Integration Theory of real valued functions

Jason Sass

June 29, 2021

Integral of a real valued function

Theorem 1. ***representation of antiderivatives***
Let F be an antiderivative of a function f defined on an interval I.
Then G is an antiderivative of f on I iff there exists a constant C such that $G(x) = F(x) + C$ for all $x \in I$.

Proof. Suppose there exists a constant C such that $G(x) = F(x) + C$ for all $x \in I$.
Let $x \in I$.
Then $G(x) = F(x) + C$, so $G'(x) = F'(x)$.
Since F is an antiderivative of f, then $F'(x) = f(x)$.
Hence, $G'(x) = F'(x) = f(x)$, so $G'(x) = f(x)$.
Therefore, G is an antiderivative of f.

Conversely, suppose G is an antiderivative of f.
Then $G'(x) = f(x)$ for all $x \in I$.
Since F is an antiderivative of f, then $F'(x) = f(x)$ for all $x \in I$.
Let $x \in I$.
Then $G'(x) = f(x) = F'(x)$, so $G'(x) = F'(x)$ for all $x \in I$.
Since functions with the same derivative on an interval differ by a constant by MVT, then there exists a real constant C such that $G - F = C$.
Thus, there exists a constant C such that $G(x) - F(x) = C$, so there exists a constant C such that $G(x) = F(x) + C$.
Therefore, there exists a constant C such that $G(x) = F(x)+C$ for all $x \in I$, as desired. ∎

Definite Integral of a real valued function

Lemma 2. *If $f \in \mathcal{R}[a, b]$, then $\int_a^a f = 0 = \int_b^b f$.*

Proof. Let $f \in \mathcal{R}[a, b]$.
Then $f : [a, b] \to \mathbb{R}$ is integrable on $[a, b]$, so $\int_a^b f$ exists.
Since $a \leq a \leq b$, then $\int_a^b f = \int_a^a f + \int_a^b f$, so $0 = \int_a^a f$.

Since $a \le b \le b$, then $\int_a^b f = \int_a^b f + \int_b^b f$, so $0 = \int_b^b f$.
Therefore, $\int_a^a f = 0 = \int_b^b f$. ☐

Proposition 3. *Let $f \in \mathcal{R}[a, b]$.*
For every $c \in [a, b]$, $\int_c^c f = 0$.

Proof. Let $c \in [a, b]$ be arbitrary.
Then either $c = a$ or $c = b$ or $a < c < b$.
We consider these cases separately.
Case 1: Suppose $c = a$.
Then $\int_c^c f = \int_a^a f = 0$.
Case 2: Suppose $c = b$.
Then $\int_c^c f = \int_b^b f = 0$.
Case 3: Suppose $a < c < b$.
Then $c \in (a, b)$.
Since $f \in \mathcal{R}[a, b]$, then $f \in \mathcal{R}[a, c]$ and $f \in \mathcal{R}[c, b]$, so $f \in \mathcal{R}[a, c]$.
Hence, by the previous lemma, $\int_c^c f = 0$.
Therefore, in all cases, $\int_c^c f = 0$. ☐

Fundamental Theorem of Calculus

Theorem 4. *Let $f : [a, b] \to \mathbb{R}$ be an integrable function.*
Let $F : [a, b] \to \mathbb{R}$ be defined by $F(x) = \int_a^x f$ for $x \in [a, b]$.
The function F is continuous.

Proof. Since $f : [a, b] \to \mathbb{R}$ is integrable, then $f \in \mathcal{R}[a, b]$.
We first prove F is a function.
Let $x \in [a, b]$.
Then either $x = a$ or $x \in (a, b)$ or $x = b$.
If $x = a$, then $F(x) = F(a) = \int_a^a f = 0$.
If $x = b$, then $F(x) = F(b) = \int_a^b f$.
If $x \in (a, b)$, then $f \in \mathcal{R}[a, b]$ iff $f \in \mathcal{R}[a, x]$ and $f \in \mathcal{R}[x, b]$.
Since $f \in \mathcal{R}[a, b]$, then $f \in \mathcal{R}[a, x]$ and $f \in \mathcal{R}[x, b]$, so $f \in \mathcal{R}[a, x]$.
Thus, $\int_a^x f$ exists as a unique real number and $F(x) = \int_a^x f$.
Therefore, $F : [a, b] \to \mathbb{R}$ is a function.

We prove F is continuous.
Since $f \in \mathcal{R}[a, b]$, then f is bounded, so there exists $M \in \mathbb{R}$ such that $|f(x)| \le M$ for all $x \in [a, b]$.
Let $x \in [a, b]$.
Then $|f(x)| \le M$.
Since $M \ge |f(x)| \ge 0$, then $M \ge 0$, so either $M > 0$ or $M = 0$.
We consider these cases separately.
Case 1: Suppose $M > 0$.
Let $c \in [a, b]$.

