Differentiation of real valued functions Notes

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Sets of Numbers

$\mathbb{R} =$ set of all real numbers
$\mathbb{R}^+ = \{x \in \mathbb{R} : x > 0\} = (0, \infty) =$ set of all positive real numbers
$\mathbb{R}^* = \{x \in \mathbb{R} : x \neq 0\} = (-\infty, 0) \cup (0, \infty) =$ set of all nonzero real numbers

Derivative of a real valued function

The derivative of a function represents the instantaneous rate of change of a function.

It represents the slope of the line tangent to the graph of a function at some point on the function.

Definition 1. derivative of a function at a point

Let $E \subset \mathbb{R}$.
Let $f : E \to \mathbb{R}$ be a function.
Let $c \in E$ be an accumulation point of E.
Let $q : E - \{c\} \to \mathbb{R}$ be a function defined by $q(x) = \frac{f(x)-f(c)}{x-c}$.

We say that f **is differentiable at** c iff $\lim_{x\to c} \frac{f(x)-f(c)}{x-c}$ exists and we write $f'(c) = \lim_{x\to c} \frac{f(x)-f(c)}{x-c}$.

We say that $f'(c)$ is the **derivative of** f **at** c.

Let $E \subset \mathbb{R}$.
Let $f : E \to \mathbb{R}$ be a function.
Let $c \in E$ be an accumulation point of E.
Let $q : E - \{c\} \to \mathbb{R}$ be a function defined by $q(x) = \frac{f(x)-f(c)}{x-c}$ for all $x \in E - \{c\}$.

We call $\frac{f(x)-f(c)}{x-c}$ the **difference quotient**.

Since c is an accumulation point of E, then c is an accumulation point of $E - \{c\}$, the domain of q.

For each $x \in E - \{c\}$, we have $x \in E$ and $x \neq c$.

Let $E \subset \mathbb{R}$.
Let $f : E \to \mathbb{R}$ be a function.
Suppose f is differentiable at c.
Then $c \in domf$ and c is an accumulation point of $domf$ and $f'(c) = \lim_{x \to c} \frac{f(x)-f(c)}{x-c}$ exists and $f'(c) \in \mathbb{R}$.
Let $q : E - \{c\} \to \mathbb{R}$ be a function defined by $q(x) = \frac{f(x)-f(c)}{x-c}$ for all $x \in E - \{c\}$.
For each $x \in E - \{c\}$, we have $x \in E$ and $x \neq c$, so $x \in domf$ and $x \neq c$.

Definition 2. Alternate definition of derivative of a function
Let E be an open subset of $\mathbb{R}$.
Let $f : E \to \mathbb{R}$ be a function.
Let $c \in E$.
Since E is open, then c is an interior point of E, so there exists $\delta > 0$ such that $N(c; \delta) \subset E$.
Let $Q : (0, \delta) \to \mathbb{R}$ be a function defined by $Q(h) = \frac{f(c+h)-f(c)}{h}$.
We say that f **is differentiable at** c iff $\lim_{h \to 0} \frac{f(c+h)-f(c)}{h}$ exists and we write $f'(c) = \lim_{h \to 0} Q(h) = \lim_{h \to 0} \frac{f(c+h)-f(c)}{h}$.
We say that $f'(c)$ is the **derivative of** f **at** c.

Let E be an open subset of $\mathbb{R}$.
Let $f : E \to \mathbb{R}$ be a function.
Let $c \in E$.
Since E is open and $c \in E$, then c is an interior point of E, so there exists $\delta > 0$ such that $N(c; \delta) \subset E$.
Let $Q : (0, \delta) \to \mathbb{R}$ be a function defined by $Q(h) = \frac{f(c+h)-f(c)}{h}$.
Suppose f is differentiable at c.
Then $f'(c) = \lim_{x \to c} \frac{f(x)-f(c)}{x-c} = \lim_{h \to 0} \frac{f(c+h)-f(c)}{h}$.

