Differentiation of real valued functions Exercises

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Derivative of a real valued function

Exercise 1. the cube function is differentiable
Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by $f(x) = x^3$.
Then $f'(x) = 3x^2$ for all $x \in \mathbb{R}$.

Proof. We prove $f'(x) = 3x^2$ for all $x \in \mathbb{R}$ using the definition of derivative.
Let $c \in \mathbb{R}$ be given.
We must prove $f'(c) = 3c^2$.
Let $q : \mathbb{R} - \{c\} \to \mathbb{R}$ be a function defined by $q(x) = \frac{f(x)-f(c)}{x-c}$ for all $x \in \mathbb{R} - \{c\}$.
Since $c \in \mathbb{R}$ and every real number is an accumulation point of $\mathbb{R}$, then c is an accumulation point of $\mathbb{R}$, so c is an accumulation point of $\mathbb{R} - \{c\}$, the domain of q.
For $x \in \mathbb{R} - \{c\}$, we have $x \in \mathbb{R}$ and $x \neq c$, so so $x - c \neq 0$.
Observe that

$$\begin{aligned}
\lim_{x\to c} q(x) &= \lim_{x\to c} \frac{f(x)-f(c)}{x-c} \\
&= \lim_{x\to c} \frac{x^3-c^3}{x-c} \\
&= \lim_{x\to c} \frac{(x-c)(x^2+cx+c^2)}{x-c} \\
&= \lim_{x\to c} (x^2+cx+c^2) \\
&= c^2+c^2+c^2 \\
&= 3c^2.
\end{aligned}$$

Thus, $f'(c) = \lim_{x\to c} \frac{f(x)-f(c)}{x-c} = 3c^2$, as desired. ∎

Exercise 2. Let $f : (0,1) \to \mathbb{R}$ be the function defined by $f(x) = \sqrt{2x^2 - 3x + 6}$.
Then $f'(x) = \frac{4x-3}{2\sqrt{2x^2-3x+6}}$ for all $x \in (0,1)$.

Proof. To prove $f'(x) = \frac{4x-3}{2\sqrt{2x^2-3x+6}}$ for all $x \in (0,1)$, let $c \in (0,1)$ be given.
We must prove $f'(c) = \frac{4c-3}{2\sqrt{2c^2-3c+6}}$.

Let $g : (0,1) \to \mathbb{R}$ be the function defined by $g(x) = 2x^2 - 3x + 6$.
Let $h : [0,\infty) \to \mathbb{R}$ be the function defined by $h(x) = \sqrt{x}$.

We first prove $dom f = dom(h \circ g)$.
Since $dom(h \circ g) = \{x \in dom g : g(x) \in dom h\} = \{x \in (0,1) : g(x) \in [0,\infty)\}$, then $dom(h \circ g) \subset (0,1)$.

Let $x \in (0,1)$.
Then $0 < x < 1$ and $g(x) = 2x^2 - 3x + 6$.
Since $0 < x < 1$, then either $x = \frac{3}{4}$ or $x \neq \frac{3}{4}$.
If $x = \frac{3}{4}$, then $g(x) = g(\frac{3}{4}) = \frac{39}{8} > 0$.
If $x \neq \frac{3}{4}$, then $x - \frac{3}{4} \neq 0$, so $(x - \frac{3}{4})^2 > 0$.
Since $\frac{9}{16} - 3 = \frac{-39}{16} < 0$, then $\frac{9}{16} - 3 < 0$.
Since $(x - \frac{3}{4})^2 > 0$ and $0 > \frac{9}{16} - 3$, then $(x - \frac{3}{4})^2 > \frac{9}{16} - 3$, so $x^2 - \frac{3x}{2} + \frac{9}{16} > \frac{9}{16} - 3$.
Thus, $x^2 - \frac{3x}{2} > -3$, so $2x^2 - 3x > -6$.
Hence $g(x) = 2x^2 - 3x + 6 > 0$, so $g(x) > 0$.
Therefore, in either case, $g(x) > 0$.
Consequently, $g(x) \in (0,\infty)$, so $g(x) \in (0,\infty)$ for all $x \in (0,1)$.
Since $g(x) \in (0,\infty)$ and $(0,\infty) \subset [0,\infty)$, then $g(x) \in [0,\infty)$.
Since $x \in (0,1)$ and $g(x) \in [0,\infty)$, then $x \in dom(h \circ g)$, so $(0,1) \subset dom(h \circ g)$.
Since $(0,1) \subset dom(h \circ g)$ and $dom(h \circ g) \subset (0,1)$, then $(0,1) = dom(h \circ g)$.
Thus, $dom f = dom(h \circ g)$.

We next prove $f(x) = (h \circ g)(x)$ for all $x \in (0,1)$.
Let $x \in (0,1)$ be given.
Then

$$
\begin{aligned}
(h \circ g)(x) &= h(g(x)) \\
&= h(2x^2 - 3x + 6) \\
&= \sqrt{2x^2 - 3x + 6} \\
&= f(x).
\end{aligned}
$$

Thus, $f(x) = (h \circ g)(x)$, so $f(x) = (h \circ g)(x)$ for all $x \in (0,1)$.
Since $dom f = dom(h \circ g)$ and $f(x) = (h \circ g)(x)$ for all $x \in (0,1)$, then $f = h \circ g$.

We next prove $rng g \subset dom h$.
Let $y \in rng g$.
Then $y = g(x)$ for some $x \in (0,1)$.
Since $x \in (0,1)$, then $g(x) \in (0,\infty)$.
Since $(0,\infty) \subset [0,\infty)$, then $g(x) \in [0,\infty)$, so $y \in [0,\infty)$.
Since $[0,\infty) = dom h$, then $y \in dom h$, so $rng g \subset dom h$.

Since $g'(x) = 4x - 3$ for all $x \in \mathbb{R}$ and $c \in \mathbb{R}$, then $g'(c) = 4c - 3$, so g is differentiable at c.

Since $c \in (0, 1)$, then $g(c) = 2c^2 - 3c + 6$ and $g(c) \in (0, \infty)$.

Since $g(c) \in (0, \infty)$, then $g(c) > 0$.

Since $h'(x) = \frac{1}{2\sqrt{x}}$ for all $x > 0$ and $g(c) > 0$, then $h'(g(c)) = \frac{1}{2\sqrt{g(c)}}$, so h is differentiable at $g(c)$.

Hence, by the chain rule,

$$\begin{aligned} f'(c) &= (h \circ g)'(c) \\ &= h'(g(c)) \cdot g'(c) \\ &= \frac{1}{2\sqrt{g(c)}} \cdot (4c - 3) \\ &= \frac{4c - 3}{2\sqrt{2c^2 - 3c + 6}}. \end{aligned}$$

Therefore, $f'(c) = \frac{4c-3}{2\sqrt{2c^2-3c+6}}$, as desired. $\square$

Exercise 3. Let $a, b, c, d, p, q \in \mathbb{R}$ such that $a < b$ and $c < d$ and $p < q$.

Let $f : [a, b] \to [c, d]$ and $g : [c, d] \to [p, q]$ and $h : [p, q] \to \mathbb{R}$ be functions such that f is differentiable at $x_0 \in [a, b]$ and g is differentiable at $f(x_0)$ and h is differentiable at $g(f(x_0))$.

Then $[h \circ (g \circ f)]'(x_0) = h'[(g \circ f)(x_0)] \cdot (g \circ f)'(x_0)$.

Proof. Since the range of f is a subset of its codomain, the interval $[c, d]$, and the interval $[c, d]$ is the domain of g, then the range of f is a subset of the domain of g.

Since f is differentiable at x_0 and g is differentiable at $f(x_0)$, then by the chain rule, the function $g \circ f$ is differentiable at x_0 and $(g \circ f)'(x_0) = g'(f(x_0)) \cdot f'(x_0)$.

Since the range of $g \circ f$ is a subset of its codomain, the interval $[p, q]$, and the interval $[p, q]$ is the domain of h, then the range of $g \circ f$ is a subset of the domain of h.

Since $g \circ f$ is differentiable at x_0 and h is differentiable at $(g \circ f)(x_0)$, then by the chain rule, the function $h \circ (g \circ f)$ is differentiable at x_0 and $[h \circ (g \circ f)]'(x_0) = h'[(g \circ f)(x_0)] \cdot (g \circ f)'(x_0)$. $\square$

Lemma 4. *Let $f : \mathbb{R} \to \mathbb{R}$ be a function.*

Let $a \in \mathbb{R}$.

If $\lim_{x \to a} |f(x)| = 0$, then $\lim_{x \to a} f(x) = 0$.

Proof. Suppose $\lim_{x \to a} |f(x)| = 0$.

Then $0 = -0 = -(\lim_{x \to a} |f(x)|) = \lim_{x \to a} -|f(x)|$.

Let $x \in \mathbb{R}$.

Then $f(x) \in \mathbb{R}$, so $-|f(x)| \leq f(x) \leq |f(x)|$.

Thus, $-|f(x)| \leq f(x) \leq |f(x)|$ for all $x \in \mathbb{R}$.

Observe that a is an accumulation point of $\mathbb{R}$, the domain of f.

Since $-|f(x)| \leq f(x) \leq |f(x)|$ for all $x \in \mathbb{R}$ and $\lim_{x\to a} -|f(x)| = 0 = \lim_{x\to a} |f(x)|$, then by the squeeze rule for limits, $\lim_{x\to a} f(x) = 0$. $\square$

Exercise 5. Let $f : (a, b) \to \mathbb{R}$ be a function.

