Integration of real valued functions Examples

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Darboux Integral of a real valued function

Example 1. The constant function is Darboux integrable.
Let $k \in \mathbb{R}$ be fixed.
Then $\int_a^b k = k(b-a)$.

Proof. Let $f : [a,b] \to \mathbb{R}$ be the function defined by $f(x) = k$.
Let $x \in [a,b]$.
Then $|f(x)| = |k|$, so $|f(x)| = |k|$ for all $x \in [a,b]$.
Hence, f is a bounded function, so the upper and lower Darboux integrals exist.
Let $P = \{a, x_1, ..., x_{n-1}, b\}$ be a partition of $[a,b]$.
Since f is a bounded function, then the upper Riemann sum is $U(f,P) = \sum_{i=1}^n \sup f(I_i)\Delta_i$ and the lower Riemann sum is $L(f,P) = \sum_{i=1}^n \inf f(I_i)\Delta_i$ where $I_i = [x_{i-1}, x_i]$ and $\Delta_i = x_i - x_{i-1}$ for each $i = 1, 2, ..., n$.
Let $i \in \{1, 2, ..., n\}$.
Then $f(I_i) = f([x_{i-1}, x_i]) = \{k\}$, so $\sup f(I_i) = \sup\{k\} = k$ and $\inf f(I_i) = \inf\{k\} = k$.
Thus, $\sup f(I_i) = k$ and $\inf f(I_i) = k$ for each $i = 1, 2, ..., n$.
Observe that

$$\begin{aligned} U(f,P) &= \sum_{i=1}^n \sup f(I_i)\Delta_i \\ &= \sum_{i=1}^n k\Delta_i \\ &= k\sum_{i=1}^n \Delta_i \\ &= k\sum_{i=1}^n (x_i - x_{i-1}) \\ &= k(x_n - x_0) \\ &= k(b-a). \end{aligned}$$

Therefore, $U(f,P) = k(b-a)$.

Since P is an arbitrary partition, then $U(f,P) = k(b-a)$ for every partition of $[a,b]$.

Let $S = \{U(f,P) : P \text{ is a partition of } [a,b]\}$.

Then $S = \{k(b-a)\}$.

The upper Darboux integral is

$$\begin{aligned}
\overline{\int_a^b} f &= \inf\{U(f,P) : P \text{ is a partition of } [a,b]\} \\
&= \inf S \\
&= \inf\{k(b-a)\} \\
&= k(b-a).
\end{aligned}$$

Observe that

$$\begin{aligned}
L(f,P) &= \sum_{i=1}^{n} \inf f(I_i)\Delta_i \\
&= \sum_{i=1}^{n} k\Delta_i \\
&= k\sum_{i=1}^{n} \Delta_i \\
&= k\sum_{i=1}^{n} (x_i - x_{i-1}) \\
&= k(x_n - x_0) \\
&= k(b-a).
\end{aligned}$$

Therefore, $L(f,P) = k(b-a)$.

Since P is an arbitrary partition, then $L(f,P) = k(b-a)$ for every partition of $[a,b]$.

Let $T = \{L(f,P) : P \text{ is a partition of } [a,b]\}$.

Then $T = \{k(b-a)\}$.

The lower Darboux integral is

$$\begin{aligned}
\underline{\int_a^b} f &= \sup\{L(f,P) : P \text{ is a partition of } [a,b]\} \\
&= \sup T \\
&= \sup\{k(b-a)\} \\
&= k(b-a).
\end{aligned}$$

Since $\underline{\int_a^b} f = k(b-a) = \overline{\int_a^b} f$, then f is Darboux integrable on $[a,b]$, so the Darboux integral of f over $[a,b]$ is $\int_a^b k = k(b-a)$. $\square$

Example 2. Dirichlet function is not Darboux integrable

Let $f : [0, 1] \to \mathbb{R}$ be a function defined by

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$$

Then f is not integrable on $[0, 1]$.

Proof. Since the range of f is the finite set $f([0, 1]) = \{0, 1\}$, then f is a bounded function, so the upper integral $\overline{\int_0^1} f$ and lower integral $\underline{\int_0^1} f$ exist.

Let P be an arbitrary partition of $[0, 1]$.

Then there exists a positive integer n such that $P = \{0, x_1, ..., x_{n-1}, 1\}$ and $x_0 = 0$ and $x_n = 1$ and $x_{k-1} < x_k$ for $k = 1, 2, ..., n$.

Let $I_k = [x_{k-1}, x_k]$ and $\Delta_k = x_k - x_{k-1}$ for each $k = 1, 2, ..., n$.

Since I_k is a subinterval of the partition P, then $I_k \subset [0, 1]$ for each $k = 1, 2, ..., n$.

Let $k \in \{1, 2, ..., n\}$.

