Differentiation of real valued functions Examples

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May 19, 2023

Differentiation of real valued functions

Example 1. the square function is differentiable
Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by $f(x) = x^2$.
Then $f'(x) = 2x$ for all $x \in \mathbb{R}$.
Therefore, $f'(x) = \frac{d}{dx}(x^2) = 2x$.

Proof. We prove $f'(x) = 2x$ for all $x \in \mathbb{R}$ using the definition of derivative.
Let $c \in \mathbb{R}$ be given.
We must prove $f'(c) = 2c$.
Let $q : \mathbb{R} - \{c\} \to \mathbb{R}$ be a function defined by $q(x) = \frac{f(x)-f(c)}{x-c}$ for all $x \in \mathbb{R} - \{c\}$.
Since $c \in \mathbb{R}$ and every real number is an accumulation point of $\mathbb{R}$, then c is an accumulation point of $\mathbb{R}$, so c is an accumulation point of $\mathbb{R} - \{c\}$, the domain of q.
For $x \in \mathbb{R} - \{c\}$, we have $x \in \mathbb{R}$ and $x \neq c$, so $x - c \neq 0$.
Observe that

$$\begin{aligned} \lim_{x\to c} q(x) &= \lim_{x\to c} \frac{f(x) - f(c)}{x - c} \\ &= \lim_{x\to c} \frac{x^2 - c^2}{x - c} \\ &= \lim_{x\to c} \frac{(x - c)(x + c)}{x - c} \\ &= \lim_{x\to c} (x + c) \\ &= 2c. \end{aligned}$$

Thus, $f'(c) = \lim_{x\to c} \frac{f(x)-f(c)}{x-c} = 2c$, as desired. □

Example 2. the reciprocal function is differentiable for nonzero x
Let $f : \mathbb{R}^* \to \mathbb{R}$ be the function defined by $f(x) = \frac{1}{x}$ for all nonzero real x.
Then $f'(x) = \frac{-1}{x^2}$ for all nonzero real x.
Therefore, $f'(x) = \frac{d}{dx}(\frac{1}{x}) = \frac{-1}{x^2}$ if $x \neq 0$.

Proof. We prove $f'(x) = \frac{-1}{x^2}$ for all nonzero real x using the definition of derivative.

Let c be a nonzero real number.

We must prove $f'(c) = \frac{-1}{c^2}$.

Let $q : \mathbb{R}^* - \{c\} \to \mathbb{R}$ be a function defined by $q(x) = \frac{f(x)-f(c)}{x-c}$ for all $x \in \mathbb{R}^* - \{c\}$.

Since $\mathbb{R}^* = \{x \in \mathbb{R} : x \neq 0\} = (-\infty, 0) \cup (0, \infty)$ is a union of open intervals, then $\mathbb{R}^*$ is an open set, so every point of $\mathbb{R}^*$ is an interior point of $\mathbb{R}^*$.

Since $c \in \mathbb{R}^*$, then c is an interior point of $\mathbb{R}^*$, so c is an accumulation point of $\mathbb{R}^*$.

Hence, c is an accumulation point of $\mathbb{R}^* - \{c\}$, the domain of q.

For $x \in \mathbb{R}^* - \{c\}$, we have $x \in \mathbb{R}^*$ and $x \neq c$, so $x - c \neq 0$.

Observe that

$$\begin{aligned}
\lim_{x\to c} q(x) &= \lim_{x\to c} \frac{f(x) - f(c)}{x - c} \\
&= \lim_{x\to c} \frac{\frac{1}{x} - \frac{1}{c}}{x - c} \\
&= \lim_{x\to c} \frac{c - x}{xc(x - c)} \\
&= \lim_{x\to c} \frac{-1}{xc} \\
&= \frac{-1}{c^2}.
\end{aligned}$$

Thus, $f'(c) = \lim_{x\to c} \frac{f(x)-f(c)}{x-c} = \frac{-1}{c^2}$, as desired. □

Example 3. the absolute value function is not differentiable at zero

Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by $f(x) = |x|$ for all $x \in \mathbb{R}$.

Then f is not differentiable at 0.