If f is differentiable at $x \in (a, b)$, then $\lim_{h\to 0} \frac{f(x+h)-f(x-h)}{2h} = f'(x)$.

Proof. Suppose f is differentiable at $x \in (a, b)$.

Then there is a real number L such that $L = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = f'(x)$.

We first prove $\lim_{h\to 0} \frac{f(x)-f(x-h)}{h} = L$.

Let $\epsilon > 0$ be given.

Since $L = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$, then there is a real $\delta > 0$ such that for all $h \in \mathbb{R}$, if $0 < |h| < \delta$, then $|\frac{f(x+h)-f(x)}{h} - L| < \epsilon$.

Let $h \in \mathbb{R}$ such that $0 < |h| < \delta$.

Then $0 < |h|$ and $|h| < \delta$.

Since $h \in \mathbb{R}$, then $-h \in \mathbb{R}$.

Since $|h| > 0$, then $h \neq 0$, so $-h \neq 0$.

Hence, $|-h| > 0$.

Since $|-h| = |h| < \delta$, then we have $0 < |-h| < \delta$.

Thus, $|\frac{f(x-h)-f(x)}{-h} - L| < \epsilon$, so $|\frac{-f(x-h)+f(x)}{h} - L| < \epsilon$.

Therefore, $|\frac{f(x)-f(x-h)}{h} - L| < \epsilon$, so $\lim_{h\to 0} \frac{f(x)-f(x-h)}{h} = L$.

We must prove $\lim_{h\to 0} \frac{f(x+h)-f(x-h)}{2h} = L$.

Observe that

$$\begin{aligned}
2L &= L + L \\
&= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} + \lim_{h\to 0} \frac{f(x)-f(x-h)}{h} \\
&= \lim_{h\to 0} [\frac{f(x+h)-f(x)}{h} + \frac{f(x)-f(x-h)}{h}] \\
&= \lim_{h\to 0} \frac{f(x+h)-f(x)+f(x)-f(x-h)}{h} \\
&= \lim_{h\to 0} \frac{f(x+h)-f(x-h)}{h}.
\end{aligned}$$

Thus, $2L = \lim_{h\to 0} \frac{f(x+h)-f(x-h)}{h}$, so $L = \frac{1}{2} \lim_{h\to 0} \frac{f(x+h)-f(x-h)}{h} = \lim_{h\to 0} \frac{f(x+h)-f(x-h)}{2h}$.

Therefore, $L = \lim_{h\to 0} \frac{f(x+h)-f(x-h)}{2h}$, as desired. $\square$

Exercise 6. Let $\delta > 0$.

Let $f : (-\delta, \delta) \to \mathbb{R}$ be a function.

If $|f(x)| \leq x^2$ for all $x \in (-\delta, \delta)$, then $f'(0) = 0$.

Proof. Let $I = (-\delta, \delta)$.

Suppose $|f(x)| \leq x^2$ for all $x \in I$.

Since $0 \in I$, then $|f(0)| \leq 0^2 = 0$.

Since $f(0) \in \mathbb{R}$ and $|f(0)| \geq 0$ and $|f(0)| \leq 0$, then $|f(0)| = 0$, so $f(0) = 0$.

Let $x \in I$ and $x \neq 0$.

Since $x \in I$, then $|f(x)| \leq x^2$.

Since $x \neq 0$, then $|x| > 0$.

Hence, $|\frac{f(x)}{x}| = \frac{|f(x)|}{|x|} \leq \frac{x^2}{|x|} = \frac{|x|^2}{|x|} = |x|$, so $0 \leq |\frac{f(x)}{x}| \leq |x|$.

Thus, $0 \leq |\frac{f(x)}{x}| \leq |x|$ for all $x \in I - \{0\}$.

Observe that 0 is an accumulation point of the deleted δ neighborhood of 0, the set $N'(0;\delta) = I - \{0\}$.

Since $0 \leq |\frac{f(x)}{x}| \leq |x|$ for all $x \in I - \{0\}$ and $\lim_{x\to 0} 0 = 0 = \lim_{x\to 0} |x|$, then by the squeeze rule for limits, $\lim_{x\to 0} |\frac{f(x)}{x}| = 0$.

Hence, by the previous lemma, $\lim_{x\to 0} \frac{f(x)}{x} = 0$.

Therefore, $0 = \lim_{x\to 0} \frac{f(x)}{x} = \lim_{x\to 0} \frac{f(x)-f(0)}{x-0} = f'(0)$, so $f'(0) = 0$, as desired. ☐

Exercise 7. Let $f : (a, b) \to \mathbb{R}$ be a function continuous on (a, b) and differentiable at $c \in (a, b)$.

Let $g : (a, b) \to \mathbb{R}$ be a function defined by

$$g(x) = \begin{cases} \frac{f(x)-f(c)}{x-c} & \text{if } x \in (a,b) - \{c\} \\ f'(c) & \text{if } x = c \end{cases}$$

Then g is continuous on (a, b).

Proof. Let $x_0 \in (a, b)$.

Either $x_0 = c$ or $x_0 \neq c$.

We consider these cases separately.

Case 1: Suppose $x_0 = c$.

Since f is differentiable at c, then $f'(c)$ exists, so c is an accumulation point of (a, b).

Thus, $\lim_{x\to x_0} g(x) = \lim_{x\to c} g(x) = \lim_{x\to c} \frac{f(x)-f(c)}{x-c} = f'(c) = g(c) = g(x_0)$, so g is continuous at x_0.

Case 2: Suppose $x_0 \neq c$.

Let $s : \mathbb{R} \to \mathbb{R}$ be the function defined by $s(x) = f(c)$.

Let $f - s : (a, b) \to \mathbb{R}$ be the function defined by $(f - s)(x) = f(x) - s(x) = f(x) - f(c)$.

Since f is continuous on (a, b), then f is continuous.

Since s is a constant function and every constant function is continuous, then s is continuous.

Since f is continuous and s is continuous, then $f - s$ is continuous.

Let $I : \mathbb{R} \to \mathbb{R}$ be the function defined by $I(x) = x$.

Let $t : \mathbb{R} \to \mathbb{R}$ be the function defined by $t(x) = c$.

Let $I - t : \mathbb{R} \to \mathbb{R}$ be the function defined by $(I - t)(x) = I(x) - t(x) = x - c$.

Since I is the identity function on $\mathbb{R}$ and the identity function is continuous, then I is continuous.

Since t is a constant function and every constant function is continuous, then t is continuous.

Since I is continuous and t is continuous, then $I - t$ is continuous.

Let $\frac{f-s}{I-t}$ be the real valued function defined by $(\frac{f-s}{I-t})(x) = \frac{(f-s)(x)}{(I-t)(x)}$ for all $x \in dom(f-s) \cap dom(I-t)$ such that $(I-t)(x) \neq 0$.

Then $(\frac{f-s}{I-t})(x) = \frac{(f-s)(x)}{(I-t)(x)} = \frac{f(x)-s(x)}{I(x)-t(x)} = \frac{f(x)-f(c)}{x-c}$ for all $x \in (a,b) \cap \mathbb{R}$ such that $I(x) - t(x) = x - c \neq 0$, so $(\frac{f-s}{I-t})(x) = \frac{f(x)-f(c)}{x-c}$ for all $x \in (a,b)$ such that $x \neq c$.

Since $x_0 \in (a,b)$ and $x_0 \neq c$, then x_0 is an arbitrary element in the domain of $\frac{f-s}{I-t}$.

To prove g is continuous at x_0 when $x_0 \neq c$ is equivalent to proving that the function $\frac{f-s}{I-t}$ is continuous.

Since $f - s$ is continuous and $I - t$ is continuous, then $\frac{f-s}{I-t}$ is continuous, so g is continuous at x_0 when $x_0 \neq c$. $\square$

Exercise 8. uniform differentiability

Let $E \subset \mathbb{R}$ be non empty.

A function $f : E \to \mathbb{R}$ is said to be **uniformly differentiable** on E iff

1. f is differentiable on E.
2. $(\forall \epsilon > 0)(\exists \delta > 0)(\forall x, y \in E)(0 < |x-y| < \delta \to |\frac{f(x)-f(y)}{x-y} - f'(x)| < \epsilon)$.

Let $f : E \to \mathbb{R}$ be a function uniformly differentiable on a nonempty set $E \subset \mathbb{R}$.

Then f' is continuous on E.

Proof. Since f is uniformly differentiable on E, then f is differentiable on E.

Hence, $f' : E \to \mathbb{R}$ is a function defined by $f'(c) = \lim_{x \to c} \frac{f(x)-f(c)}{x-c}$ for each $c \in E$.

To prove f' is continuous on E, let $c \in E$.

We must prove f' is continuous at c.

To prove f' is continuous at c, let $\epsilon > 0$ be given.

We must prove there exists $\delta > 0$ such that for all $x \in E$, if $|x - c| < \delta$, then $|f'(x) - f'(c)| < \epsilon$.

Since $c \in E$, then $f'(c) = \lim_{x \to c} \frac{f(x)-f(c)}{x-c}$.

Since $\epsilon > 0$, then $\frac{\epsilon}{2} > 0$, so there exists $\delta_1 > 0$ such that for every $x \in E - \{c\}$, if $0 < |x - c| < \delta_1$, then $|\frac{f(x)-f(c)}{x-c} - f'(c)| < \frac{\epsilon}{2}$.