To prove F is continuous, we must prove F is continuous at c.
Let $\epsilon > 0$ be given.
Let $\delta = \frac{\epsilon}{M}$.
Since $\epsilon > 0$ and $M > 0$, then $\delta > 0$.
Let $x \in [a, b]$ such that $|x - c| < \delta$.
Then $|x - c| < \frac{\epsilon}{M}$.

Let $A, B \in \mathbb{R}$ such that $[A, B] \subset [a, b]$.
Since $|f(x)| \leq M$ for all $x \in [a, b]$, then $-M \leq f(x) \leq M$ for all $x \in [a, b]$.
Since $[A, B] \subset [a, b]$, then $-M \leq f(x) \leq M$ for all $x \in [A, B]$.
Thus, $\int_A^B (-M) \leq \int_A^B f \leq \int_A^B M$, so $(-M)(B - A) \leq \int_A^B f \leq M(B - A)$.
Hence, $|\int_A^B f| \leq M(B - A)$, so $|\int_A^B f| \leq M(B - A)$ for all $A, B \in \mathbb{R}$ such that $[A, B] \subset [a, b]$.

We prove $[c, x] \subset [a, b]$.
Let $t \in [c, x]$.
Then $c \leq t \leq x$, so $c \leq t$ and $t \leq x$.
Since $c \in [a, b]$, then $a \leq c \leq b$, so $a \leq c$.
Since $x \in [a, b]$, then $a \leq x \leq b$, so $x \leq b$.
Thus, $a \leq c \leq t \leq x \leq b$, so $a \leq t \leq b$.
Hence, $t \in [a, b]$, so $[c, x] \subset [a, b]$.

Since $[c, x] \subset [a, b]$, then $|\int_c^x f| \leq M(x - c)$.
Since $|F(x) - F(c)| = |\int_a^x f - \int_a^c f| = |\int_c^x f| \leq M(x - c) \leq M|x - c| < M(\frac{\epsilon}{M}) = \epsilon$, then $|F(x) - F(c)| < \epsilon$, so F is continuous at c, as desired.
Case 2: Suppose $M = 0$.
Then $|f(x)| \leq 0$ for all $x \in [a, b]$.
Let $x \in [a, b]$.
Then $|f(x)| \leq 0$.
Since $|f(x)| \geq 0$ and $|f(x)| \leq 0$, then $|f(x)| = 0$, so $f(x) = 0$.
Thus, $f(x) = 0$ for all $x \in [a, b]$, so f is the constant zero function.
Observe that $F(x) = \int_a^x f = \int_a^x 0 = 0(x - a) = 0$, so $F(x) = 0$.
Since x is arbitrary, then $F(x) = 0$ for all $x \in [a, b]$, so F is a constant function.
Since every constant function is continuous, then F is continuous. ☐

Theorem 5. ***Fundamental Theorem of Calculus (derivative of an integral***

Let $f : [a, b] \to \mathbb{R}$ be an integrable function.
Let $F : [a, b] \to \mathbb{R}$ be defined by $F(x) = \int_a^x f$ for all $x \in [a, b]$.
If f is continuous at x, then F is differentiable at x and $F'(x) = f(x)$.

Proof. Let $c \in [a, b]$.
Suppose f is continuous at c.
We must prove F is differentiable at c and $F'(c) = f(c)$.

Let $\epsilon > 0$ be given.

Then $\frac{\epsilon}{2} > 0$.

Since f is continuous at c, then there exists $\delta > 0$ such that for all $x \in [a,b]$, if $|x-c| < \delta$, then $|f(x) - f(c)| < \frac{\epsilon}{2}$.

Let $x \in [a,b]$ such that $0 < |x-c| < \delta$.

Then $0 < |x-c|$ and $|x-c| < \delta$.

Since $x \in [a,b]$ and $|x-c| < \delta$, then $|f(x) - f(c)| < \frac{\epsilon}{2}$.

Thus, $-\frac{\epsilon}{2} < f(x) - f(c) < \frac{\epsilon}{2}$, so $f(c) - \frac{\epsilon}{2} < f(x) < f(c) + \frac{\epsilon}{2}$.

Hence, $f(c) - \frac{\epsilon}{2} \leq f(x) \leq f(c) + \frac{\epsilon}{2}$.

Since $f : [a,b] \to \mathbb{R}$ is integrable, then $f \in \mathcal{R}[a,b]$, so $\int_c^x [f(c) - \frac{\epsilon}{2}] \leq \int_c^x f \leq \int_c^x [f(c) + \frac{\epsilon}{2}]$.

Thus, $[f(c) - \frac{\epsilon}{2}](x-c) \leq \int_c^x f \leq [f(c) + \frac{\epsilon}{2}](x-c)$.