Proof. Let $q : \mathbb{R} - \{0\} \to \mathbb{R}$ be a function defined by $q(x) = \frac{f(x)-f(0)}{x-0} = \frac{|x|}{x}$ for all $x \in \mathbb{R} - \{0\} = \mathbb{R}^*$.

Since $0 \in \mathbb{R}$ and every real number is an accumulation point of $\mathbb{R}$, then 0 is an accumulation point of $\mathbb{R}$, so 0 is an accumulation point of $\mathbb{R} - \{0\} = \mathbb{R}^*$, the domain of q.

We consider the limit $\lim_{x\to 0} q(x) = \lim_{x\to 0} \frac{|x|}{x}$.

For $x \in \mathbb{R}^*$, we have $x \in \mathbb{R}$ and $x \neq 0$, so either $x > 0$ or $x < 0$.

If $x > 0$, then $\lim_{x\to 0} q(x) = \lim_{x\to 0} \frac{|x|}{x} = \lim_{x\to 0} \frac{x}{x} = \lim_{x\to 0} 1 = 1$.

If $x < 0$, then $\lim_{x\to 0} q(x) = \lim_{x\to 0} \frac{|x|}{x} = \lim_{x\to 0} \frac{-x}{x} = \lim_{x\to 0} -1 = -1$.

Therefore, $\lim_{x\to 0} q(x)$ does not exist, so f is not differentiable at 0. □

Example 4. the square root function is differentiable for positive real x

Let $f : [0, \infty) \to \mathbb{R}$ be the function defined by $f(x) = \sqrt{x}$.

Then $f'(x) = \frac{1}{2\sqrt{x}}$ for all positive real x and f is not differentiable at 0.

Therefore, $f'(x) = \frac{d}{dx}(\sqrt{x}) = \frac{1}{2\sqrt{x}}$ if $x > 0$ and f is not differentiable at 0.

Proof. We prove f is not differentiable at 0.

Observe that $0 \in [0, \infty)$ and 0 is an accumulation point of $[0, \infty)$, the domain of f.

We prove $\lim_{x \to 0} \frac{\sqrt{x}}{x}$ does not exist.

Let $g : \mathbb{R}^+ \to \mathbb{R}$ be the function defined by $g(x) = \frac{\sqrt{x}}{x}$.

Let $x \in \mathbb{R}^+$.

Then $x \in \mathbb{R}$ and $x > 0$, so $g(x) = \frac{\sqrt{x}}{x} = \frac{1}{\sqrt{x}}$.

Thus, $g(x) = \frac{1}{\sqrt{x}}$ for all $x \in \mathbb{R}^+$.

We prove there is no real number L such that $\lim_{x \to 0} g(x) = L$.

Observe that

$$\begin{aligned}
\neg(\exists L \in \mathbb{R})(\lim_{x \to 0} g(x) = L) \quad &\Leftrightarrow \\
\neg(\exists L \in \mathbb{R})(\forall \epsilon > 0)(\exists \delta > 0)(\forall x \in \mathbb{R}^+)(0 < |x| < \delta \to |g(x) - L| < \epsilon) \quad &\Leftrightarrow \\
(\forall L \in \mathbb{R})(\exists \epsilon > 0)(\forall \delta > 0)(\exists x \in \mathbb{R}^+)(0 < |x| < \delta \wedge |g(x) - L| \geq \epsilon).
\end{aligned}$$

Thus, we prove $(\forall L \in \mathbb{R})(\exists \epsilon > 0)(\forall \delta > 0)(\exists x \in \mathbb{R}^+)(0 < |x| < \delta \wedge |\frac{1}{\sqrt{x}} - L| \geq \epsilon)$.

Let L be an arbitrary real number.

Let $\epsilon = |L| + 1$.

Since $|L| \geq 0$, then $\epsilon = |L| + 1 \geq 1 > 0$, so $\epsilon > 0$.

Let $\delta > 0$ be given.

We must prove there exists $x > 0$ such that $0 < |x| < \delta$ and $|\frac{1}{\sqrt{x}} - L| \geq \epsilon$.

Let $x = \min\{\frac{\delta}{2}, \frac{1}{(|L|+\epsilon)^2}\}$.

Then $x \leq \frac{\delta}{2}$ and $x \leq \frac{1}{(|L|+\epsilon)^2}$ and either $x = \frac{\delta}{2}$ or $x = \frac{1}{(|L|+\epsilon)^2}$.