Since f is uniformly differentiable on E and $\frac{\epsilon}{2} > 0$, then there exists $\delta_2 > 0$ such that for every $x, y \in E$, if $0 < |x - y| < \delta_2$, then $|\frac{f(x)-f(y)}{x-y} - f'(x)| < \frac{\epsilon}{2}$.

Let $\delta = \min\{\delta_1, \delta_2\}$.

Then $\delta \leq \delta_1$ and $\delta \leq \delta_2$.

Since $\delta_1 > 0$ and $\delta_2 > 0$, then $\delta > 0$.

Let $x \in E$ such that $|x - c| < \delta$.

Either $x = c$ or $x \neq c$.

We consider these cases separately.

Case 1: Suppose $x = c$.

Then $|f'(x) - f'(c)| = |f'(c) - f'(c)| = 0 < \epsilon$.

Case 2: Suppose $x \neq c$.
Then $|x - c| > 0$, so $0 < |x - c| < \delta$.
Since $0 < |x - c| < \delta \leq \delta_1$, then $0 < |x - c| < \delta_1$.
Since $x \in E$ and $x \neq c$, then $x \in E - \{c\}$.
Since $x \in E - \{c\}$ and $0 < |x - c| < \delta_1$, then $|\frac{f(x)-f(c)}{x-c} - f'(c)| < \frac{\epsilon}{2}$.
Since $0 < |x - c| < \delta \leq \delta_2$, then $0 < |x - c| < \delta_2$.
Since $x \in E$ and $c \in E$ and $0 < |x - c| < \delta_2$, then $|\frac{f(x)-f(c)}{x-c} - f'(x)| < \frac{\epsilon}{2}$.
Observe that

$$\begin{aligned}
|f'(x) - f'(c)| &= |f'(x) - \frac{f(x) - f(c)}{x - c} + \frac{f(x) - f(c)}{x - c} - f'(c)| \\
&\leq |f'(x) - \frac{f(x) - f(c)}{x - c}| + |\frac{f(x) - f(c)}{x - c} - f'(c)| \\
&= |\frac{f(x) - f(c)}{x - c} - f'(x)| + |\frac{f(x) - f(c)}{x - c} - f'(c)| \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&= \epsilon.
\end{aligned}$$

Thus, $|f'(x) - f'(c)| < \epsilon$. ☐

Exercise 9. Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that f is differentiable at c and $f(c) = 0$.
Let $g : \mathbb{R} \to \mathbb{R}$ be a function defined by $g(x) = |f(x)|$.
Then g is differentiable at c iff $f'(c) = 0$.

Proof. We first prove if $f'(c) = 0$, then g is differentiable at c.
Suppose $f'(c) = 0$.
Then $0 = f'(c) = \lim_{x\to c} \frac{f(x)-f(c)}{x-c} = \lim_{x\to c} \frac{f(x)-0}{x-c} = \lim_{x\to c} \frac{f(x)}{x-c}$.
Let $h : \mathbb{R} - \{c\} \to \mathbb{R}$ be the function defined by $h(x) = \frac{|f(x)|}{x-c}$ for all $x \neq c$.
Observe that c is an accumulation point of the set $\mathbb{R} - \{c\}$, the domain of h.
We first prove $\lim_{x\to c} h(x) = 0$.
Let $\epsilon > 0$ be given.
Since $\lim_{x\to c} \frac{f(x)}{x-c} = 0$, then there exists $\delta > 0$ such that for all $x \in \mathbb{R} - \{c\}$, if $0 < |x - c| < \delta$, then $|\frac{f(x)}{x-c}| < \epsilon$.
Let $x \in \mathbb{R} - \{c\}$ such that $0 < |x - c| < \delta$.
Then $|\frac{f(x)}{x-c}| < \epsilon$.
Thus, $|h(x)| = |\frac{|f(x)|}{x-c}| = \frac{||f(x)||}{|x-c|} = \frac{|f(x)|}{|x-c|} = |\frac{f(x)}{x-c}| < \epsilon$.
Hence, $\lim_{x\to c} h(x) = 0$.
Since $0 = \lim_{x\to c} h(x) = \lim_{x\to c} \frac{|f(x)|}{x-c} = \lim_{x\to c} \frac{|f(x)|-0}{x-c} = \lim_{x\to c} \frac{|f(x)|-|f(c)|}{x-c} = \lim_{x\to c} \frac{g(x)-g(c)}{x-c}$, then $0 = g'(c)$, so g is differentiable at c. ☐

Proof. Conversely, we prove if g is differentiable at c, then $f'(c) = 0$.
Suppose g is differentiable at c.
Then there is a real number L such that $g'(c) = L$.

Hence, $L = g'(c) = \lim_{x \to c} \frac{g(x)-g(c)}{x-c} = \lim_{x \to c} \frac{|f(x)|-|f(c)|}{x-c} = \lim_{x \to c} \frac{|f(x)|}{x-c}$.
Either $L > 0$ or $L = 0$ or $L < 0$.

Suppose $L > 0$.
Since $\lim_{x \to c} \frac{|f(x)|}{x-c} = L$, then there is a $\delta > 0$ such that for all $x \neq c$, if $0 < |x - c| < \delta$, then $|\frac{|f(x)|}{x-c} - L| < L$.
Let $x = c - \frac{\delta}{2}$.
Then $0 < |x - c| = |\frac{-\delta}{2}| = \frac{\delta}{2} < \delta$, so $0 < |x - c| < \delta$.
Hence, $|\frac{|f(x)|}{x-c} - L| < L$.
Thus, $-L < \frac{|f(x)|}{x-c} - L < L$, so $-L < \frac{|f(x)|}{x-c} - L$.
Consequently, $0 < \frac{|f(x)|}{x-c}$.
Since $x - c = \frac{-\delta}{2} < 0$, then $x - c < 0$, so $0 > |f(x)|$.
Therefore, $|f(x)| < 0$, a contradiction since the absolute value of a real number is non-negative.
Hence, L cannot be positive.

Suppose $L < 0$.
Since $\lim_{x \to c} \frac{|f(x)|}{x-c} = L$, then there is a $\delta > 0$ such that for all $x \neq c$, if $0 < |x - c| < \delta$, then $|\frac{|f(x)|}{x-c} - L| < L$.
Let $x = c + \frac{\delta}{2}$.
Then $0 < |x - c| = |\frac{\delta}{2}| = \frac{\delta}{2} < \delta$, so $0 < |x - c| < \delta$.
Hence, $|\frac{|f(x)|}{x-c} - L| < L$.
Thus, $-L < \frac{|f(x)|}{x-c} - L < L$, so $\frac{|f(x)|}{x-c} - L < L$.
Consequently, $\frac{|f(x)|}{x-c} < 2L$.
Since $L < 0$, then $2L < 0$, so $\frac{|f(x)|}{x-c} < 2L < 0$.
Hence, $\frac{|f(x)|}{x-c} < 0$.
Since $x - c = \frac{\delta}{2} > 0$, then $x - c > 0$, so $|f(x)| < 0$.
Therefore, $|f(x)| < 0$, a contradiction since the absolute value of a real number is non-negative.
Hence, L cannot be negative.

Since L is a real number and L is neither positive nor negative, then L must be zero.
Therefore, $0 = L = \lim_{x \to c} \frac{|f(x)|}{x-c}$.

We next prove $\lim_{x \to c} \frac{f(x)}{x-c} = 0$ using the $\epsilon - \delta$ definition of a limit.
Let $\epsilon > 0$ be given.
Since $\lim_{x \to c} \frac{|f(x)|}{x-c} = 0$, then there is a $\delta > 0$ such that for all real numbers $x \neq c$, if $0 < |x - c| < \delta$, then $|\frac{|f(x)|}{x-c}| < \epsilon$.
Let $x \neq c$ be an arbitrary real number such that $0 < |x - c| < \delta$.

Then $|\frac{|f(x)|}{x-c}| < \epsilon$.
Since $|\frac{f(x)}{x-c}| = \frac{|f(x)|}{|x-c|} = \frac{||f(x)||}{|x-c|} = |\frac{|f(x)|}{x-c}| < \epsilon$, then $\lim_{x\to c} \frac{f(x)}{x-c} = 0$.

Therefore, $0 = \lim_{x\to c} \frac{f(x)}{x-c} = \lim_{x\to c} \frac{f(x)-f(c)}{x-c} = f'(c)$, so $f'(c) = 0$, as desired. $\square$

Exercise 10. Let $f : \mathbb{R} \to \mathbb{R}$ be a function defined by $f(x) = \frac{1}{1+x^2}$.
Then f has an absolute maximum.

Proof. Let $x \in \mathbb{R}$ be arbitrary.
Then $x^2 \geq 0$, so $1 + x^2 \geq 1$.
Hence, $1 \geq \frac{1}{1+x^2} = f(x)$, so $1 \geq f(x)$ for all $x \in \mathbb{R}$.
Since $f(0) = 1 \geq f(x)$, then $f(0) = 1$ is an absolute maximum of f on $\mathbb{R}$. $\square$

Mean Value Theorem

Exercise 11. Let $f : [-1, 1] \to \mathbb{R}$ be the function defined by $f(x) = 2x^3 + 3x^2 - 36x + 5$.
Then f is one to one.