Since $|x-c| > 0$, then $x - c \neq 0$, so $f(c) - \frac{\epsilon}{2} \leq \frac{\int_c^x f}{x-c} \leq f(c) + \frac{\epsilon}{2}$.

Hence, $-\frac{\epsilon}{2} \leq \frac{\int_c^x f}{x-c} - f(c) \leq \frac{\epsilon}{2}$, so $|\frac{\int_c^x f}{x-c} - f(c)| \leq \frac{\epsilon}{2} < \epsilon$.

Therefore, $|\frac{\int_c^x f}{x-c} - f(c)| < \epsilon$, so $\lim_{x \to c} \frac{\int_c^x f}{x-c} = f(c)$.

Observe that

$$\begin{aligned} f(c) &= \lim_{x \to c} \frac{\int_c^x f}{x-c} \\ &= \lim_{x \to c} \frac{\int_a^x f - \int_a^c f}{x-c} \\ &= \lim_{x \to c} \frac{F(x) - F(c)}{x-c} \\ &= F'(c). \end{aligned}$$

Therefore, $F'(c) = f(c)$, so F is differentiable at c. $\square$

Theorem 6. ***Fundamental Theorem of Calculus (integral of a derivative)***

Let $F : [a,b] \to \mathbb{R}$ be a differentiable function.

If F' is continuous, then F' is integrable and $\int_a^b F' = F(b) - F(a)$.

Proof. Suppose F' is continuous.

Then $F' : [a,b] \to \mathbb{R}$ is continuous, so $F' \in \mathcal{R}[a,b]$.

Hence, F' is integrable and $\int_a^b F'$ exists.

We must prove $\int_a^b F' = F(b) - F(a)$.

Since F' is the derivative of F, then F is an antiderivative of F'.

Since $F' : [a,b] \to \mathbb{R}$ is integrable, let $\mathcal{F} : [a,b] \to \mathbb{R}$ be the function defined by $\mathcal{F}(x) = \int_a^x F'$ for all $x \in [a,b]$.

Since F' is continuous, then by FTC(derivative of an integral), $\mathcal{F}$ is an antiderivative of F'.

Since $\mathcal{F}$ is an antiderivative of F' and F is an antiderivative of F', then there exists a constant C such that $F(x) = \mathcal{F}(x) + C$ for all $x \in [a,b]$.

Observe that

$$\begin{aligned} F(b) - F(a) &= (\mathcal{F}(b) + C) - (\mathcal{F}(a) + C) \\ &= \mathcal{F}(b) - \mathcal{F}(a) \\ &= \int_a^b F' - \int_a^a F' \\ &= \int_a^b F' - 0 \\ &= \int_a^b F'. \end{aligned}$$

Therefore, $\int_a^b F' = F(b) - F(a)$, as desired. □

Darboux Integral of a real valued function

Lemma 7. ***The existence of the supremum and infimum of the direct image of each subinterval of a partition for a bounded function.***

Let $f : [a, b] \to \mathbb{R}$ be a bounded function.

Let n be a fixed positive integer.

Let $P = \{a, x_1, ..., x_{n-1}, b\}$ be a partition of $[a, b]$.

For each $k = 1, 2, ..., n$ the supremum and infimum of the set $\{f(x) : x \in [x_{k-1}, x_k]\}$ exist.

Proof. Let $k \in \{1, 2, ..., n\}$.

Let $I_k = [x_{k-1}, x_k]$.

The direct image of the subinterval I_k under f is the set $f(I_k) = \{f(x) : x \in I_k\}$.

The direct image of f is the set $f([a, b]) = \{f(x) : x \in [a, b]\}$.

Since P is a partition of $[a, b]$, then $x_{k-1} < x_k$ and $I_k \subset [a, b]$.

Since $x_{k-1} < x_k$, then the interval I_k is not empty.

Let $x \in I_k$.

Then $f(x)$ exists, so $f(x) \in f(I_k)$.

Hence, the set $f(I_k)$ is not empty.

Let $y \in f(I_k)$.

Then there exists $x \in I_k$ such that $y = f(x)$.

Since $x \in I_k$ and $I_k \subset [a, b]$, then $x \in [a, b]$.

Thus, $f(x) = y \in f([a, b])$, so $f(I_k) \subset f([a, b])$.

Since f is a bounded function, then the set $f([a, b])$ is bounded.

Since $f([a, b])$ is bounded and $f(I_k) \subset f([a, b])$, then $f(I_k)$ is bounded, so $f(I_k)$ is bounded above and below in $\mathbb{R}$.

Since $f(I_k)$ is not empty and bounded above in $\mathbb{R}$, then by completeness of $\mathbb{R}$, $\sup f(I_k)$ exists.