Since $|L| \geq 0$ and $\epsilon > 0$, then $|L| + \epsilon > 0$, so $(|L| + \epsilon)^2 > 0$.

Hence, $\frac{1}{(|L|+\epsilon)^2} > 0$.

Since $\delta > 0$, then $\frac{\delta}{2} > 0$.

Since either $x = \frac{\delta}{2}$ or $x = \frac{1}{(|L|+\epsilon)^2}$ and $\frac{\delta}{2} > 0$ and $\frac{1}{(|L|+\epsilon)^2} > 0$, then $x > 0$.

Since $\delta > 0$ and $0 < x = |x| \leq \frac{\delta}{2} < \delta$, then $0 < |x| < \delta$.

Since $x \leq \frac{1}{(|L|+\epsilon)^2}$ and $(|L| + \epsilon)^2 > 0$, then $x(|L| + \epsilon)^2 \leq 1$.

Since $x > 0$, then $0 < (|L| + \epsilon)^2 \leq \frac{1}{x}$.

Thus, $0 < ||L| + \epsilon| \leq \sqrt{\frac{1}{x}} = \frac{1}{\sqrt{x}}$, so $0 < |L| + \epsilon \leq \frac{1}{\sqrt{x}}$.

Hence, $|L| + \epsilon \leq \frac{1}{\sqrt{x}}$, so $\epsilon \leq \frac{1}{\sqrt{x}} - |L|$.

Therefore, $|\frac{1}{\sqrt{x}} - L| \geq |\frac{1}{\sqrt{x}}| - |L| = \frac{1}{\sqrt{x}} - |L| \geq \epsilon$, so $|\frac{1}{\sqrt{x}} - L| \geq \epsilon$.

Thus, there exists $x > 0$ such that $0 < |x| < \delta$ and $|\frac{1}{\sqrt{x}} - L| \geq \epsilon$.

Consequently, there is no real number L such that $\lim_{x \to 0} g(x) = L$, so $\lim_{x \to 0} g(x)$ does not exist.

Since $\lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{\sqrt{x}}{x} = \lim_{x \to 0} g(x)$ does not exist, then f is not differentiable at 0. □

Proof. We prove $f'(x) = \frac{1}{2\sqrt{x}}$ for all positive real x using the definition of derivative.

Let c be a positive real number.

Then $c \in \mathbb{R}^+$, so $c \in \mathbb{R}$ and $c > 0$.

Since $c > 0$, then $\sqrt{c} > 0$.

We must prove $f'(c) = \frac{1}{2\sqrt{c}}$.

Let $q : \mathbb{R}^+ - \{c\} \to \mathbb{R}$ be a function defined by $q(x) = \frac{f(x)-f(c)}{x-c}$ for all $x \in \mathbb{R}^+ - \{c\}$.

Since $\mathbb{R}^+ = \{x \in \mathbb{R} : x > 0\} = (0, \infty)$ is an open set, then every point of $\mathbb{R}^+$ is an interior point of $\mathbb{R}^+$.

Since $c \in \mathbb{R}^+$, then c is an interior point of $\mathbb{R}^+$, so c is an accumulation point of $\mathbb{R}^+$.

Hence, c is an accumulation point of $\mathbb{R}^+ - \{c\}$, the domain of q.

For $x \in \mathbb{R}^+ - \{c\}$, we have $x \in \mathbb{R}^+$ and $x \neq c$, so $x > 0$.

Since $x > 0$ and $c > 0$, then $\sqrt{x} = \sqrt{c}$ iff $x = c$ iff $x - c = 0$.

Since $x \neq c$, then $\sqrt{x} \neq \sqrt{c}$, so $\sqrt{x} - \sqrt{c} \neq 0$.

Since $c > 0$, then $\lim_{x\to c}(\sqrt{x} + \sqrt{c}) = \lim_{x\to c} \sqrt{x} + \lim_{x\to c} \sqrt{c} = \sqrt{c} + \sqrt{c} = 2\sqrt{c} > 0$, so $\lim_{x\to c}(\sqrt{x} + \sqrt{c}) \neq 0$.

