# Integration of real valued functions Exercises

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## Darboux Integral of a real valued function

**Exercise 1.** Let $f : [0, 3] \to \mathbb{R}$ be the function defined by

$$f(x) = \begin{cases} 2 & \text{if } 0 \leq x \leq 1 \\ 3 & \text{if } 1 < x \leq 3 \end{cases}$$

Then $\underline{\int_0^3} f = \overline{\int_0^3} f = \int_0^3 f = 8$.

*Proof.* We prove $f$ is a bounded function.
Let $x \in dom f$.
Then either $0 \leq x \leq 1$ or $1 < x \leq 3$.
We consider these cases separately.
**Case 1:** Suppose $0 \leq x \leq 1$.
Then $|f(x)| = |2| = 2 < 3$.
**Case 2:** Suppose $1 < x \leq 3$.
Then $|f(x)| = |3| = 3$.
Thus, in all cases, $|f(x)| \leq 3$.
Hence, $|f(x)| \leq 3$ for all $x \in dom f$, so $f$ is bounded.

Therefore, the upper Darboux integral $\overline{\int_0^3} f$ and the lower Darboux integral $\underline{\int_0^3} f$ exist and the upper Riemann sum $U(f, P)$ and lower Riemann sum $L(f, P)$ exist for every partition $P$ of the interval $[0, 3]$.
Let $S = \{L(f, P) : P \text{ is a partition of } [0, 3]\}$.
Let $T = \{U(f, P) : P \text{ is a partition of } [0, 3]\}$.
Then $\underline{\int_0^3} f = \sup S$ and $\overline{\int_0^3} f = \inf T$ and $\underline{\int_0^3} f \leq \overline{\int_0^3} f$.

□

*Proof.* Let $P_2 = \{0, 1, 3\}$ be a partition of $[0, 3]$.
The upper Riemann sum is

$$
\begin{aligned}
U(f,P_2) &= \sum_{k=1}^{2} \sup f(I_k)\Delta_k \\
&= \sup f(I_1)\Delta_1 + \sup f(I_2)\Delta_2 \\
&= \sup f([0,1])(1-0) + \sup f([1,3])(3-1) \\
&= \sup\{2\}(1) + \sup\{2,3\}(2) \\
&= 2(1) + 3(2) \\
&= 8.
\end{aligned}
$$

Since $P_2$ is a partition of $[0,3]$ and $U(f,P_2) = 8$, then $8 \in T$.

Since $\overline{\int_0^3} f$ is a lower bound of $T$ and $8 \in T$, then $\overline{\int_0^3} f \leq 8$. □

*Proof.* We prove for every $\epsilon$ if $0 < \epsilon < 2$, then $8 - \epsilon \in S$.

Let $\epsilon$ be an arbitrary real number such that $0 < \epsilon < 2$.

Since $0 < \epsilon < 2$, then $1 < 1 + \epsilon < 3$.

Since $0 < 1 < 1 + \epsilon < 3$, let $P_\epsilon = \{0, 1, 1 + \epsilon, 3\}$ be a partition of the interval $[0,3]$.

The lower Riemann sum is

$$
\begin{aligned}
L(f,P_\epsilon) &= \sum_{k=1}^{3} \inf f(I_k)\Delta_k \\
&= \inf f(I_1)\Delta_1 + \inf f(I_2)\Delta_2 + \inf f(I_3)\Delta_3 \\
&= \inf f([0,1])(1-0) + \inf f([1,1+\epsilon])[(1+\epsilon)-1] + \inf f([1+\epsilon,3])[3-(1+\epsilon)] \\
&= \inf\{2\}(1) + \inf\{2,3\}\epsilon + \inf\{3\}(2-\epsilon) \\
&= 2*1 + 2\epsilon + 3(2-\epsilon) \\
&= 8 - \epsilon.
\end{aligned}
$$

Since $P_\epsilon$ is a partition of $[0,3]$ and $L(f,P_\epsilon) = 8 - \epsilon$, then $8 - \epsilon \in S$.