Since $f(I_k)$ is not empty and bounded below in $\mathbb{R}$, then by completeness of $\mathbb{R}$, $\inf f(I_k)$ exists.

Since k is arbitrary, then the supremum and infimum of $f(I_k)$ exist for each $k = 1, 2, ..., n$. □

Lemma 8. ***A lower Riemann sum is smaller than an upper Riemann sum for a given partition.***

Let $f : [a, b] \to \mathbb{R}$ be a bounded function.

Let P be a partition of $[a, b]$.

Let $U(f, P)$ be an upper Riemann sum.

Let $L(f, P)$ be a lower Riemann sum.

Then $L(f, P) \leq U(f, P)$.

Proof. Since P is a partition of $[a, b]$, then $P = \{a, x_1, ..., x_{n-1}, b\}$ for a fixed positive integer n.

Let $I_k = [x_{k-1}, x_k]$ and $\Delta_k = x_k - x_{k-1}$ for each $k = 1, 2, ..., n$.

Let $k \in \{1, 2, ..., n\}$.

Since f is a bounded function and P is a partition of $[a, b]$, then by the previous lemma, the supremum and infimum of the set $\{f(x) : x \in I_k\}$ exist, so $\sup f(I_k)$ and $\inf f(I_k)$ exist.

Thus, $\inf f(I_k) \leq \sup f(I_k)$.

Since P is a partition of $[a, b]$, then $x_{k-1} < x_k$, so $\Delta_k = x_k - x_{k-1} > 0$.

Thus, $\inf f(I_k)\Delta_k \leq \sup f(I_k)\Delta_k$.

Since k is arbitrary, then $\inf f(I_k)\Delta_k \leq \sup f(I_k)\Delta_k$ for each $k = 1, 2, ..., n$.

Therefore,

$$\begin{aligned} L(f, P) &= \sum_{k=1}^{n} \inf f(I_k)\Delta_k \\ &\leq \sum_{k=1}^{n} \sup f(I_k)\Delta_k \\ &= U(f, P). \end{aligned}$$

□

Lemma 9. ***Refining a partition increases lower Riemann sums and decreases upper Riemann sums.***

Let $f : [a, b] \to \mathbb{R}$ be a bounded function.

If P is a partition of $[a, b]$ and Q is a refinement of P, then $L(f, P) \leq L(f, Q)$ and $U(f, P) \geq U(f, Q)$.

Proof. Suppose P is a partition of $[a, b]$ and Q is a refinement of P.

Since Q is a refinement of P, then $P \subset Q$.

We first prove $L(f, P) \leq L(f, Q)$.

For simplicity, assume $P = (a, x_1, b)$ and $Q = (a, x_1, x_2, b)$.

We need to re-work this proof. We will do this later.

Then Q is a refinement of P.

We have $\Delta_{p1} = x_1 - a = \Delta_{q1}$ and $\Delta_{q1} = x_1 - a$ and $\Delta_{q2} = x_2 - x_1$ and $\Delta_{q3} = b - x_2$.

Since $\Delta_{p1} = x_1 - a = \Delta_{q1}$, then $\Delta_{p1} = \Delta_{q1}$.

Since $\Delta_{q2}+\Delta_{q3} = (x_2-x_1)+(b-x_2) = b-x_1 = \Delta_{p2}$, then $\Delta_{q2}+\Delta_{q3} = \Delta_{p2}$.

Since $[x_1, x_2] \subset [x_1, b]$, then $f([x_1, x_2]) \subset f([x_1, b])$.

Since f is bounded in $\mathbb{R}$, then $\inf f([x_1, x_2])$ exists and $\inf f([x_1, b])$ exists, so by the comparison property of infima, $\inf f([x_1, b]) \leq \inf f([x_1, x_2])$.

Since $\Delta_{q2} > 0$, then this implies $\inf f([x_1, b])\Delta_{q2} \leq \inf f([x_1, x_2])\Delta_{q2}$.

Similarly, since $[x_2, b] \subset [x_1, b]$, then $f([x_2, b]) \subset f([x_1, b])$.

Since f is bounded in $\mathbb{R}$, then $\inf f([x_2, b])$ exists and $\inf f([x_1, b])$ exists, so by the comparison property of infima, $\inf f([x_1, b]) \leq \inf f([x_2, b])$.

Since $\Delta_{q3} > 0$, then this implies $\inf f([x_1, b])\Delta_{q3} \leq \inf f([x_2, b])\Delta_{q3}$.