Observe that

$$\begin{aligned}
\lim_{x\to c} q(x) &= \lim_{x\to c} \frac{f(x) - f(c)}{x - c} \\
&= \lim_{x\to c} \frac{\sqrt{x} - \sqrt{c}}{x - c} \\
&= \lim_{x\to c} \frac{\sqrt{x} - \sqrt{c}}{(\sqrt{x} - \sqrt{c})(\sqrt{x} + \sqrt{c})} \\
&= \lim_{x\to c} \frac{1}{\sqrt{x} + \sqrt{c}} \\
&= \frac{1}{\lim_{x\to c}(\sqrt{x} + \sqrt{c})} \\
&= \frac{1}{2\sqrt{c}}.
\end{aligned}$$

Therefore, $f'(c) = \lim_{x\to c} \frac{f(x)-f(c)}{x-c} = \frac{1}{2\sqrt{c}}$, as desired. □

Lemma 5. $\lim_{x\to 0} \frac{1-\cos x}{x} = 0$ *and* $\lim_{x\to 0} \frac{\cos x - 1}{x} = 0$.

Proof. We first prove $\lim_{x\to 0} \frac{1-\cos x}{x} = 0$.

Since $\cos x$ is continuous, then $\cos x$ is continuous at 0.

Since 0 is an accumulation point of $\mathbb{R}$, the domain of $\cos x$, then by the characterization of continuity, $\lim_{x\to 0} \cos x = \cos 0 = 1$.

Since $\sin x$ is continuous, then $\sin x$ is continuous at 0.

Since 0 is an accumulation point of $\mathbb{R}$, the domain of $\sin x$, then by the characterization of continuity, $\lim_{x\to 0} \sin x = \sin 0 = 0$.

We use the unproven fact that $\lim_{x\to 0} \frac{\sin x}{x} = 1$.

Let $g(x) = \frac{1+\cos x}{1+\cos x}$.

Since $1 + \cos x = 0$ iff $\cos x = -1$ iff x is an odd integer multiple of π, then the domain of g is the set $\mathbb{R} - \{x : \ x \text{ is an odd integer multiple of } \pi \}$.

Let $x \in dom g$.

Then $x \in \mathbb{R}$ and x is not an odd integer multiple of π, so $1 + \cos x \neq 0$.

Thus, $g(x) = \frac{1+\cos x}{1+\cos x} = 1$, so $g(x) = 1$ for all $x \in dom g$.

Hence, $1 = \lim_{x\to 0} 1 = \lim_{x\to 0} g(x) = \lim_{x\to 0} \frac{1+\cos x}{1+\cos x}$.

Observe that

$$
\begin{aligned}
0 &= \frac{0}{2} \\
&= \frac{0}{1+1} \\
&= \frac{\lim_{x\to 0} \sin x}{\lim_{x\to 0} 1 + \lim_{x\to 0} \cos x} \\
&= \frac{\lim_{x\to 0} \sin x}{\lim_{x\to 0}(1 + \cos x)} \\
&= \lim_{x\to 0} \frac{\sin x}{1+\cos x} \\
&= 1 \cdot \lim_{x\to 0} \frac{\sin x}{1+\cos x} \\
&= \lim_{x\to 0} \frac{\sin x}{x} \cdot \lim_{x\to 0} \frac{\sin x}{1+\cos x} \\
&= \lim_{x\to 0} \frac{\sin x}{x} \cdot \frac{\sin x}{1+\cos x} \\
&= \lim_{x\to 0} \frac{\sin^2 x}{x(1+\cos x)} \\
&= \lim_{x\to 0} \frac{1-\cos^2 x}{x(1+\cos x)} \\
&= \lim_{x\to 0} \frac{(1-\cos x)(1+\cos x)}{x(1+\cos x)} \\
&= \lim_{x\to 0} \frac{1-\cos x}{x} \cdot \frac{1+\cos x}{1+\cos x} \\
&= \lim_{x\to 0} \frac{1-\cos x}{x} \cdot \lim_{x\to 0} \frac{1+\cos x}{1+\cos x} \\
&= \lim_{x\to 0} \frac{1-\cos x}{x} \cdot 1 \\
&= \lim_{x\to 0} \frac{1-\cos x}{x}.
\end{aligned}
$$

Therefore, $\lim_{x\to 0} \frac{1-\cos x}{x} = 0$. $\square$

Proof. We next prove $\lim_{x\to 0} \frac{\cos x - 1}{x} = 0$.