A function differentiable on a set is differentiable at each point of the set.

Definition 3. function differentiable on a set
A function f is **differentiable on a set** $E \subset domf$ iff f is differentiable at c for all $c \in E$.

Definition 4. differentiable function
A function f is **differentiable** iff f is differentiable on $domf$.

Therefore, a function f is differentiable iff f is differentiable at c for all $c \in domf$.

Let $E \subset \mathbb{R}$.
Let f be a real valued function differentiable on E.
Then f is differentiable at c for each $c \in E$.
Hence, $f'(c) = \lim_{x \to c} \frac{f(x)-f(c)}{x-c}$ exists and $f'(c) \in \mathbb{R}$.

Since c is arbitrary, then $f'(c) \in \mathbb{R}$ for each $c \in E$.

Thus, $f' : E \to \mathbb{R}$ is a function defined by $f'(c) = \lim_{x \to c} \frac{f(x)-f(c)}{x-c}$ for each $c \in E$.

Hence, if f is a real valued function differentiable on E, then f' is a function defined on E.

Therefore, the derivative is a function.

Let $E \subset \mathbb{R}$.

Let f be a real valued function differentiable on E.

If $y = f(x)$, we denote the derivative of f at $x \in E$ by $y' = f'(x) = \frac{dy}{dx} = \frac{d}{dx} f(x) = D_x y$.

Equivalently, $f' = \frac{df}{dx} = Df$.

Proposition 5. ***the derivative of a constant is zero***

Let $k \in \mathbb{R}$ be fixed.

Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by $f(x) = k$.

Then $f'(x) = 0$ for all $x \in \mathbb{R}$.

Let $k \in \mathbb{R}$.

If $f(x) = k$, then $f'(x) = \frac{d}{dx}(k) = 0$.

Proposition 6. ***the derivative of the identity function is 1***

Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by $f(x) = x$.

Then $f'(x) = 1$ for all $x \in \mathbb{R}$.

If $f(x) = x$, then $f'(x) = \frac{d}{dx}(x) = 1$.

Example 7. the square function is differentiable

Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by $f(x) = x^2$.

Then $f'(x) = 2x$ for all $x \in \mathbb{R}$.

Therefore, $f'(x) = \frac{d}{dx}(x^2) = 2x$.

Example 8. the reciprocal function is differentiable for nonzero x

Let $f : \mathbb{R}^* \to \mathbb{R}$ be the function defined by $f(x) = \frac{1}{x}$ for all nonzero real x.

Then $f'(x) = \frac{-1}{x^2}$ for all nonzero real x.

Therefore, $f'(x) = \frac{d}{dx}(\frac{1}{x}) = \frac{-1}{x^2}$ if $x \neq 0$.

Example 9. the absolute value function is not differentiable at zero

Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by $f(x) = |x|$ for all $x \in \mathbb{R}$.

Then f is not differentiable at 0.

Example 10. the square root function is differentiable for positive real x

Let $f : [0, \infty) \to \mathbb{R}$ be the function defined by $f(x) = \sqrt{x}$.

Then $f'(x) = \frac{1}{2\sqrt{x}}$ for all positive real x and f is not differentiable at 0.

Therefore, $f'(x) = \frac{d}{dx}(\sqrt{x}) = \frac{1}{2\sqrt{x}}$ if $x > 0$ and f is not differentiable at 0.

Lemma 11. $\lim_{x \to 0} \frac{1-\cos x}{x} = 0$ *and* $\lim_{x \to 0} \frac{\cos x - 1}{x} = 0$.

Proposition 12. *If $f(x) = \sin x$, then $f'(x) = \cos x$.*
Therefore, $\frac{d}{dx} \sin x = \cos x$.

Proposition 13. *If $f(x) = \cos x$, then $f'(x) = -\sin x$.*
Therefore, $\frac{d}{dx} \cos x = -\sin x$.