Observe that

$$
\begin{aligned}
L(f, P) &= \sum_{k=1}^{n} \inf f(I_k)\Delta_{pk} \\
&= \inf f([a, x_1])\Delta_{p1} + \inf f([x_1, b])\Delta_{p2} \\
&= \inf f([a, x_1])\Delta_{p1} + \inf f([x_1, b])(\Delta_{q2} + \Delta_{q3}) \\
&= \inf f([a, x_1])\Delta_{p1} + \inf f([x_1, b])\Delta_{q2} + \inf f([x_1, b])\Delta_{q3} \\
&= \inf f([a, x_1])\Delta_{q1} + \inf f([x_1, b])\Delta_{q2} + \inf f([x_1, b])\Delta_{q3} \\
&\leq \inf f([a, x_1])\Delta_{q1} + \inf f([x_1, x_2])\Delta_{q2} + \inf f([x_2, b])\Delta_{q3} \\
&= \sum_{k=1}^{n} \inf f(I_k)\Delta_{qk} \\
&= L(f, Q).
\end{aligned}
$$

Therefore, $L(f, P) \leq L(f, Q)$. □

Proof. We next prove $U(f, P) \geq U(f, Q)$.

For simplicity, assume $P = (a, x_1, b)$ and $Q = (a, x_1, x_2, b)$.

Then Q is a refinement of P.

We have $\Delta_{p1} = x_1 - a = \Delta_{q1}$ and $\Delta_{q1} = x_1 - a$ and $\Delta_{q2} = x_2 - x_1$ and $\Delta_{q3} = b - x_2$.

Since $\Delta_{p1} = x_1 - a = \Delta_{q1}$, then $\Delta_{p1} = \Delta_{q1}$.

Since $\Delta_{q2}+\Delta_{q3} = (x_2-x_1)+(b-x_2) = b-x_1 = \Delta_{p2}$, then $\Delta_{q2}+\Delta_{q3} = \Delta_{p2}$.

Since $[x_1, x_2] \subset [x_1, b]$, then $f([x_1, x_2]) \subset f([x_1, b])$.

Since f is bounded in $\mathbb{R}$, then $\sup f([x_1, x_2])$ exists and $\sup f([x_1, b])$ exists, so by the comparison property of suprema, $\sup f([x_1, x_2]) \leq \sup f([x_1, b])$.

Since $\Delta_{q2} > 0$, then this implies $\sup f([x_1, x_2])\Delta_{q2} \leq \sup f([x_1, b])\Delta_{q2}$.

Similarly, since $[x_2, b] \subset [x_1, b]$, then $f([x_2, b]) \subset f([x_1, b])$.

Since f is bounded in $\mathbb{R}$, then $\sup f([x_2, b])$ exists and $\sup f([x_1, b])$ exists, so by the comparison property of suprema, $\sup f([x_2, b]) \leq \sup f([x_1, b])$.

Since $\Delta_{q3} > 0$, then this implies $\sup f([x_2, b])\Delta_{q3} \leq \sup f([x_1, b])\Delta_{q3}$.

Observe that

$$\begin{aligned}
U(f, P) &= \sum_{k=1}^{n} \sup f(I_k)\Delta_{pk} \\
&= \sup f([a, x_1])\Delta_{p1} + \sup f([x_1, b])\Delta_{p2} \\
&= \sup f([a, x_1])\Delta_{p1} + \sup f([x_1, b])(\Delta_{q2} + \Delta_{q3}) \\
&= \sup f([a, x_1])\Delta_{p1} + \sup f([x_1, b])\Delta_{q2} + \sup f([x_1, b])\Delta_{q3} \\
&= \sup f([a, x_1])\Delta_{q1} + \sup f([x_1, b])\Delta_{q2} + \sup f([x_1, b])\Delta_{q3} \\
&\geq \sup f([a, x_1])\Delta_{q1} + \sup f([x_1, x_2])\Delta_{q2} + \sup f([x_2, b])\Delta_{q3} \\
&= \sum_{k=1}^{n} \sup f(I_k)\Delta_{qk} \\
&= U(f, Q).
\end{aligned}$$

Therefore, $U(f, P) \geq U(f, Q)$. □

Proposition 10. ***Any lower Riemann sum is smaller than any upper Riemann sum.***

Let $f : [a, b] \to \mathbb{R}$ be a bounded function.

If P and Q are partitions of $[a, b]$, then $L(f, P) \leq U(f, Q)$.

Proof. Suppose P and Q are partitions of $[a, b]$.

Since $P \subset P \cup Q$, then $P \cup Q$ is a refinement of P.

Since refining a partition increases lower sums, then $L(f, P) \leq L(f, P \cup Q)$.

Since a lower sum is smaller than an upper sum and $P \cup Q$ is a partition, then $L(f, P \cup Q) \leq U(f, P \cup Q)$.

Since $Q \subset P \cup Q$, then $P \cup Q$ is a refinement of Q.

Since refining a partition decreases upper sums, then $U(f, Q) \geq U(f, P \cup Q)$.