Proof. Let $x \in [-1, 1]$.
Then $-1 \leq x \leq 1$ and $f'(x) = 6x^2+6x-36 = 6(x^2+x-6) = 6(x+3)(x-2)$.
Since $-1 \leq x \leq 1$, then $-1 \leq x$ and $x \leq 1$.
Since $x \geq -1$, then $x + 3 \geq 2 > 0$, so $x + 3 > 0$.
Since $x \leq 1$, then $x - 2 \leq -1 < 0$, so $x - 2 < 0$.
Since $x + 3 > 0$ and $x - 2 < 0$, then $f'(x) = 6(x + 3)(x - 2) < 0$.
Hence, $f'(x) < 0$ for all $x \in [-1, 1]$, so f is strictly decreasing on $[-1, 1]$.
Thus, f is monotonic, so f is one to one on the interval $[-1, 1]$. $\square$

Exercise 12. Let $A, B, C \in \mathbb{R}$ be fixed with $A \neq 0$.
Let $a, b \in \mathbb{R}$ with $a < b$.
Let $p : [a, b] \to \mathbb{R}$ be the function defined by $p(x) = Ax^2 + Bx + C$.
The value of c guaranteed by MVT is the midpoint of the interval $[a, b]$.

Proof. Since p is a polynomial function, then p is continuous, so p is continuous on $[a, b]$.
Let $x \in [a, b]$.
Then $p(x) = Ax^2 + Bx + C$, so $p'(x) = 2Ax + B$ for all $x \in [a, b]$.
Thus, p is differentiable, so p is differentiable on $[a, b]$.
Since $(a, b) \subset [a, b]$, then p is differentiable on (a, b).
Since p is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b), then by MVT, there exists $c \in (a, b)$ such that $p'(c) = \frac{p(b)-p(a)}{b-a}$.
Since $c \in (a, b)$ and $(a, b) \subset [a, b]$, then $c \in [a, b]$.

Thus,

$$\begin{aligned}
2Ac+B &= p'(c) \\
&= \frac{p(b)-p(a)}{b-a} \\
&= \frac{(Ab^2+Bb+C)-(Aa^2+Ba+C)}{b-a} \\
&= \frac{Ab^2+Bb-Aa^2-Ba}{b-a} \\
&= \frac{Ab^2-Aa^2+Bb-Ba}{b-a} \\
&= \frac{A(b^2-a^2)+B(b-a)}{b-a} \\
&= \frac{A(b-a)(b+a)+B(b-a)}{b-a} \\
&= A(b+a)+B \\
&= A(a+b)+B.
\end{aligned}$$

Hence, $2Ac + B = A(a + b) + B$, so $2Ac = A(a + b)$.
Since $A \neq 0$, then $c = \frac{A(a+b)}{2A} = \frac{a+b}{2}$, so c is the average of a and b.
Therefore, c is the midpoint of the interval $[a, b]$. $\square$

Exercise 13. Let a and b be fixed real numbers with $a > 0$ and let n be a fixed positive integer.
Let $p : \mathbb{R} \to \mathbb{R}$ be the function defined by $p(x) = x^{2n+1} + ax + b$.
Then p cannot have two real roots.

Proof. Suppose for the sake of contradiction p has two real roots.
Then there exist real numbers c_1 and c_2 with $c_1 \neq c_2$ and $p(c_1) = 0 = p(c_2)$.
Since p is a polynomial function, then p is continuous and differentiable and $p'(x) = (2n + 1)x^{2n} + a$.
Since p is continuous, then p is continuous on the closed interval $[c_1, c_2]$.
Since p is differentiable, then p is differentiable on the open interval (c_1, c_2).
Since $p(c_1) = p(c_2)$, then by Rolle's theorem, there exists $c \in (c_1, c_2)$ such that $p'(c) = 0$.
Either $c = 0$ or $c \neq 0$.
Suppose $c = 0$.
Then $0 = p'(c) = p'(0) = (2n + 1)0^{2n} + a = 0 + a = a$, so $0 = a$.
But, $a > 0$, so $c \neq 0$.
Thus, $0 = p'(c) = (2n + 1)c^{2n} + a$, so $a = -(2n + 1)c^{2n} = -(2n + 1)(c^n)^2$.
Since n is a positive integer, then $n > 0$.
Hence, $2n > 0$, so $2n + 1 > 1 > 0$.
Thus, $2n + 1 > 0$.
Since $c \neq 0$ and $n > 0$, then $c^n \neq 0$, so $(c^n)^2 > 0$.

Since $-1 < 0$ and $2n+1 > 0$ and $(c^n)^2 > 0$, then the product $-(2n+1)(c^n)^2 < 0$, so $a < 0$.

But, this contradicts the hypothesis $a > 0$.

Therefore, p cannot have two real roots. $\square$

Exercise 14. Let p be a nonconstant polynomial function of x.

Between any two consecutive zeros of p' there is at most one zero of p.

Proof. Let x_1 and x_2 be arbitrary consecutive zeros of p'.

Then $x_1 \neq x_2$ and $p'(x_1) = 0 = p'(x_2)$ and there are no zeros of p' between x_1 and x_2.

Since $x_1 \neq x_2$, then either $x_1 < x_2$ or $x_1 > x_2$.

Without loss of generality, assume $x_1 < x_2$.

To prove there is at most one zero of p between x_1 and x_2, we prove by contradiction.

Suppose there is more than one zero of p between x_1 and x_2.

Then there exist at least two zeros of p between x_1 and x_2.

Let c_1 and c_2 be two zeros of p between x_1 and x_2.

Then $c_1 \neq c_2$ and $p(c_1) = 0 = p(c_2)$ and $x_1 < c_1 < x_2$ and $x_1 < c_2 < x_2$.

Since $c_1 \neq c_2$, then either $c_1 < c_2$ or $c_1 > c_2$.

Without loss of generality, assume $c_1 < c_2$.

Then $x_1 < c_1 < c_2 < x_2$.

Since p is a polynomial function of x, then p is continuous and differentiable.

Since p is continuous, then p is continuous on the closed interval $[c_1, c_2]$.

Since p is differentiable, then p is differentiable on the open interval (c_1, c_2).

Thus, by Rolle's theorem, there exists $c \in (c_1, c_2)$ such that $p'(c) = 0$.

Since $c \in (c_1, c_2)$, then $c_1 < c < c_2$.

Hence, $x_1 < c_1 < c < c_2 < x_2$, so $x_1 < c < x_2$.

Therefore, c is between x_1 and x_2 and $p'(c) = 0$, so there is a zero of p' between x_1 and x_2.

This contradicts the fact that x_1 and x_2 are consecutive zeros of p'.

Therefore, there is not more than one zero of p between x_1 and x_2, so there is at most one zero of p between x_1 and x_2. $\square$

Exercise 15. The equation $\cos x = x^3 + x^2 + 4x$ has exactly one root on the interval $[0, \frac{\pi}{2}]$.

Proof. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = \cos x$.

Let $g : \mathbb{R} \to \mathbb{R}$ be defined by $g(x) = x^3 + x^2 + 4x$.

Then f and g are continuous.

Let $h = f - g$.

Then h is a function defined by $h(x) = f(x) - g(x) = \cos x - (x^3 + x^2 + 4x)$ for all $x \in [0, \frac{\pi}{2}]$.

Since f is continuous and g is continuous and the difference of continuous functions is continuous, then h is continuous, so h is continuous on $[0, \frac{\pi}{2}]$.

Since $h(0) = \cos 0 - (0^3 + 0^2 + 4 \cdot 0) = 1 > 0 > 0 - ((\frac{\pi}{2})^3 + (\frac{\pi}{2})^2 + 2\pi) = \cos(\frac{\pi}{2}) - ((\frac{\pi}{2})^3 + (\frac{\pi}{2})^2 + 4 \cdot \frac{\pi}{2}) = h(\frac{\pi}{2})$, then $h(0) > 0 > h(\frac{\pi}{2})$.

Since h is continuous on the closed interval $[0, \frac{\pi}{2}]$ and $h(0) > 0 > h(\frac{\pi}{2})$, then by IVT, there exists $c \in (0, \frac{\pi}{2})$ such that $h(c) = 0$.

Thus, there is at least one root of the equation on the open interval $(0, \frac{\pi}{2})$.

Since $h(0) > 0$, then 0 is not a root of the equation.

Since $h(\frac{\pi}{2}) < 0$, then $\frac{\pi}{2}$ is not a root of the equation.

Suppose c_1 and c_2 are distinct roots of the equation.

Then $c_1 \in (0, \frac{\pi}{2})$ and $h(c_1) = 0$ and $c_2 \in (0, \frac{\pi}{2})$ and $h(c_2) = 0$ and $c_1 \neq c_2$.

Since $h(c_1) = 0 = h(c_2)$, then $h(c_1) = h(c_2)$.

Since $c_1 \neq c_2$, then either $c_1 < c_2$ or $c_1 > c_2$.

Without loss of generality, assume $c_1 < c_2$.

Observe that $h'(x) = f'(x) - g'(x) = -\sin(x) - (3x^2 + 2x + 4) = -\sin(x) - 3x^2 - 2x - 4 = (-1)(\sin x + 3x^2 + 2x + 4)$ for all $x \in (0, \frac{\pi}{2})$.

Let $x \in (0, \frac{\pi}{2})$.

Then $0 < x < \frac{\pi}{2}$, so $0 < x$.

Hence, $0 < x^2$, so $0 < 3x^2$.

Since $0 < x$, then $0 < 2x$, so $0 < 3x^2 + 2x$.

Thus, $4 < 3x^2 + 2x + 4$.

Since $\sin x > 0$ for all $x \in (0, \frac{\pi}{2})$, then $\sin x > 0$.