Therefore, $8 - \epsilon \in S$ for every $\epsilon$ such that $0 < \epsilon < 2$. □

*Proof.* We prove $\sup S \geq 8$.

Suppose $\sup S < 8$.

Then $8 - \sup S > 0$, so $\frac{8-\sup S}{2} > 0$.

Let $\epsilon = \frac{8-\sup S}{2}$.

Then $\epsilon > 0$.

Since $\sup S \geq \inf f * (3-0) = 2 * (3-0) = 6 > 4$, then $\sup S > 4$, so $\sup S > 8 - 4$.

Hence, $4 > 8 - \sup S$, so $2 > \frac{8-\sup S}{2}$.

Thus, $2 > \epsilon$.

Since $0 < \epsilon$ and $\epsilon < 2$, then $0 < \epsilon < 2$.

Hence, $8 - \epsilon = 8 - \frac{8-\sup S}{2} = \frac{\sup S + 8}{2} \in S$.

Therefore, $\frac{\sup S+8}{2} \in S$.

Since $\sup S < 8$, then $2\sup S < \sup S + 8$, so $\sup S < \frac{\sup S+8}{2}$.

Thus, $\frac{\sup S+8}{2}$ is an element of $S$ that is greater than the least upper bound of $S$.

This contradicts the fact that $\sup S$ is an upper bound of $S$.

Hence, $\sup S \geq 8$, so $\underline{\int_0^3} f \geq 8$. $\square$

*Proof.* Since $8 \leq \underline{\int_0^3} f \leq \overline{\int_0^3} f \leq 8$, then $8 \leq \underline{\int_0^3} f$ and $\underline{\int_0^3} f \leq 8$ and $8 \leq \overline{\int_0^3} f$ and $\overline{\int_0^3} f \leq 8$.

Since $\underline{\int_0^3} f \leq 8$ and $8 \leq \underline{\int_0^3} f$, then $\underline{\int_0^3} f = 8$.

Since $\overline{\int_0^3} f \leq 8$ and $8 \leq \overline{\int_0^3} f$, then $\overline{\int_0^3} f = 8$.

Therefore, $f$ is Darboux integrable and the integral of $f$ over $[0,3]$ is $\int_0^3 f = \underline{\int_0^3} f = 8 = \overline{\int_0^3} f$. $\square$

**Exercise 2.** Let $f : [0,1] \to \mathbb{R}$ be the function defined by $f(x) = x$.

Then $\int_0^1 x dx = \underline{\int_0^1} x = \frac{1}{2} = \overline{\int_0^1} x$.

*Proof.* Let $x \in [0,1]$.

Then $0 \leq x \leq 1$, so $0 \leq x = f(x) \leq 1$.

Hence, $0 \leq f(x) \leq 1$, so $f$ is bounded.

Therefore, the upper Darboux integral $\overline{\int_0^1} x$ and the lower Darboux integral $\underline{\int_0^1} x$ exist and the upper Riemann sum $U(f,P)$ and lower Riemann sum $L(f,P)$ exist for every partition $P$ of the interval $[0,1]$.

Let $S = \{L(f,P) : P \text{ is a partition of } [0,1]\}$.

Let $T = \{U(f,P) : P \text{ is a partition of } [0,1]\}$.

Then $\underline{\int_0^1} x = \sup S$ and $\overline{\int_0^1} x = \inf T$ and $\underline{\int_0^1} x \leq \overline{\int_0^1} x$. $\square$

*Proof.* Let $P_n = \{0, \frac{1}{n}, \frac{2}{n}, ..., \frac{n-1}{n}, 1\}$ be a partition of $[0,1]$ for each $n \in \mathbb{Z}^+$.

For each $k = 1, 2, ..., n$ the $k^{th}$ subinterval is $I_k = [\frac{k-1}{n}, \frac{k}{n}]$ and its length is $\Delta_k = \frac{1}{n}$.

Hence, for each $k = 1, 2, ..., n$ the direct image of $I_k$ is $f(I_k) = f([\frac{k-1}{n}, \frac{k}{n}]) = [\frac{k-1}{n}, \frac{k}{n}]$.