Theorem 14. ***differentiability implies continuity***
Let f be a real valued function of a real variable x.
If f is differentiable at c, then f is continuous at c.

Thus, if f is not continuous at c, then f is not differentiable at c.
Therefore, if f is discontinuous at c, then f is not differentiable at c.

Corollary 15. ***Every differentiable function is continuous.***
Let f be a real valued function of a real variable x.
If f is differentiable, then f is continuous.

Example 16. not every continuous function is differentiable
The absolute value function is continuous on $\mathbb{R}$, but it is not differentiable at zero.

Algebraic properties of derivatives

Theorem 17. ***scalar multiple rule for derivatives***
Let f be a real valued function of a real variable x.
If f is differentiable at c, then for every $\lambda \in \mathbb{R}$, the function λf is differentiable at c and $(\lambda f)'(c) = \lambda f'(c)$.

Let f be a real valued differentiable function of a real variable x.
Let $\lambda \in \mathbb{R}$.
Then the scalar multiple λf is differentiable and $(\lambda f)' = \lambda f'$.
Equivalently, $D(\lambda f) = \lambda Df$.

Theorem 18. ***derivative of a sum equals sum of a derivative***
Let f and g be real valued functions of a real variable x.
Let c be an accumulation point of $\text{dom} f \cap \text{dom} g$.
If f is differentiable at c and g is differentiable at c, then the function $f + g$ is differentiable at c and
$(f + g)'(c) = f'(c) + g'(c)$.

Let f and g be real valued differentiable functions of a real variable x.
Then the sum $f + g$ is differentiable and $(f + g)' = f' + g'$.
Equivalently, $D(f + g) = Df + Dg$.

Corollary 19. ***derivative of a difference equals difference of a derivative***
Let f and g be real valued functions of a real variable x.
Let c be an accumulation point of $\text{dom} f \cap \text{dom} g$.

If f is differentiable at c and g is differentiable at c, then the function $f - g$ is differentiable at c and

$(f - g)'(c) = f'(c) - g'(c)$.

Let f and g be real valued differentiable functions of a real variable x.

Then the difference $f - g$ is differentiable and $(f - g)' = f' - g'$.

Equivalently, $D(f - g) = Df - Dg$.

TODO We need to write up a proposition to cover the derivative of a finite sum of functions. The proof would involve using math induction. We'll write this up later since we don't have time right now.

Theorem 20. ***product rule for derivatives***

Let f and g be real valued functions of a real variable x.

Let c be an accumulation point of $dom f \cap dom g$.

If f is differentiable at c and g is differentiable at c, then the function fg is differentiable at c and

$(fg)'(c) = f(c)g'(c) + g(c)f'(c)$.

Let f and g be real valued differentiable functions of a real variable x.

Then the product fg is differentiable and $(fg)' = fg' + gf'$.

Equivalently, $D(fg) = fDg + gDf$.

Theorem 21. ***quotient rule for derivatives***

Let f and g be real valued functions of a real variable x.

Let c be an accumulation point of $dom f \cap dom g$.

If f is differentiable at c and g is differentiable at c and $g(c) \neq 0$, then the function $\frac{f}{g}$ is differentiable at c and

$(\frac{f}{g})'(c) = \frac{g(c)f'(c) - f(c)g'(c)}{(g(c))^2}$.

Let f and g be real valued differentiable functions of a real variable x such that $g(x) \neq 0$ for all $x \in dom f \cap dom g$.

Then the quotient $\frac{f}{g}$ is differentiable and $(\frac{f}{g})' = \frac{gf' - fg'}{g^2}$.

Equivalently, $D(\frac{f}{g}) = \frac{gDf - fDg}{g^2}$.

Corollary 22. ***power rule for derivatives***

Let $n \in \mathbb{Z}$ be fixed.

Let $f : \mathbb{R}^* \to \mathbb{R}$ be the function defined by $f(x) = x^n$.

Then $f'(x) = nx^{n-1}$.