Therefore, $L(f, P) \leq L(f, P \cup Q) \leq U(f, P \cup Q) \leq U(f, Q)$, so $L(f, P) \leq U(f, Q)$. □

Lemma 11. *For every $n \in \mathbb{Z}^+$, $\sum_{i=1}^{n}(x_i - x_{i-1}) = x_n - x_0$.*

Proof. We prove by induction on n.

Let $S = \{n \in \mathbb{Z}^+ : \sum_{i=1}^{n}(x_i - x_{i-1}) = x_n - x_0\}$.

Since $1 \in \mathbb{Z}^+$ and $\sum_{i=1}^{1}(x_i - x_{i-1}) = x_1 - x_0$, then $1 \in S$.

Suppose $k \in S$.

Then $k \in \mathbb{Z}^+$ and $\sum_{i=1}^{k}(x_i - x_{i-1}) = x_k - x_0$.

Observe that

$$\begin{aligned}\sum_{i=1}^{k+1}(x_i - x_{i-1}) &= \sum_{i=1}^{k}(x_i - x_{i-1}) + (x_{k+1} - x_k) \\ &= (x_k - x_0) + (x_{k+1} - x_k) \\ &= x_{k+1} - x_0.\end{aligned}$$

Since $k \in \mathbb{Z}^+$, then $k+1 \in \mathbb{Z}^+$.

Since $k+1 \in \mathbb{Z}^+$ and $\sum_{i=1}^{k+1}(x_i - x_{i-1}) = x_{k+1} - x_0$, then $k+1 \in S$.

Thus, $k \in S$ implies $k+1 \in S$.

By the principle of mathematical induction, $S = \mathbb{Z}^+$, so $\sum_{i=1}^{n}(x_i - x_{i-1}) = x_n - x_0$ for all $n \in \mathbb{Z}^+$, as desired. □

Proposition 12. *Let $f : [a, b] \to \mathbb{R}$ be a bounded function.*

Let $M = \sup f([a, b])$.

Let $m = \inf f([a, b])$.

If P is a partition of $[a, b]$, then $m(b - a) \leq L(f, P) \leq U(f, P) \leq M(b - a)$.

Proof. Since f is a bounded function and the set $\{a, b\}$ is a partition of the interval $[a, b]$, then the suprema and infima of the set $f([a, b])$ exist.

Let $M = \sup f([a, b])$.

Let $m = \inf f([a, b])$.

Suppose P is a partition of $[a, b]$.

Then there exists a positive integer n such that $P = \{a, x_1, ..., x_{n-1}, b\}$ is a finite set of points and $x_{k-1} < x_k$ for each $k = 1, 2, ..., n$ and $x_0 = a$ and $x_n = b$.

Since f is a bounded function and P is a partition of $[a, b]$, then the lower Riemann sum $L(f, P)$ and the upper Riemann sum $U(f, P)$ exist.

Let $k \in \{1, 2, ..., n\}$ be arbitrary.

Since P is a partition of $[a, b]$, then $I_k = [x_{k-1}, x_k]$ is a subset of the interval $[a, b]$.

Since $f : [a, b] \to \mathbb{R}$ is a map and $I_k \subset [a, b]$, then $f(I_k) \subset f([a, b])$.

Since f is a bounded function and P is a partition of $[a, b]$, then $\sup f(I_k)$ and $\inf f(I_k)$ exist.

Since $f(I_k) \subset f([a, b])$ and $\inf f(I_k)$ and $\inf f([a, b])$ exist, then by the comparison property of infima, $\inf f([a, b]) \leq \inf f(I_k)$, so $m \leq \inf f(I_k)$.

Since $f(I_k) \subset f([a, b])$ and $\sup f(I_k)$ and $\sup f([a, b])$ exist, then by the comparison property of suprema, $\sup f(I_k) \leq \sup f([a, b])$, so $\sup f(I_k) \leq M$.

Since $x_{k-1} < x_k$, then $\Delta_k = x_k - x_{k-1} > 0$.

Since $m \leq \inf f(I_k)$ and $\Delta_k > 0$, then $m\Delta_k \leq \inf f(I_k)\Delta_k$.

Since $\sup f(I_k) \leq M$ and $\Delta_k > 0$, then $\sup f(I_k)\Delta_k \leq M\Delta_k$.

Since k is arbitrary, then $m\Delta_k \leq \inf f(I_k)\Delta_k$ and $\sup f(I_k)\Delta_k \leq M\Delta_k$ for each $k \in \{1, 2, ..., n\}$.

Observe that

$$\begin{aligned} m(b-a) &= m(x_n - x_0) \\ &= m\sum_{k=1}^{n}(x_k - x_{k-1}) \\ &= m\sum_{k=1}^{n}\Delta_k \\ &= \sum_{k=1}^{n} m\Delta_k \\ &\leq \sum_{k=1}^{n}\inf f(I_k)\Delta_k \\ &= L(f,P). \end{aligned}$$

Thus, $m(b-a) \leq L(f,P)$.