Since $\sin x > 0$ and $3x^2 + 2x + 4 > 4$, then $\sin x + 3x^2 + 2x + 4 > 4 > 0$, so $\sin x + 3x^2 + 2x + 4 > 0$.

Hence, $h'(x) = (-1)(\sin x + 3x^2 + 2x + 4) < 0$, so $h'(x) < 0$.

Thus, $h'(x) < 0$ for all $x \in (0, \frac{\pi}{2})$.

Therefore, h is strictly decreasing.

Since $c_1 < c_2$, then $h(c_1) > h(c_2)$.

But, this contradicts the fact that $h(c_1) = h(c_2)$.

Therefore, c_1 and c_2 are not distinct roots of the equation, so there are no distinct roots of the equation.

Hence, there is at most one root of the equation.

Since there is at least one root of the equation and there is at most one root of the equation, then there is exactly one root of the equation. $\square$

Exercise 16. Let $b \in \mathbb{R}$ be a fixed constant.

The equation $x^3 - 3x + b = 0$ has at most one root in the interval $[-1, 1]$.

Proof. Let $f : [-1, 1] \to \mathbb{R}$ be a function defined by $f(x) = x^3 - 3x + b$.

To prove there is at most one root of the equation on the interval, we must prove there is at most one zero of f in the interval $[-1, 1]$.

Let c_1 and c_2 be real zeros of f in the interval $[-1, 1]$.

Then $c_1, c_2 \in [-1, 1]$ and $f(c_1) = 0 = f(c_2)$.

We must prove $c_1 = c_2$.

Suppose for the sake of contradiction $c_1 \neq c_2$.

Then either $c_1 < c_2$ or $c_1 > c_2$.

Without loss of generality, assume $c_1 < c_2$.

Since $c_1, c_2 \in [-1, 1]$, then $-1 \leq c_1 \leq 1$ and $-1 \leq c_2 \leq 1$.
Thus, $-1 \leq c_1 < c_2 \leq 1$, so $[c_1, c_2] \subset [-1, 1]$ and $(c_1, c_2) \subset [-1, 1]$.
Since f is a polynomial function, then f is differentiable and continuous.
Since f is continuous, then f is continuous on $[-1, 1]$.
Since $[c_1, c_2] \subset [-1, 1]$, then f is continuous on $[c_1, c_2]$.
Since f is differentiable, then f is differentiable on $[-1, 1]$.
Since $(c_1, c_2) \subset [-1, 1]$, then f is differentiable on (c_1, c_2).
Since f is continuous on the closed interval $[c_1, c_2]$ and differentiable on the open interval (c_1, c_2) and $f(c_1) = f(c_2)$, then by Rolle's theorem, there exists $c \in (c_1, c_2)$ such that $f'(c) = 0$.

If $x \in [-1, 1]$, then $f(x) = x^3 - 3x + b$, so $f'(x) = 3x^2 - 3$.
Thus, $f'(x) = 3x^2 - 3$ for all $x \in [-1, 1]$.
Since $c \in (c_1, c_2)$ and $(c_1, c_2) \subset [-1, 1]$, then $c \in [-1, 1]$.
Hence, $f'(c) = 3c^2 - 3$.
Since $3c^2 - 3 = f'(c) = 0$, then $3c^2 - 3 = 0$, so $3c^2 = 3$.
Thus, $c^2 = 1$, so either $c = 1$ or $c = -1$.
Since $c \in (c_1, c_2)$, then $c_1 < c < c_2$.
Since $-1 \leq c_1 < c_2 \leq 1$ and $c_1 < c < c_2$, then $-1 \leq c_1 < c < c_2 \leq 1$, so $-1 < c < 1$.
Hence, $c \neq -1$ and $c \neq 1$.
But, this contradicts the fact that either $c = 1$ or $c = -1$.
Therefore, $c_1 = c_2$, as desired. □

Exercise 17. Let $f : [0, 2] \to \mathbb{R}$ be a differentiable function such that $f(0) = 0$ and $f(1) = 1 = f(2)$.
Then
1. There is $c_1 \in (1, 2)$ such that $f'(c_1) = 0$.
2. There is $c_2 \in (0, 1)$ such that $f'(c_2) = 1$.
3. There is $c_3 \in (0, 2)$ such that $f'(c_3) = \frac{1}{3}$.

Proof. We prove 1.
Since f is differentiable, then f is differentiable on $[0, 2]$.
Since f is differentiable on $[0, 2]$ and $[1, 2] \subset [0, 2]$, then f is differentiable on $[1, 2]$, so f is continuous on $[1, 2]$.
Since f is differentiable on $[0, 2]$ and $(1, 2) \subset [0, 2]$, then f is differentiable on $(1, 2)$.
Since $f(1) = 1 = f(2)$, then $f(1) = f(2)$.
Since f is a function continuous on the closed interval $[1, 2]$ and differentiable on the open interval $(1, 2)$ and $f(1) = f(2)$, then by Rolle's theorem, there exists $c_1 \in (1, 2)$ such that $f'(c_1) = 0$. □

Proof. We prove 2.
Since f is differentiable, then f is differentiable on $[0, 2]$.
Since f is differentiable on $[0, 2]$ and $[0, 1] \subset [0, 2]$, then f is differentiable on $[0, 1]$ so f is continuous on $[0, 1]$.

Since f is differentiable on $[0, 1]$ and $(0, 1) \subset [0, 1]$, then f is differentiable on $(0, 1)$.

Since f is a function continuous on the closed interval $[0, 1]$ and differentiable on the open interval $(0, 1)$, then by MVT, there exists $c_2 \in (0, 1)$ such that $f'(c_2) = \frac{f(1)-f(0)}{1-0} = \frac{1-0}{1-0} = 1$. □

Proof. We prove 3.

Observe that there exists $c_1 \in (1, 2)$ such that $f'(c_1) = 0$ and there exists $c_2 \in (0, 1)$ such that $f'(c_2) = 1$.

Since $c_2 \in (0, 1)$ and $c_1 \in (1, 2)$, then $0 < c_2 < 1$ and $1 < c_1 < 2$.

Hence, $0 < c_2 < 1 < c_1 < 2$, so $c_2 < c_1$ and $[c_2, c_1] \subset [0, 2]$ and $(c_2, c_1) \subset (0, 2)$.

Since f is differentiable on $[0, 2]$ and $[c_2, c_1] \subset [0, 2]$, then f is differentiable on $[c_2, c_1]$.

Since $c_2 < c_1$ and f is differentiable on the closed interval $[c_2, c_1]$ and $f'(c_2) = 1 > \frac{1}{3} > 0 = f'(c_1)$, then by the intermediate value property of derivatives, there exists $c \in (c_2, c_1)$ such that $f'(c) = \frac{1}{3}$.

Since $c \in (c_2, c_1)$ and $(c_2, c_1) \subset (0, 2)$, then $c \in (0, 2)$.

Therefore, there exists $c \in (0, 2)$ such that $f'(c) = \frac{1}{3}$, as desired. □

Exercise 18. Let $n \in \mathbb{N}$ and $0 < a < b$.

Then $na^{n-1}(b - a) \leq b^n - a^n \leq nb^{n-1}(b - a)$.

Proof. Let $f : [a, b] \to \mathbb{R}$ be a function defined by $f(x) = x^n$.

Let $x \in [a, b]$.

Then $f(x) = x^n$, so $f'(x) = nx^{n-1}$.

Hence, $f'(x) = nx^{n-1}$ for all $x \in \mathbb{R}$, so f is differentiable.

Since f is a polynomial function in x, then f is continuous, so f is continuous on $[a, b]$.

Since f is differentiable and $(a, b) \subset [a, b]$, then f is differentiable on (a, b).

Since f is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b), then by MVT, there exists $c \in (a, b)$ such that $f'(c) = \frac{f(b)-f(a)}{b-a}$.

Since $\frac{b^n-a^n}{b-a} = \frac{f(b)-f(a)}{b-a} = f'(c) = nc^{n-1}$, then $\frac{b^n-a^n}{b-a} = nc^{n-1}$.

Since $c \in (a, b)$, then $a < c < b$, so $0 < a < c < b$.

Since $n \in \mathbb{N}$, then $n \geq 1$, so $n - 1 \geq 0$.

Thus, $a^{n-1} \leq c^{n-1} \leq b^{n-1}$.

Since $n \geq 1 > 0$, then $n > 0$, so $na^{n-1} \leq nc^{n-1} \leq nb^{n-1}$.

Hence, $na^{n-1} \leq \frac{b^n-a^n}{b-a} \leq nb^{n-1}$.

Since $a < b$, then $b - a > 0$, so $na^{n-1}(b - a) \leq b^n - a^n \leq nb^{n-1}(b - a)$, as desired. □

Exercise 19. For all real $x \geq 0$, $-x \leq \sin x \leq x$.

Proof. Let $x \geq 0$ be given.

Then either $x > 0$ or $x = 0$.

We consider these cases separately.

Case 1: Suppose $x = 0$.

Then $-x = -0 = 0 = \sin 0 = \sin x = 0 = x$, so $-x = \sin x = x$.

Case 2: Suppose $x > 0$.

Let $f : \mathbb{R} \to \mathbb{R}$ be the function $f(x) = \sin x$.

Since f is continuous and $[0, x] \subset \mathbb{R}$, then f is continuous on $[0, x]$.

Since $f'(x) = \cos x$, then f is differentiable on $\mathbb{R}$.

Since $(0, x) \subset \mathbb{R}$, then f is differentiable on $(0, x)$.