Let $n \in \mathbb{Z}$ be fixed.

Let $f : \mathbb{R}^* \to \mathbb{R}$ be the function defined by $f(x) = x^n$.

Then $f'(x) = nx^{n-1}$ for all $x \in \mathbb{R}^*$.

Therefore, $f'(x) = \frac{d}{dx}(x^n) = nx^{n-1}$.

Proposition 23. ***derivatives of trig functions***

1. $\frac{d}{dx}(\tan x) = \sec^2 x$.
2. $\frac{d}{dx}(\cot x) = -\csc^2 x$.
3. $\frac{d}{dx}(\sec x) = \sec x \tan x$.
4. $\frac{d}{dx}(\csc x) = -\csc x \cot x$.

Theorem 24. ***chain rule for derivatives***

Let f and g be real valued functions such that $rngf \subset domg$.

If f is differentiable at c and g is differentiable at $f(c)$, then the function $g \circ f$ is differentiable at c and

$(g \circ f)'(c) = g'(f(c)) \cdot f'(c)$.

Let f and g be real valued functions such that $rngf \subset domg$.

If f is differentiable at x and g is differentiable at $f(x)$, then the function $g \circ f$ is differentiable at x and

$(g \circ f)'(x) = g'(f(x)) \cdot f'(x)$.

Let $y = g(u)$ be a differentiable function of u and let $u = f(x)$ be a differentiable function of x such that $rngf \subset domg$.

Then $y = g(u) = g(f(x)) = (g \circ f)(x)$ is a differentiable function of x.

Since $y = g(u)$ is a differentiable function of u, then $\frac{dy}{du} = g'(u)$.

Since $u = f(x)$ is a differentiable function of x, then $\frac{du}{dx} = f'(x)$.

Hence, $\frac{dy}{dx} = \frac{d}{dx}(g \circ f)(x) = (g \circ f)'(x) = g'(f(x)) \cdot f'(x) = g'(u) \cdot f'(x) = \frac{dy}{du} \cdot \frac{du}{dx}$.

Therefore, $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$.

Mean Value Theorem

Lemma 25. *Let $E \subset \mathbb{R}$.*

Let $f : E \to \mathbb{R}$ be a function.

Let $c \in E$.

1. If $f(c)$ is a relative maximum and f is differentiable at c, then $f'(c) = 0$.

2. If $f(c)$ is a relative minimum and f is differentiable at c, then $f'(c) = 0$.

Let $E \subset \mathbb{R}$.

Let $f : E \to \mathbb{R}$ be a function.

Suppose $f(c)$ is a relative maximum.

Either f is not differentiable at c or f is differentiable at c.

If f is differentiable at c, then $f'(c) = 0$.

Thus, either f is not differentiable at c or $f'(c) = 0$.

Therefore, if $f(c)$ is a relative maximum, then either f is not differentiable at c or $f'(c) = 0$.

Suppose $f(c)$ is a relative minimum.

Either f is not differentiable at c or f is differentiable at c.

If f is differentiable at c, then $f'(c) = 0$.

Thus, either f is not differentiable at c or $f'(c) = 0$.

Therefore, if $f(c)$ is a relative minimum, then either f is not differentiable at c or $f'(c) = 0$.

Therefore, if $f(c)$ is a relative extremum, then either f is not differentiable at c or $f'(c) = 0$.