Since $\lim_{x\to 0} \frac{1-\cos x}{x} = 0$, then by the scalar multiple rule for limits, we have

$$
\begin{aligned}
0 &= -1 \cdot 0 \\
&= -1 \cdot \lim_{x \to 0} \frac{1 - \cos x}{x} \\
&= \lim_{x \to 0} (-1 \cdot \frac{1 - \cos x}{x}) \\
&= \lim_{x \to 0} \frac{-1 + \cos x}{x} \\
&= \lim_{x \to 0} \frac{\cos x - 1}{x}.
\end{aligned}
$$

Therefore, $\lim_{x \to 0} \frac{\cos x - 1}{x} = 0$. □

Proposition 6. *If* $f(x) = \sin x$*, then* $f'(x) = \cos x$*.*

Therefore, $\frac{d}{dx} \sin x = \cos x$*.*

Proof. We use the unproven fact $\lim_{h \to 0} \frac{\sin h}{h} = 1$.

The domain of f is the open set $\mathbb{R}$.

For $x \in \mathbb{R}$, we have

$$
\begin{aligned}
\cos x &= 0 + \cos x \\
&= \sin x \cdot 0 + \cos x \cdot 1 \\
&= \sin x \cdot \lim_{h \to 0} \frac{\cos h - 1}{h} + \cos x \cdot \lim_{h \to 0} \frac{\sin h}{h} \\
&= \lim_{h \to 0} \sin x \cdot \frac{\cos h - 1}{h} + \lim_{h \to 0} \cos x \cdot \frac{\sin h}{h} \\
&= \lim_{h \to 0} [\sin x \cdot \frac{\cos h - 1}{h} + \cos x \cdot \frac{\sin h}{h}] \\
&= \lim_{h \to 0} [\frac{\sin x \cos h - \sin x}{h} + \frac{\cos x \sin h}{h}] \\
&= \lim_{h \to 0} [\frac{\sin x \cos h - \sin x + \cos x \sin h}{h}] \\
&= \lim_{h \to 0} [\frac{\sin x \cos h + \cos x \sin h - \sin x}{h}] \\
&= \lim_{h \to 0} [\frac{\sin(x + h) - \sin x}{h}] \\
&= \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} \\
&= f'(x).
\end{aligned}
$$

Therefore, $f'(x) = \cos x$. □

Proposition 7. *If* $f(x) = \cos x$*, then* $f'(x) = -\sin x$*.*

Therefore, $\frac{d}{dx} \cos x = -\sin x$*.*

Proof. We use the unproven fact $\lim_{h\to 0} \frac{\sin h}{h} = 1$.

The domain of f is the open set $\mathbb{R}$.

For $x \in \mathbb{R}$, we have

$$\begin{aligned}
-\sin x &= 0 - \sin x \\
&= \cos x \cdot 0 - \sin x \cdot 1 \\
&= \cos x \cdot \lim_{h\to 0} \frac{\cos h - 1}{h} - \sin x \cdot \lim_{h\to 0} \frac{\sin h}{h} \\
&= \lim_{h\to 0} \cos x \cdot \frac{\cos h - 1}{h} - \lim_{h\to 0} \sin x \cdot \frac{\sin h}{h} \\
&= \lim_{h\to 0} [\cos x \cdot \frac{\cos h - 1}{h} - \sin x \cdot \frac{\sin h}{h}] \\
&= \lim_{h\to 0} [\frac{\cos x \cos h - \cos x}{h} - \frac{\sin x \sin h}{h}] \\
&= \lim_{h\to 0} [\frac{\cos x \cos h - \cos x - \sin x \sin h}{h}] \\
&= \lim_{h\to 0} [\frac{\cos x \cos h - \sin x \sin h - \cos x}{h}] \\
&= \lim_{h\to 0} [\frac{\cos(x+h) - \cos x}{h}] \\
&= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= f'(x).
\end{aligned}$$

Therefore, $f'(x) = -\sin x$. □