Since f is continuous on the closed interval $[0, x]$ and differentiable on the open interval $(0, x)$, then by MVT, there exists $c \in (0, x)$ such that $f'(c) = \frac{f(x)-f(0)}{x-0}$.

Since $c \in (0, x)$, then $f'(c) = \cos c$.

Since $c \in (0, x)$ and $(0, x) \subset \mathbb{R}$, then $c \in \mathbb{R}$.

Since $-1 \leq \cos x \leq 1$ for all $x \in \mathbb{R}$ and $c \in \mathbb{R}$, then $-1 \leq \cos c \leq 1$.

Hence, $-1 \leq \cos c = f'(c) = \frac{f(x)-f(0)}{x-0} \leq 1$.

Thus, $-1 \leq \frac{\sin x}{x} \leq 1$.

Since $x > 0$, then $-x \leq \sin x \leq x$. □

Exercise 20. The inequality $|\sin x - \sin y| \leq |x - y|$ is true for all real x and y.

Proof. Let $x, y \in \mathbb{R}$ be arbitrary.

Then either $x = y$ or $x \neq y$.

We consider these cases separately.

Case 1: Suppose $x = y$.

Then $|\sin x - \sin y| = |\sin x - \sin x| = 0 = |0 - 0| = |x - y|$, so $|\sin x - \sin y| = |x - y|$.

Case 2: Suppose $x \neq y$.

Then either $x < y$ or $x > y$.

Without loss of generality, assume $x < y$.

Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by $f(x) = \sin x$.

Since the sin function is continuous on $\mathbb{R}$ and $[x, y] \subset \mathbb{R}$, then f is continuous on the interval $[x, y]$.

Since $f'(x) = \cos x$ for all $x \in \mathbb{R}$, then f is differentiable on $\mathbb{R}$.

Since $(x, y) \subset \mathbb{R}$, then f is differentiable on (x, y).

Since f is continuous on the closed interval $[x, y]$ and differentiable on the open interval (x, y), then by MVT, there exists $c \in (x, y)$ such that $f'(c) = \frac{f(y)-f(x)}{y-x}$.

Since $|\cos x| \leq 1$ for all $x \in \mathbb{R}$, then we have $1 \geq |\cos c| = |f'(c)| = |\frac{f(y)-f(x)}{y-x}| = |\frac{\sin y - \sin x}{y-x}| = \frac{|\sin y - \sin x|}{|y-x|} = \frac{|\sin x - \sin y|}{|x-y|}$.

Thus, $1 \geq \frac{|\sin x - \sin y|}{|x-y|}$.

Since $x \neq y$, then $x - y \neq 0$, so $|x - y| > 0$.

Therefore, $|x - y| \geq |\sin x - \sin y|$, so $|\sin x - \sin y| \leq |x - y|$. □

Exercise 21. For all real x, $e^x \geq 1 + x$.

Proof. Let $x \in \mathbb{R}$ be arbitrary.

Then either $x > 0$ or $x = 0$ or $x < 0$.

We consider these cases separately.

Case 1: Suppose $x = 0$.

Then $e^x = e^0 = 1 = 1 + 0 = 1 + x$, so $e^x = 1 + x$.

Case 2: Suppose $x > 0$.

Let $f : \mathbb{R} \to \mathbb{R}$ be the function $f(x) = e^x$.

Since f is continuous and $[0, x] \subset \mathbb{R}$, then f is continuous on $[0, x]$.

Since $f'(x) = e^x$, then f is differentiable on $\mathbb{R}$.

Since $(0, x) \subset \mathbb{R}$, then f is differentiable on $(0, x)$.

Since f is continuous on the closed interval $[0, x]$ and differentiable on the open interval $(0, x)$, then by MVT, there exists $c \in (0, x)$ such that $f'(c) = \frac{f(x)-f(0)}{x-0}$.

Since $c \in (0, x)$, then $0 < c < x$, so $0 < c$.

Since $f'(x) > 1$ for all $x > 0$ and $c > 0$, then $f'(c) > 1$.

Thus, $1 < f'(c) = \frac{f(x)-f(0)}{x-0} = \frac{e^x-1}{x}$, so $1 < \frac{e^x-1}{x}$.

Since $x > 0$, then $x < e^x - 1$, so $1 + x < e^x$.

Hence, $e^x > 1 + x$.

Case 3: Suppose $x < 0$.

Let $f : \mathbb{R} \to \mathbb{R}$ be the function $f(x) = e^x$.

Since f is continuous and $[x, 0] \subset \mathbb{R}$, then f is continuous on $[x, 0]$.

Since $f'(x) = e^x$, then f is differentiable on $\mathbb{R}$.

Since $(x, 0) \subset \mathbb{R}$, then f is differentiable on $(x, 0)$.

Since f is continuous on the closed interval $[x, 0]$ and differentiable on the open interval $(x, 0)$, then by MVT, there exists $c \in (x, 0)$ such that $f'(c) = \frac{f(0)-f(x)}{0-x}$.

Since $c \in (x, 0)$, then $x < c < 0$, so $c < 0$.

Since $f'(x) < 1$ for all $x < 0$ and $c < 0$, then $f'(c) < 1$.

Thus, $1 > f'(c) = \frac{f(0)-f(x)}{0-x} = \frac{1-e^x}{-x} = \frac{e^x-1}{x}$, so $1 > \frac{e^x-1}{x}$.

Since $x < 0$, then $x < e^x - 1$, so $1 + x < e^x$.

Hence, $e^x > 1 + x$.

Therefore, in all cases, either $e^x > 1 + x$ or $e^x = 1 + x$, so $e^x \geq 1 + x$. $\square$

Exercise 22. Bernoulli's Inequality

Let $p > 1$.

Then $(1 + x)^p > 1 + px$ for all real $x > 0$.

Proof. Let $x > 0$ be given.

We must prove $(1 + x)^p > 1 + px$.

Let $f : \mathbb{R} \to \mathbb{R}$ be the function $f(x) = (1 + x)^p$.

Since the power function is continuous and the linear function $1 + x$ is continuous, then the composition f is continuous.

Since f is continuous and $[0, x] \subset \mathbb{R}$, then f is continuous on $[0, x]$.

Since $f'(x) = p(1 + x)^{p-1}$, then f is differentiable on $\mathbb{R}$.

Since $(0, x) \subset \mathbb{R}$, then f is differentiable on $(0, x)$.

Since f is continuous on the closed interval $[0, x]$ and differentiable on the open interval $(0, x)$, then by MVT, there exists $c \in (0, x)$ such that $f'(c) = \frac{f(x)-f(0)}{x-0}$.

Since $c \in (0, x)$, then $0 < c < x$ and $f'(c) = p(1 + c)^{p-1}$.

Hence, $p(1+c)^{p-1} = f'(c) = \frac{f(x)-f(0)}{x-0} = \frac{(1+x)^p-1}{x}$, so $p(1+c)^{p-1} = \frac{(1+x)^p-1}{x}$.

Since $x > 0$, then $px(1+c)^{p-1} = (1+x)^p - 1$, so $1+px(1+c)^{p-1} = (1+x)^p$.

Since $c \in (0, x)$, then $0 < c < x$, so $0 < c$.

Hence, $c > 0$, so $1 + c > 1$.

Since $p > 1$, then $p - 1 > 0$, so $(1 + c)^{p-1} > 1$.

Since $p > 0$ and $x > 0$, then $px > 0$, so $px(1 + c)^{p-1} > px$.

Therefore, $(1 + x)^p = 1 + px(1 + c)^{p-1} > 1 + px$, so $(1 + x)^p > 1 + px$, as desired. □

Exercise 23. For all real $x > 0$, $\sqrt{1+x} < 1 + \frac{x}{2}$.

Proof. Let $x > 0$ be given.

Then $x^2 > 0$, so $\frac{x^2}{4} > 0$.

Since $0 < 1$, then $x < 1 + x$.

Since $0 < x$ and $x < 1 + x$, then $0 < 1 + x$.

Thus, $0 < 1 + x < 1 + x + \frac{x^2}{4} = (1 + \frac{x}{2})^2$, so $0 < 1 + x < (1 + \frac{x}{2})^2$.

Since $-2 < 0 < x$, then $-2 < x$, so $-1 < \frac{x}{2}$.

Hence, $0 < 1 + \frac{x}{2} = |1 + \frac{x}{2}|$.

Therefore, $0 < \sqrt{1+x} < \sqrt{(1 + \frac{x}{2})^2} = |1+\frac{x}{2}| = 1+\frac{x}{2}$, so $\sqrt{1+x} < 1+\frac{x}{2}$. □

Proof. Alternate proof using MVT.

To prove $\sqrt{1+x} < 1 + \frac{x}{2}$ for all $x > 0$, let $h > 0$.

We must prove $\sqrt{1+h} < 1 + \frac{h}{2}$.

Let $f : [0, h] \to \mathbb{R}$ be a function defined by $f(x) = \sqrt{1+x}$.

Since the square root function is continuous and the linear function $1 + x$ is continuous, then the composition f is continuous, so f is continuous on $[0, h]$.

Observe that $f'(x) = \frac{1}{2\sqrt{1+x}}$ for all $x \geq 0$.

Thus, f is differentiable on $[0, \infty)$.

Since $(0, h) \subset [0, \infty)$, then f is differentiable on $(0, h)$.

Since f is continuous on the closed interval $[0, h]$ and differentiable on the open interval $(0, h)$, then by MVT, there exists $c \in (0, h)$ such that $f'(c) = \frac{f(h)-f(0)}{h-0}$.