Then $I_k = [x_{k-1}, x_k]$ and $x_{k-1} < x_k$ and $I_k \subset [0, 1]$ and $\Delta_k = x_k - x_{k-1}$.

Since $\mathbb{Q}$ is dense in $\mathbb{R}$ and $x_{k-1} < x_k$, then there exists a rational s such that $x_{k-1} < s < x_k$.

Hence, $s \in [x_{k-1}, x_k]$, so $s \in I_k$.

Since $I_k \subset [0, 1]$, then $s \in [0, 1]$.

Since s is rational, then $f(s) = 1$.

Since $\mathbb{R} - \mathbb{Q}$ is dense in $\mathbb{R}$ and $x_{k-1} < x_k$, then there exists an irrational t such that $x_{k-1} < t < x_k$.

Hence, $t \in [x_{k-1}, x_k]$, so $t \in I_k$.

Since $I_k \subset [0, 1]$, then $t \in [0, 1]$.

Since t is irrational, then $f(t) = 0$.

Since $f(s) = 1$ and $f(t) = 0$, then $0 \in f(I_k)$ and $1 \in f(I_k)$, so $\{0, 1\} \subset f(I_k)$.

Since $I_k \subset [0, 1]$, then $f(I_k) \subset f([0, 1])$.

Since $f(I_k) \subset f([0, 1])$ and $f([0, 1]) = \{0, 1\}$, then $f(I_k) \subset \{0, 1\}$.

Since $f(I_k) \subset \{0, 1\}$ and $\{0, 1\} \subset f(I_k)$, then $f(I_k) = \{0, 1\}$, so $\sup f(I_k) = 1$ and $\inf f(I_k) = 0$.

Hence, $\inf f(I_k)\Delta_k = 0\Delta_k = 0$ and $\sup f(I_k)\Delta_k = 1 \cdot \Delta_k = \Delta_k$.

Since k is arbitrary, then $\inf f(I_k)\Delta_k = 0$ and $\sup f(I_k)\Delta_k = \Delta_k$ for each $k = 1, 2, ..., n$.

The upper Riemann sum is

$$\begin{aligned} U(f,P) &= \sum_{k=1}^{n} \sup f(I_k)\Delta_k \\ &= \sum_{k=1}^{n} \Delta_k \\ &= \sum_{k=1}^{n}(x_k - x_{k-1}) \\ &= x_n - x_0 \\ &= 1 - 0 \\ &= 1. \end{aligned}$$

The lower Riemann sum is

$$\begin{aligned} L(f,P) &= \sum_{k=1}^{n} \inf f(I_k)\Delta_k \\ &= \sum_{k=1}^{n} 0 \\ &= 0. \end{aligned}$$

Therefore, $U(f,P) = 1$ and $L(f,P) = 0$ for any partition P of $[0,1]$.
The upper Darboux integral is

$$\begin{aligned} \overline{\int_0^1} f &= \inf\{U(f,P) : P \text{ is a partition of } [a,b]\} \\ &= \inf\{1\} \\ &= 1. \end{aligned}$$

The lower Darboux integral is

$$\begin{aligned} \underline{\int_0^1} f &= \sup\{L(f,P) : P \text{ is a partition of } [a,b]\} \\ &= \inf\{0\} \\ &= 0. \end{aligned}$$

Since $\underline{\int_0^1} f = 0 < 1 = \overline{\int_0^1} f$, then f is not Darboux integrable on $[0,1]$. $\square$

Riemann Integral of a real valued function

Example 3. Let $k \in \mathbb{R}$ be fixed.
Then $\int_a^b kdx = k(b-a)$.

Proof. Let $f : [a, b] \to \mathbb{R}$ be the function defined by $f(x) = k$.

Let $\epsilon > 0$ be given.

Let $\delta = 1$.

Let $\dot{P} = \{([x_{i-1}, x_i], t_i) : i \in \mathbb{Z}^+, 1 \le i \le n\}$ be an arbitrary tagged partition of $[a, b]$ with $||\dot{P}|| < 1$.

Since $\dot{P}$ is a partition, then $x_0 = a$ and $x_n = b$.

Observe that

$$\begin{aligned}
|S(f; \dot{P}) - k(b-a)| &= |\sum_{i=1}^{n} f(t_i)(x_i - x_{i-1}) - k(b-a)| \\
&= |\sum_{i=1}^{n} k(x_i - x_{i-1}) - k(b-a)| \\
&= |k\sum_{i=1}^{n}(x_i - x_{i-1}) - k(b-a)| \\
&= |k(x_n - x_0) - k(b-a)| \\
&= |k(b-a) - k(b-a)| \\
&= 0 \\
&< \epsilon.
\end{aligned}$$

Therefore, $\int_a^b k dx = k(b-a)$. □