentiable, so g is continuous.

Thus, g is continuous on $[a, b]$.

Since $a < b$, then the closed interval $[a, b]$ is not empty, so by EVT, g attains a minimum on $[a, b]$.

Let $g(c)$ be a minimum of g on $[a, b]$.

Then there exists $c \in [a, b]$ such that $g(c) \leq g(x)$ for all $x \in [a, b]$.

Since $c \in [a, b]$, then $a \leq c \leq b$, so $a \leq c$ and $c \leq b$.

We prove $c < b$.

Since $b \in [a, b]$, then $g'(b) = f'(b) - k > 0$, so g is differentiable at b.

Thus, by the previous lemma, there exists $\delta_1 > 0$ such that $g(b) > g(x)$ for all $x \in (b - \delta_1, b) \cap [a, b]$.

Let $x \in (b - \delta_1, b) \cap [a, b]$.

Then $x \in (b - \delta_1, b)$ and $x \in [a, b]$ and $g(b) > g(x)$.

Since $x \in [a, b]$ and $g(c)$ is a minimum of g on $[a, b]$, then $g(c) \leq g(x)$.

Thus, $g(c) \leq g(x)$ and $g(x) < g(b)$, so $g(c) < g(b)$.

Since $c = b$ implies $g(c) = g(b)$, then $g(c) \neq g(b)$ implies $c \neq b$.

Since $g(c) < g(b)$, then $g(c) \neq g(b)$, so $c \neq b$.

Since $c \leq b$ and $c \neq b$, then $c < b$.

We prove $a < c$.

Since $a \in [a, b]$, then $g'(a) = f'(a) - k < 0$, so g is differentiable at a.

Thus, by the previous lemma, there exists $\delta_2 > 0$ such that $g(a) > g(x)$ for all $x \in (a, a + \delta_2) \cap [a, b]$.

Let $y \in (a, a + \delta_2) \cap [a, b]$.

Then $y \in (a, a + \delta_2)$ and $y \in [a, b]$ and $g(a) > g(y)$.

Since $y \in [a, b]$ and $g(c)$ is a minimum of g on $[a, b]$, then $g(c) \leq g(y)$.

Thus, $g(c) \leq g(y)$ and $g(y) < g(a)$, so $g(c) < g(a)$.

Since $c = a$ implies $g(c) = g(a)$, then $g(c) \neq g(a)$ implies $c \neq a$.

Since $g(c) < g(a)$, then $g(c) \neq g(a)$, so $c \neq a$.

Since $c \geq a$ and $c \neq a$, then $c > a$.

Since $a < c$ and $c < b$, then $a < c < b$, so $c \in (a, b)$.

Since the open interval (a, b) is an open set, then c is an interior point of (a, b).

Hence, there exists $\delta > 0$ such that $N(c; \delta) \subset (a, b)$.

Since $N(c; \delta) \subset (a, b)$ and $(a, b) \subset [a, b]$, then $N(c; \delta) \subset [a, b]$.

Thus, there exists $\delta > 0$ such that $N(c; \delta) \subset [a, b]$.

Let $p \in N(c; \delta)$.

Since $N(c; \delta) \subset [a, b]$, then $p \in [a, b]$.

Since $g(c)$ is a minimum of g on $[a, b]$, then $g(c) \leq g(p)$.

Hence, $g(c) \leq g(p)$ for all $p \in N(c; \delta)$.

Since there exists $\delta > 0$ such that $N(c; \delta) \subset [a, b]$ and $g(c) \leq g(p)$ for all $p \in N(c; \delta)$, then $g(c)$ is a relative minimum of g.

Since $c \in [a, b]$, then $g'(c) = f'(c) - k$, so g is differentiable at c.

Since $g(c)$ is a relative minimum of g and g is differentiable at c, then by a previous lemma to Rolle's theorem, $g'(c) = 0$.

Thus, $0 = g'(c) = f'(c) - k$, so $f'(c) = k$.

Therefore, there exists $c \in (a, b)$ such that $f'(c) = k$, as desired. $\square$

L'Hopital's Rule

Theorem 23. ***Cauchy Mean Value Theorem***
Let $a, b \in \mathbb{R}$ with $a < b$.
Let f be a real valued function continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b).
Let g be a real valued function continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b).
Then there exists $c \in (a, b)$ such that $f'(c)(g(b) - g(a)) = g'(c)(f(b) - f(a))$.

Proof. Let $h : [a, b] \to \mathbb{R}$ be a function defined by $h(x) = f(x)(g(b) - g(a)) - g(x)(f(b) - f(a))$.
Then $h = f(g(b) - g(a)) - g(f(b) - f(a))$.
Since f is continuous on $[a, b]$, then the scalar multiple $f(g(b) - g(a))$ is continuous on $[a, b]$.
Since g is continuous on $[a, b]$, then the scalar multiple $g(f(b) - f(a))$ is continuous on $[a, b]$.
Hence, the difference $h = f(g(b) - g(a)) - g(f(b) - f(a))$ is continuous on $[a, b]$.
Let $x \in (a, b)$.
Since f is differentiable on (a, b) and g is differentiable on (a, b), then $h'(x) = f'(x)(g(b) - g(a)) - g'(x)(f(b) - f(a))$, so h is differentiable on (a, b).

Observe that

$$\begin{aligned}
h(a) &= f(a)(g(b) - g(a)) - g(a)(f(b) - f(a)) \\
&= f(a)g(b) - f(a)g(a) - g(a)f(b) + g(a)f(a) \\
&= f(a)g(b) - g(a)f(b) \\
&= f(b)g(b) - g(a)f(b) - g(b)f(b) + f(a)g(b) \\
&= f(b)(g(b) - g(a)) - g(b)(f(b) - f(a)) \\
&= h(b).
\end{aligned}$$

Since h is continuous on $[a, b]$ and differentiable on (a, b) and $h(a) = h(b)$, then by Rolle's theorem, there exists $c \in (a, b)$ such that $h'(c) = 0$.
Thus, $0 = h'(c) = f'(c)(g(b) - g(a)) - g'(c)(f(b) - f(a))$, so $g'(c)(f(b) - f(a)) = f'(c)(g(b) - g(a))$.
Therefore, there exists $c \in (a, b)$ such that $f'(c)(g(b) - g(a)) = g'(c)(f(b) - f(a))$, as desired. □