Since $f$ is an increasing function, then $\sup f(I_k) = \frac{k}{n}$ and $\inf f(I_k) = \frac{k-1}{n}$ and $\Delta_k = \frac{1}{n}$ for each $k = 1, 2, ..., n$.

For every positive integer $n$ the lower Riemann sum is

$$\begin{aligned}
L(f, P_n) &= \sum_{k=1}^{n} \inf f(I_k)\Delta_k \\
&= \sum_{k=1}^{n} \frac{k-1}{n}\frac{1}{n} \\
&= \sum_{k=1}^{n} \frac{k-1}{n^2} \\
&= \frac{1}{n^2}\sum_{k=1}^{n}(k-1) \\
&= \frac{1}{n^2}[\sum_{k=1}^{n} k - \sum_{k=1}^{n} 1] \\
&= \frac{1}{n^2}[\frac{n(n+1)}{2} - n] \\
&= \frac{n-1}{2n}.
\end{aligned}$$

Therefore, $L(f, P_n) = \frac{n-1}{2n}$ for all $n \in \mathbb{Z}^+$.

Let $A = \{L(f, P_n) : n \in \mathbb{Z}^+\}$.

Then $A = \{\frac{n-1}{2n} : n \in \mathbb{Z}^+\}$, so $A$ is the range of the sequence $(a_n)$ defined by $a_n = \frac{n-1}{2n}$.

Let $x \in A$.

Then $x = L(f, P_n)$ for some positive integer $n$.

Since $P_n$ is a partition of $[0, 1]$, then $x \in S$.

Hence, $A \subset S$.

Let $n \in \mathbb{Z}^+$ be given.

Then $a_n = \frac{n-1}{2n} \in A$.

Since $A \subset S$, then $a_n \in S$.

Since $\sup S$ is an upper bound of $S$, then $a_n \leq \sup S$.

Since $n$ is arbitrary, then $a_n \leq \sup S$ for all $n \in \mathbb{Z}^+$, so $\sup S$ is an upper bound of $A$.

Since $\lim_{n\to\infty} \frac{n-1}{2n} = \frac{1}{2}$, then $(a_n)$ is a convergent sequence, so $\frac{1}{2} = \lim_{n\to\infty} \frac{n-1}{2n} \leq \sup S$.

Hence, $\frac{1}{2} \leq \underline{\int_0^1} x$. □

*Proof.* For every positive integer $n$ the upper Riemann sum is

$$\begin{aligned} U(f, P_n) &= \sum_{k=1}^{n} \sup f(I_k) \Delta_k \\ &= \sum_{k=1}^{n} \frac{k}{n} \frac{1}{n} \\ &= \sum_{k=1}^{n} \frac{k}{n^2} \\ &= \frac{1}{n^2} \sum_{k=1}^{n} k \\ &= \frac{1}{n^2} [\frac{n(n+1)}{2}] \\ &= \frac{n+1}{2n}. \end{aligned}$$

Therefore, $U(f, P_n) = \frac{n+1}{2n}$ for all $n \in \mathbb{Z}^+$.

Let $B = \{U(f, P_n) : n \in \mathbb{Z}^+\}$.

Then $B = \{\frac{n+1}{2n} : n \in \mathbb{Z}^+\}$, so $B$ is the range of the sequence $(b_n)$ defined by $b_n = \frac{n+1}{2n}$.

Let $y \in B$.

Then $y = U(f, P_n)$ for some positive integer $n$.

Since $P_n$ is a partition of $[0, 1]$, then $y \in T$.

Hence, $B \subset T$.

Let $n \in \mathbb{Z}^+$ be given.

Then $b_n = \frac{n+1}{2n} \in B$.

Since $B \subset T$, then $b_n \in T$.

Since $\inf T$ is a lower bound of $T$, then $\inf T \leq b_n$.

Since $n$ is arbitrary, then $\inf T \leq b_n$ for all $n \in \mathbb{Z}^+$, so $\inf T$ is a lower bound of $B$.