Let $a, b \in \mathbb{R}$ with $a < b$.
Let f be a real valued function continuous on $[a, b]$.
Then by EVT, f has a maximum on $[a, b]$.
Let $f(c)$ be a maximum of f on $[a, b]$.
Then there exists $c \in [a, b]$ such that $f(c) \geq f(x)$ for all $x \in [a, b]$.
Either c is an endpoint of $[a, b]$ or not.
If c is an endpoint of $[a, b]$, then either $c = a$ or $c = b$, so either $f(a)$ or $f(b)$ is a maximum of f.
If c is not an endpoint of $[a, b]$, then $c \neq a$ and $c \neq b$.
Since $c \in [a, b]$ and $c \neq a$ and $c \neq b$, then $c \in (a, b)$.
Since the open interval (a, b) is an open set, then c is an interior point of (a, b).
Hence, there exists $\delta > 0$ such that $N(c; \delta) \subset (a, b)$.
Let $x \in N(c; \delta)$.
Since $N(c; \delta) \subset (a, b)$ and $(a, b) \subset [a, b]$, then $x \in [a, b]$ and $N(c; \delta) \subset [a, b]$.
Since $x \in [a, b]$, then $f(c) \geq f(x)$, so $f(c) \geq f(x)$ for all $x \in N(c; \delta)$.
Since there exists $\delta > 0$ such that $N(c; \delta) \subset [a, b]$ and $f(c) \geq f(x)$ for all $x \in N(c; \delta)$, then $f(c)$ is a relative maximum.
Thus, either f is not differentiable at c or $f'(c) = 0$.

Let f be a real valued function continuous on the interval $[a, b]$.
If $f(c)$ is a maximum of f on $[a, b]$, then
1. either $f(a)$ or $f(b)$ is a maximum or
2. $c \in (a, b)$ and f is not differentiable at c or
3. $c \in (a, b)$ and $f'(c) = 0$.

Let $a, b \in \mathbb{R}$ with $a < b$.
Let f be a real valued function continuous on $[a, b]$.
Then by EVT, f has a minimum on $[a, b]$.
Let $f(c)$ be a minimum of f on $[a, b]$.
Then there exists $c \in [a, b]$ such that $f(c) \leq f(x)$ for all $x \in [a, b]$.
Either c is an endpoint of $[a, b]$ or not.
If c is an endpoint of $[a, b]$, then either $c = a$ or $c = b$, so either $f(a)$ or $f(b)$ is a minimum of f.
If c is not an endpoint of $[a, b]$, then $c \neq a$ and $c \neq b$.
Since $c \in [a, b]$ and $c \neq a$ and $c \neq b$, then $c \in (a, b)$.
Since the open interval (a, b) is an open set, then c is an interior point of (a, b).
Hence, there exists $\delta > 0$ such that $N(c; \delta) \subset (a, b)$.
Let $x \in N(c; \delta)$.
Since $N(c; \delta) \subset (a, b)$ and $(a, b) \subset [a, b]$, then $x \in [a, b]$ and $N(c; \delta) \subset [a, b]$.
Since $x \in [a, b]$, then $f(c) \leq f(x)$, so $f(c) \leq f(x)$ for all $x \in N(c; \delta)$.

Since there exists $\delta > 0$ such that $N(c;\delta) \subset [a,b]$ and $f(c) \leq f(x)$ for all $x \in N(c;\delta)$, then $f(c)$ is a relative minimum.

Thus, either f is not differentiable at c or $f'(c) = 0$.

Let f be a real valued function continuous on the interval $[a,b]$.

If $f(c)$ is a minimum of f on $[a,b]$, then

1. either $f(a)$ or $f(b)$ is a minimum or
2. $c \in (a,b)$ and f is not differentiable at c or
3. $c \in (a,b)$ and $f'(c) = 0$.

Let f be a real valued function continuous on the interval $[a,b]$.

If $f(c)$ is an extreme value of f on $[a,b]$, then

1. either $f(a)$ or $f(b)$ is an extreme value or
2. $c \in (a,b)$ and f is not differentiable at c or
3. $c \in (a,b)$ and $f'(c) = 0$

Theorem 26. ***Rolle's Theorem***

Let $a, b \in \mathbb{R}$ with $a < b$.

Let f be a real valued function continuous on the closed interval $[a,b]$ and differentiable on the open interval (a,b).

If $f(a) = f(b)$, then there exists $c \in (a,b)$ such that $f'(c) = 0$.

Theorem 27. ***Mean Value Theorem***

Let $a, b \in \mathbb{R}$ with $a < b$.