Observe that

$$\begin{aligned} U(f,P) &= \sum_{k=1}^{n}\sup f(I_k)\Delta_k \\ &\leq \sum_{k=1}^{n} M\Delta_k \\ &= M\sum_{k=1}^{n}\Delta_k \\ &= M\sum_{k=1}^{n}(x_k - x_{k-1}) \\ &= M(x_n - x_0) \\ &= M(b-a). \end{aligned}$$

Thus, $U(f,P) \leq M(b-a)$.

Since a lower Riemann sum is smaller than an upper Riemann sum and P is a partition, then $L(f,P) \leq U(f,P)$.

Therefore, $m(b-a) \leq L(f,P) \leq U(f,P) \leq M(b-a)$. □

Theorem 13. ***The upper and lower Darboux integrals exist for a bounded function.***

Let $f : [a,b] \to \mathbb{R}$ be a bounded function.

Then the lower integral $\underline{\int_a^b} f$ and upper integral $\overline{\int_a^b} f$ exist and $\underline{\int_a^b} f \leq \overline{\int_a^b} f$.

Proof. Let S be the set of all lower Riemann sums of all the partitions of $[a,b]$.

Then $S = \{L(f,P) : P \text{ is a partition of } [a,b]\}$.

Let T be the set of all upper Riemann sums of all the partitions of $[a,b]$.

Then $T = \{U(f,P) : P \text{ is a partition of } [a,b]\}$.

Let $P_0 = \{a,b\}$ be a partition of the interval $[a,b]$.

Since f is bounded, then the upper Riemann sum $U(f,P_0)$ exists and the lower Riemann sum $L(f,P_0)$ exists, so $U(f,P_0) \in T$ and $L(f,P_0) \in S$.

Hence, $T \neq \emptyset$ and $S \neq \emptyset$.

Let $x \in S$.

Then there exists a partition P_1 of $[a,b]$ such that $x = L(f,P_1)$.

Since any lower Riemann sum is smaller than any upper Riemann sum, then $x = L(f,P_1) \leq U(f,P_0)$.

Since x is arbitrary, then $x \leq U(f,P_0)$ for every $x \in S$.

Hence, $U(f,P_0)$ is an upper bound of S, so S is bounded above in $\mathbb{R}$.

Since $S \neq \emptyset$ and bounded above in $\mathbb{R}$, then by completeness of $\mathbb{R}$, $\sup S$ exists.

Therefore, $\underline{\int_a^b} f = \sup\{L(f,P) : P \text{ is a partition of } [a,b]\} = \sup S$ exists.

Let $y \in T$.

Then there exists a partition P_2 of $[a,b]$ such that $y = U(f,P_2)$.

Since any lower Riemann sum is smaller than any upper Riemann sum, then $L(f,P_0) \leq U(f,P_2) = y$.

Since y is arbitrary, then $L(f,P_0) \leq y$ for every $y \in T$.

Hence, $L(f,P_0)$ is a lower bound of T, so T is bounded below in $\mathbb{R}$.

Since $T \neq \emptyset$ and bounded below in $\mathbb{R}$, then by completeness of $\mathbb{R}$, $\inf T$ exists.

Therefore, $\overline{\int_a^b} f = \inf\{U(f,P) : P \text{ is a partition of } [a,b]\} = \inf T$ exists.

We next prove $\sup S \leq \inf T$.

Let $t \in T$.

Then $t = U(f,P)$ for some partition P of $[a,b]$.

Let $s \in S$.

Then $s = L(f,Q)$ for some partition Q of $[a,b]$.

Since any lower Riemann sum is smaller than any upper Riemann sum, then $s \leq t$.

Since s is arbitrary, then $s \leq t$ for all $s \in S$, so t is an upper bound of S.

Since $\sup S$ is the least upper bound of S, then $\sup S \leq t$.

Since t is arbitrary, then $\sup S \leq t$ for all $t \in T$.

Hence, $\sup S$ is a lower bound of T.

Since $\inf T$ is the greatest lower bound of T, then $\sup S \leq \inf T$.

Therefore, $\underline{\int_a^b} f \leq \overline{\int_a^b} f$. $\square$

Proposition 14. *Let $f : [a,b] \to \mathbb{R}$ be a bounded function.*

Let $M = \sup f([a,b])$.

Let $m = \inf f([a,b])$.

Then $m(b-a) \leq \underline{\int_a^b} f$ and $\overline{\int_a^b} f \leq M(b-a)$.

Proof. Since f is a bounded function and the set $\{a, b\}$ is a partition of the interval $[a, b]$, then the suprema and infima of the set $f([a, b])$ exist.

Let $M = \sup f([a, b])$.

Let $m = \inf f([a, b])$.