Since $c \in (0, h)$, then $f'(c) = \frac{1}{2\sqrt{1+c}}$.

Thus, $\frac{1}{2\sqrt{1+c}} = f'(c) = \frac{f(h)-f(0)}{h-0} = \frac{f(h)-1}{h}$, so $\frac{1}{2\sqrt{1+c}} = \frac{f(h)-1}{h}$.

Hence, $f(h) = 1 + \frac{h}{2\sqrt{1+c}}$.

Since $c \in (0, h)$, then $0 < c < h$, so $0 < c$.

Thus, $1 < 1 + c$, so $1 < \sqrt{1+c}$.

Consequently, $\frac{1}{\sqrt{1+c}} < 1$.

Since $\frac{h}{2} > 0$, then $\frac{h}{2\sqrt{1+c}} < \frac{h}{2}$.

Therefore, $\sqrt{1+h} = f(h) = 1 + \frac{h}{2\sqrt{1+c}} < 1 + \frac{h}{2}$, so $\sqrt{1+h} < 1 + \frac{h}{2}$, as desired. $\square$

Exercise 24. Let $0 < p < 1$.

Then $(1+x)^p < 1 + px$ for all real $x > 0$.

Proof. To prove $(1+x)^p < 1 + px$ for all real $x > 0$, let $h > 0$.

We must prove $(1+h)^p < 1 + ph$.

Let $f : [0,h] \to \mathbb{R}$ be a function defined by $f(x) = (1+x)^p$.

Since the power function is continuous and the linear function $1 + x$ is continuous, then the composition f is continuous, so f is continuous on $[0,h]$.

Observe that $f'(x) = p(1+x)^{p-1}$ for all $x \geq 0$.

Thus, f is differentiable on $[0,\infty)$.

Since $(0,h) \subset [0,\infty)$, then f is differentiable on $(0,h)$.

Since f is continuous on the closed interval $[0,h]$ and differentiable on the open interval $(0,h)$, then by MVT, there exists $c \in (0,h)$ such that $f'(c) = \frac{f(h)-f(0)}{h-0}$.

Since $c \in (0,h)$, then $f'(c) = p(1+c)^{p-1}$.

Thus, $p(1+c)^{p-1} = f'(c) = \frac{f(h)-f(0)}{h-0} = \frac{f(h)-1}{h}$, so $p(1+c)^{p-1} = \frac{f(h)-1}{h}$.

Hence, $f(h) = 1 + ph(1+c)^{p-1}$.

Since $0 < p < 1$, then $0 < p$ and $p < 1$, so $1 - p > 0$.

Thus, $f(h) = 1 + \frac{ph}{(1+c)^{1-p}}$.

Since $c \in (0,h)$, then $0 < c < h$, so $0 < c$.

Thus, $1 < 1 + c$, so $1 < (1+c)^{1-p}$.

Consequently, $\frac{1}{(1+c)^{1-p}} < 1$.

Since $p > 0$ and $h > 0$, then $ph > 0$, so $\frac{ph}{(1+c)^{1-p}} < ph$.

Therefore, $(1+h)^p = f(h) = 1 + \frac{ph}{(1+c)^{1-p}} < 1 + ph$, so $(1+h)^p < 1 + ph$, as desired. $\square$

Exercise 25. Let f be a real valued function differentiable on an interval I.

If $f'(x) \neq 0$ for all $x \in I$, then either $f'(x) > 0$ for all $x \in I$ or $f'(x) < 0$ for all $x \in I$.

Proof. We prove by contrapositive.

Suppose there exists $a \in I$ such that $f'(a) \leq 0$ and there exists $b \in I$ such that $f'(b) \geq 0$.

We must prove there exists $c \in I$ such that $f'(c) = 0$.

Since $f'(a) \leq 0$, then either $f'(a) < 0$ or $f'(a) = 0$.

Since $f'(b) \geq 0$, then either $f'(b) > 0$ or $f'(b) = 0$.

Thus, there are 4 cases.

We consider these cases separately.

Case 1: Suppose $f'(a) = 0$ and $f'(b) = 0$.

Then there exists $a \in I$ such that $f'(a) = 0$.

Case 2: Suppose $f'(a) = 0$ and $f'(b) > 0$.

Then there exists $a \in I$ such that $f'(a) = 0$.

Case 3: Suppose $f'(a) < 0$ and $f'(b) = 0$.

Then there exists $b \in I$ such that $f'(b) = 0$.
Case 4: Suppose $f'(a) < 0$ and $f'(b) > 0$.
Since $f'(a) < 0$ and $0 < f'(b)$, then $f'(a) < 0 < f'(b)$, so $f'(a) < f'(b)$.
Hence, $a \neq b$, so either $a < b$ or $a > b$.
We consider these cases separately.
Case 4a: Suppose $a < b$.
Since f is differentiable on the closed interval $[a, b]$ and $a < b$ and $f'(a) < 0 < f'(b)$, then by the intermediate value property of derivatives, there exists $c \in (a, b)$ such that $f'(c) = 0$.
Since $c \in (a, b)$ and $(a, b) \subset I$, then $c \in I$.
Thus, there exists $c \in I$ such that $f'(c) = 0$, as desired.
Case 4b: Suppose $a > b$.
Since f is differentiable on the closed interval $[b, a]$ and $b < a$ and $f'(b) > 0 > f'(a)$, then by the intermediate value property of derivatives, there exists $c \in (b, a)$ such that $f'(c) = 0$.
Since $c \in (b, a)$ and $(b, a) \subset I$, then $c \in I$.
Thus, there exists $c \in I$ such that $f'(c) = 0$, as desired. □

Exercise 26. Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function.
Let $g : \mathbb{R} \to \mathbb{R}$ be a differentiable function.
Let $f(0) = g(0)$ and $f'(x) > g'(x)$ for all $x \in \mathbb{R}$.
Then $f(x) > g(x)$ for all $x \in (0, \infty)$.

Proof. Let $h = f - g$ for all $x \in \mathbb{R}$.
Then h is a real valued function defined by $h(x) = f(x) - g(x)$ for all $x \in \mathbb{R}$.
Since f is differentiable and g is differentiable, then the difference $f - g$ is differentiable, so h is differentiable.
Thus, h is differentiable on $\mathbb{R}$, so $h'(x) = f'(x) - g'(x)$ for all $x \in \mathbb{R}$.

Let $c \in \mathbb{R}$ be arbitrary.
Then $f'(c) > g'(c)$ and $h'(c) = f'(c) - g'(c)$.
Hence, $h'(c) = f'(c) - g'(c) > 0$, so $h'(c) > 0$.
Thus, $h'(c) > 0$ for all $c \in \mathbb{R}$, so $h'(x) > 0$ for all $x \in \mathbb{R}$.
Consequently, h is strictly increasing on the interval $(-\infty, \infty) = \mathbb{R}$.

Let $x \in (0, \infty)$.
Then $0 < x$.
Since h is strictly increasing on $\mathbb{R}$ and $0 \in \mathbb{R}$ and $x \in \mathbb{R}$ and $0 < x$, then $h(0) < h(x)$, so $h(x) > h(0)$.
Thus, $f(x) - g(x) = h(x) > h(0) = f(0) - g(0) = f(0) - f(0) = 0$, so $f(x) - g(x) > 0$.
Hence, $f(x) > g(x)$.
Therefore, $f(x) > g(x)$ for all $x \in (0, \infty)$, as desired. □

Exercise 27. Let $f : [a, b] \to \mathbb{R}$ be function.
Let $c \in (a, b)$.

If f is differentiable at c and $f'(c) > 0$, then there exists $x \in (c, b)$ such that $f(x) > f(c)$.

Proof. Suppose f is differentiable at c and $f'(c) > 0$.

Then $f'(c) = \lim_{x \to c} \frac{f(x)-f(c)}{x-c} > 0$.

Let $q : [a, b] - \{c\} \to \mathbb{R}$ be a function defined by $q(x) = \frac{f(x)-f(c)}{x-c}$.

Then $f'(c) = \lim_{x \to c} q(x) > 0$, so the limit of q at c exists and is positive.

Hence, by a previous lemma, there exists $\delta > 0$ such that $q(x) > 0$ for all $x \in N'(c; \delta) \cap [a, b]$.

Thus, there exists $\delta > 0$ such that $\frac{f(x)-f(c)}{x-c} > 0$ for all $x \in N'(c; \delta) \cap [a, b]$.

Let $K = \min\{c + \delta, b\}$.

Then $K \leq c + \delta$ and $K \leq b$ and either $K = c + \delta$ or $K = b$.

Since $c \in (a, b)$, then $a < c < b$, so $a < c$ and $c < b$.

Let $x = \frac{c+K}{2}$.

We prove $c < x$.

Since $\delta > 0 = c - c$, then $c + \delta > c$.

Since $c < c + \delta$ and $c < b$ and either $K = c + \delta$ or $K = b$, then $c < K$.

Thus, $2c = c + c < c + K$, so $2c < c + K$.

Hence, $c < \frac{c+K}{2} = x$, so $c < x$.

We prove $x < b$.

Since $K \leq b$, then $c + K \leq c + b < b + b = 2b$, so $c + K < 2b$.

Thus, $x = \frac{c+K}{2} < b$, so $x < b$.

We prove $x \in [a, b]$.

Since $c < x$ and $x < b$, then $c < x < b$, so $x \in (c, b)$.