Let f be a real valued function continuous on the closed interval $[a,b]$ and differentiable on the open interval (a,b).

Then there exists $c \in (a,b)$ such that $f'(c) = \frac{f(b)-f(a)}{b-a}$.

Let $a, b \in \mathbb{R}$ with $a < b$.

Let f be a real valued function continuous on the closed interval $[a,b]$ and differentiable on the open interval (a,b).

Then by MVT, there exists $c \in (a,b)$ such that $f'(c) = \frac{f(b)-f(a)}{b-a}$.

Therefore, MVT implies that there exists a point $c \in (a,b)$ at which the instantaneous rate of change equals the average rate of change of f over the interval $[a,b]$.

Corollary 28. *Let f be a real valued function differentiable on an interval I. Then*

1. *If $f'(x) = 0$ for all $x \in I$, then f is constant on I.*
2. *If $f'(x) > 0$ for all $x \in I$, then f is strictly increasing on I.*
3. *If $f'(x) < 0$ for all $x \in I$, then f is strictly decreasing on I.*

Corollary 29. ***functions with the same derivative on an interval differ by a constant***

Let f and g be real valued functions differentiable on an interval I.

If $f'(x) = g'(x)$ for all $x \in I$, then there exists $C \in \mathbb{R}$ such that $f - g = C$.

Theorem 30. ***First Derivative Test for relative extrema of a function continuous on an open interval***

Let $E \subset \mathbb{R}$.

Let $f : E \to \mathbb{R}$ be a function.

Let $c \in E$.

Let there exist $\delta > 0$ such that $(c - \delta, c + \delta) \subset E$.

Suppose f is continuous on $(c - \delta, c + \delta)$ and differentiable on $(c - \delta, c)$ and $(c, c + \delta)$.

1. If $f'(x) < 0$ for all $x \in (c - \delta, c)$ and $f'(x) > 0$ for all $x \in (c, c + \delta)$, then $f(c)$ is a relative minimum.

2. If $f'(x) > 0$ for all $x \in (c - \delta, c)$ and $f'(x) < 0$ for all $x \in (c, c + \delta)$, then $f(c)$ is a relative maximum.

Lemma 31. *Let $E \subset \mathbb{R}$.*

Let $f : E \to \mathbb{R}$ be a function.

Let c be a point.

1. If the limit of f at c exists and is positive, then there exists $\delta > 0$ such that $f(x) > 0$ for all $x \in N'(c; \delta) \cap E$.

2. If the limit of f at c exists and is negative, then there exists $\delta > 0$ such that $f(x) < 0$ for all $x \in N'(c; \delta) \cap E$.

Lemma 32. *Let $E \subset \mathbb{R}$.*

Let $f : E \to \mathbb{R}$ be a function.

Let f be differentiable at $c \in E$.

1. If $f'(c) > 0$, then there exists $\delta > 0$ such that $f(c) > f(x)$ for all $x \in (c - \delta, c) \cap E$.

2. If $f'(c) < 0$, then there exists $\delta > 0$ such that $f(c) > f(x)$ for all $x \in (c, c + \delta) \cap E$.

Theorem 33. ***Intermediate Value property of Derivatives***

Let $a, b \in \mathbb{R}$ with $a < b$.

Let f be a real valued function differentiable on the closed interval $[a, b]$.

For every real number k such that $f'(a) < k < f'(b)$, there exists $c \in (a, b)$ such that $f'(c) = k$.

Therefore, the derivative of a function differentiable on a closed bounded interval has the intermediate value property.

L'Hopital's Rule

Theorem 34. ***Cauchy Mean Value Theorem***

Let $a, b \in \mathbb{R}$ with $a < b$.

Let f be a real valued function continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b).

Let g be a real valued function continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b).

Then there exists $c \in (a, b)$ such that $f'(c)(g(b) - g(a)) = g'(c)(f(b) - f(a))$.