Since f is a bounded function, then the upper integral $\overline{\int_a^b} f$ and the lower integral $\underline{\int_a^b} f$ exist.

Let $S = \{L(f, P) : P \text{ is a partition of } [a, b]\}$.

Let $T = \{U(f, P) : P \text{ is a partition of } [a, b]\}$.

Then $\underline{\int_a^b} f = \sup S$ and $\overline{\int_a^b} f = \inf T$.

Let P be a partition of $[a, b]$.

Then $L(f, P) \in S$ and $U(f, P) \in T$.

Since $\sup S$ is an upper bound of S and $L(f, P) \in S$, then $L(f, P) \leq \sup S$.

Since f is a bounded function and $m = \inf f([a, b])$ and P is a partition of $[a, b]$, then $m(b - a) \leq L(f, P)$.

Since $\inf T$ is a lower bound of T and $U(f, P) \in T$, then $\inf T \leq U(f, P)$.

Since f is a bounded function and $M = \sup f([a, b])$ and P is a partition of $[a, b]$, then $U(f, P) \leq M(b - a)$.

Since $m(b - a) \leq L(f, P)$ and $L(f, P) \leq \sup S$, then $m(b - a) \leq \sup S$, so $m(b - a) \leq \underline{\int_a^b} f$.

Since $\inf T \leq U(f, P)$ and $U(f, P) \leq M(b - a)$, then $\inf T \leq M(b - a)$, so $\overline{\int_a^b} f \leq M(b - a)$. □

Theorem 15. ***Darboux integrability criterion***

Let $f : [a, b] \to \mathbb{R}$ be a bounded function.

Then f is Darboux integrable on $[a, b]$ iff for every $\epsilon > 0$ there exists a partition P of $[a, b]$ such that $U(f, P) - L(f, P) < \epsilon$.

Proof. □

Riemann Integral of a real valued function

Theorem 16. ***Integral of a Riemann integrable function is unique.***

If $f : [a, b] \to \mathbb{R}$ is a Riemann integrable function, then the value of the integral is unique.

Proof. Let $f : [a, b] \to \mathbb{R}$ be a Riemann integrable function.

Then there exists $L \in \mathbb{R}$ such that for every $\epsilon > 0$, there exists $\delta > 0$ such that if $\dot{P}$ is any tagged partition of $[a, b]$ with $||\dot{P}|| < \delta$, then $|S(f; \dot{P}) - L| < \epsilon$.

To prove L is unique, let $L_1, L_2 \in \mathbb{R}$ such that L_1 and L_2 satisfy the definition of a Riemann integrable function.

We must prove $L_1 = L_2$.

Let $\epsilon > 0$ be given.

Then $\frac{\epsilon}{2} > 0$.

Since L_1 satisfies the definition, then there exists $\delta_1 > 0$ such that if $\dot{P}$ is any tagged partition of $[a, b]$ with $||\dot{P}|| < \delta_1$, then $|S(f; \dot{P}) - L_1| < \frac{\epsilon}{2}$.

Since L_2 satisfies the definition, then there exists $\delta_2 > 0$ such that if $\dot{P}$ is any tagged partition of $[a, b]$ with $||\dot{P}|| < \delta_2$, then $|S(f; \dot{P}) - L_2| < \frac{\epsilon}{2}$.

Let $\delta = \min\{\delta_1, \delta_2\}$.

Then $\delta \leq \delta_1$ and $\delta \leq \delta_2$.

Since $\delta_1 > 0$ and $\delta_2 > 0$, then $\delta > 0$.

Let $\dot{P}$ be an arbitrary tagged partition of $[a, b]$ with $||\dot{P}|| < \delta$.

Since $||\dot{P}|| < \delta$ and $\delta \leq \delta_1$, then $||\dot{P}|| < \delta_1$, so $|S(f; \dot{P}) - L_1| < \frac{\epsilon}{2}$.

Since $||\dot{P}|| < \delta$ and $\delta \leq \delta_2$, then $||\dot{P}|| < \delta_2$, so $|S(f; \dot{P}) - L_2| < \frac{\epsilon}{2}$.

Observe that

$$\begin{aligned}
|L_1 - L_2| &= |L_1 - S(f; \dot{P}) + S(f; \dot{P}) - L_2| \\
&\leq |L_1 - S(f; \dot{P})| + |S(f; \dot{P}) - L_2| \\
&= |S(f; \dot{P}) - L_1| + |S(f; \dot{P}) - L_2| \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&= \epsilon.
\end{aligned}$$

Thus, $|L_1 - L_2| < \epsilon$.

Since ϵ is arbitrary, then $|L_1 - L_2| < \epsilon$ for every $\epsilon > 0$.

Therefore, by a previous lemma, $L_1 - L_2 = 0$, so $L_1 = L_2$, as desired. $\square$