Since $(c, b) \subset [a, b]$, then $x \in [a, b]$.

We prove $x \in N'(c; \delta)$.

Since $c < K$ and $K \leq c+\delta$, then $c < K \leq c+\delta$, so $0 < K - c = |K - c| \leq \delta$.

Hence, $|K - c| \leq \delta$.

Observe that

$$\begin{aligned}
d(x,c) &= |x-c| \\
&= |\frac{c+K}{2} - c| \\
&= |\frac{c+K-2c}{2}| \\
&= |\frac{K-c}{2}| \\
&= \frac{1}{2}|K-c| \\
&\leq \frac{\delta}{2} \\
&< \delta.
\end{aligned}$$

Thus, $d(x,c) < \delta$, so $x \in N(c;\delta)$.
Since $x > c$, then $x \neq c$, so $x \in N'(c;\delta)$.

Since $x \in N'(c;\delta)$ and $x \in [a,b]$, then $x \in N'(c;\delta) \cap [a,b]$.
Hence, $\frac{f(x)-f(c)}{x-c} > 0$.
Since $x > c$, then $x - c > 0$, so $f(x) - f(c) > 0$.
Thus, $f(x) > f(c)$.
Therefore, there exists $x \in (c,b)$ such that $f(x) > f(c)$, as desired. □

Exercise 28. Let $a, b \in \mathbb{R}$ with $a < b$.
Let $f : [a,b] \to \mathbb{R}$ be a function continuous on $[a,b]$ and differentiable on (a,b).
If there is a real number L such that $\lim_{x\to a} f'(x) = L$, then $f'(a) = L$.

Proof. Suppose there is a real number L such that $\lim_{x\to a} f'(x) = L$.
Let $\epsilon > 0$ be given.
Then there is a $\delta_1 > 0$ such that for all $x \in [a,b]$, if $0 < |x-a| < \delta_1$, then $|f'(x) - L| < \epsilon$.
Let $\delta = \min\{\delta_1, b-a\}$.
Then $\delta \leq \delta_1$ and $\delta \leq b - a$.
Since $a < b$, then $b - a > 0$.
Since $\delta_1 > 0$ and $b - a > 0$, then $\delta > 0$.
Let $x \in [a,b]$ such that $0 < |x-a| < \delta$.
Since $0 < |x-a| < \delta$ and $\delta \leq \delta_1$, then $0 < |x-a| < \delta_1$.
Since $x \in [a,b]$ and $0 < |x-a| < \delta_1$, then $|f'(x) - L| < \epsilon$.
Since $x \in [a,b]$, then $a \leq x \leq b$, so $a \leq x$.
Since $0 < |x-a| < \delta$, then $0 < |x-a|$ and $|x-a| < \delta$.
Since $|x-a| > 0$, then $x - a \neq 0$, so $x \neq a$.
Since $x \geq a$ and $x \neq a$, then $x > a$.
Since $0 < |x-a| < \delta$ and $\delta \leq b-a$, then $0 < |x-a| < b-a$, so $|x-a| < b-a$.
Hence, $x - a < b - a$, so $x < b$.

Since $a < x$ and $x < b$, then $a < x < b$, so $[a, x] \subset [a, b]$ and $(a, x) \subset (a, b)$.

Since f is continuous on $[a, b]$ and $[a, x] \subset [a, b]$, then f is continuous on $[a, x]$.

Since f is differentiable on (a, b) and $(a, x) \subset (a, b)$, then f is differentiable on (a, x).

Since f is continuous on $[a, x]$ and differentiable on (a, x), then by MVT, there is $c \in (a, x)$ such that $f'(c) = \frac{f(x)-f(a)}{x-a}$.

Since $c \in (a, x)$ and $(a, x) \subset (a, b) \subset [a, b]$, then $c \in [a, b]$.

Since $c \in (a, x)$, then $a < c < x$, so $a < c$.

Since $c > a$, then $c - a > 0$, so $|c - a| > 0$.

Since $|x - a| < \delta$, then $x - a < \delta$, so $x < a + \delta$.

Since $a < c < x$ and $x < a + \delta$, then $a < c < x < a + \delta$, so $c < a + \delta$.

Hence, $c - a < \delta$.

Since $0 < c - a = |c - a| < \delta$, then $0 < |c - a| < \delta$.

Since $0 < |c - a| < \delta$ and $\delta \leq \delta_1$, then $0 < |c - a| < \delta_1$.

Since $c \in [a, b]$ and $0 < |c-a| < \delta_1$, then $|f'(c)-L| < \epsilon$, so $|\frac{f(x)-f(a)}{x-a} - L| < \epsilon$.

Thus, for every $\epsilon > 0$, there is $\delta > 0$ such that for all $x \in [a, b]$, if $0 < |x - a| < \delta$, then $|\frac{f(x)-f(a)}{x-a} - L| < \epsilon$, so $\lim_{x \to a} \frac{f(x)-f(a)}{x-a} = L$.

Therefore, there is a real number L such that $L = \lim_{x \to a} \frac{f(x)-f(a)}{x-a} = f'(a)$, so $f'(a) = L$, as desired. $\square$

Exercise 29. Let f be a real valued function of a real variable.

Let I be an interval of at least two elements such that $I \subset dom f$.

Let f be differentiable on I. Then

1. If f is strictly increasing on I, then $f'(x) \geq 0$ for all $x \in I$.
2. If f is strictly decreasing on I, then $f'(x) \leq 0$ for all $x \in I$.

Proof. We prove 1.

Suppose f is strictly increasing on I.

To prove $f'(x) \geq 0$ for all $x \in I$, let $c \in I$ be arbitrary.

We must prove $f'(c) \geq 0$.

Let $q : I - \{c\} \to \mathbb{R}$ be a function defined by $q(x) = \frac{f(x)-f(c)}{x-c}$ for all $x \in I - \{c\}$.

Let $x \in I - \{c\}$.

Then $x \in I$ and $x \neq c$ and $q(x) = \frac{f(x)-f(c)}{x-c}$.

Since $x \in \mathbb{R}$ and $x \neq c$, then either $x > c$ or $x < c$.

We consider these cases separately.

Case 1: Suppose $x > c$.

Then $c < x$.

Since f is strictly increasing on I and $c \in I$ and $x \in I$, then $f(c) < f(x)$.

Since $x > c$, then $x - c > 0$.

Since $f(c) < f(x)$, then $f(x) - f(c) > 0$.

Thus, $q(x) = \frac{f(x)-f(c)}{x-c} > 0$, so $q(x) > 0$.

Case 2: Suppose $x < c$.

Since f is strictly increasing on I and $x \in I$ and $c \in I$, then $f(x) < f(c)$.

Since $x < c$, then $x - c < 0$.
Since $f(x) < f(c)$, then $f(x) - f(c) < 0$.
Thus, $q(x) = \frac{f(x)-f(c)}{x-c} > 0$, so $q(x) > 0$.

Hence, in all cases, $q(x) > 0$, so $q(x) \geq 0$.
Consequently, $0 \leq q(x)$, so $0 \leq q(x)$ for all $x \in I - \{c\}$.
Observe that c is an accumulation point of $\mathbb{R} \cap I - \{c\} = I - \{c\}$.
Since a limit preserves a non-strict inequality, then $0 = \lim_{x\to c} 0 \leq \lim_{x\to c} q(x) = \lim_{x\to c} \frac{f(x)-f(c)}{x-c} = f'(c)$.
Therefore, $0 \leq f'(c)$, so $f'(c) \geq 0$, as desired. □

Proof. We prove 2.
Suppose f is strictly decreasing on I.
To prove $f'(x) \leq 0$ for all $x \in I$, let $c \in I$ be arbitrary.
We must prove $f'(c) \leq 0$.
Let $q : I - \{c\} \to \mathbb{R}$ be a function defined by $q(x) = \frac{f(x)-f(c)}{x-c}$ for all $x \in I - \{c\}$.
Let $x \in I - \{c\}$.
Then $x \in I$ and $x \neq c$ and $q(x) = \frac{f(x)-f(c)}{x-c}$.
Since $x \in \mathbb{R}$ and $x \neq c$, then either $x > c$ or $x < c$.
We consider these cases separately.
Case 1: Suppose $x > c$.
Then $c < x$.
Since f is strictly decreasing on I and $c \in I$ and $x \in I$, then $f(c) > f(x)$.
Since $x > c$, then $x - c > 0$.
Since $f(x) < f(c)$, then $f(x) - f(c) < 0$.
Thus, $q(x) = \frac{f(x)-f(c)}{x-c} < 0$, so $q(x) < 0$.
Case 2: Suppose $x < c$.
Since f is strictly decreasing on I and $x \in I$ and $c \in I$, then $f(x) > f(c)$.
Since $x < c$, then $x - c < 0$.
Since $f(x) > f(c)$, then $f(x) - f(c) > 0$.
Thus, $q(x) = \frac{f(x)-f(c)}{x-c} < 0$, so $q(x) < 0$.

Hence, in all cases, $q(x) < 0$, so $q(x) \leq 0$.
Consequently, $q(x) \leq 0$ for all $x \in I - \{c\}$.
Observe that c is an accumulation point of $\mathbb{R} \cap I - \{c\} = I - \{c\}$.
Since a limit preserves a non-strict inequality, then $f'(c) = \lim_{x\to c} \frac{f(x)-f(c)}{x-c} = \lim_{x\to c} q(x) \leq \lim_{x\to c} 0 = 0$.
Therefore, $f'(c) \leq 0$, as desired. □