Since $\lim_{n\to\infty} \frac{n+1}{2n} = \frac{1}{2}$, then $(b_n)$ is a convergent sequence, so $\inf T \leq \lim_{n\to\infty} \frac{n+1}{2n} = \frac{1}{2}$.

Since $\inf T \leq \frac{1}{2}$, then $\overline{\int_0^1} x \leq \frac{1}{2}$. ∎

*Proof.* Since $\frac{1}{2} \leq \underline{\int_0^1} x$ and $\underline{\int_0^1} x \leq \overline{\int_0^1} x$ and $\overline{\int_0^1} x \leq \frac{1}{2}$, then $\frac{1}{2} \leq \underline{\int_0^1} x \leq \overline{\int_0^1} x \leq \frac{1}{2}$, so $\underline{\int_0^1} x \leq \frac{1}{2}$ and $\frac{1}{2} \leq \overline{\int_0^1} x$.

Since $\underline{\int_0^1} x \leq \frac{1}{2}$ and $\frac{1}{2} \leq \underline{\int_0^1} x$, then $\underline{\int_0^1} x = \frac{1}{2}$.

Since $\overline{\int_0^1} x \leq \frac{1}{2}$ and $\frac{1}{2} \leq \overline{\int_0^1} x$, then $\overline{\int_0^1} x = \frac{1}{2}$.

Therefore, $f$ is Darboux integrable and the integral of $x$ over $[0, 1]$ is $\int_0^1 x = \underline{\int_0^1} x = \frac{1}{2} = \overline{\int_0^1} x$. ∎

**Exercise 3.** Let $f : [0,1] \to \mathbb{R}$ be the function defined by $f(x) = x^2$.

Then $\int_0^1 x^2 dx = \underline{\int_0^1} x^2 = \frac{1}{3} = \overline{\int_0^1} x^2$.

*Proof.* Let $x \in [0,1]$.

Then $0 \leq x \leq 1$, so $0 \leq x^2 = f(x) \leq 1$.

Hence, $0 \leq f(x) \leq 1$, so $f$ is bounded.

Therefore, the upper Darboux integral $\overline{\int_0^1} x^2$ and the lower Darboux integral $\underline{\int_0^1} x^2$ exist and the upper Riemann sum $U(f,P)$ and lower Riemann sum $L(f,P)$ exist for every partition $P$ of the interval $[0,1]$.

Let $S = \{L(f,P) : P \text{ is a partition of } [0,1]\}$.

Let $T = \{U(f,P) : P \text{ is a partition of } [0,1]\}$.

Then $\underline{\int_0^1} x^2 = \sup S$ and $\overline{\int_0^1} x^2 = \inf T$ and $\underline{\int_0^1} x^2 \leq \overline{\int_0^1} x^2$. □

*Proof.* Let $P_n = \{0, \frac{1}{n}, \frac{2}{n}, ..., \frac{n-1}{n}, 1\}$ be a partition of $[0,1]$ for each $n \in \mathbb{Z}^+$.

For each $k = 1, 2, ..., n$ the $k^{th}$ subinterval is $I_k = [\frac{k-1}{n}, \frac{k}{n}]$ and its length is $\Delta_k = \frac{1}{n}$.

Hence, for each $k = 1, 2, ..., n$ the direct image of $I_k$ is $f(I_k) = f([\frac{k-1}{n}, \frac{k}{n}]) = [(\frac{k-1}{n})^2, (\frac{k}{n})^2]$, so $\sup f(I_k) = (\frac{k}{n})^2$ and $\inf f(I_k) = (\frac{k-1}{n})^2$ and $\Delta_k = \frac{1}{n}$ for each $k = 1, 2, ..., n$.

For every positive integer $n$ the lower Riemann sum is

$$\begin{aligned}
L(f,P_n) &= \sum_{k=1}^{n} \inf f(I_k)\Delta_k \\
&= \sum_{k=1}^{n} (\frac{k-1}{n})^2 \frac{1}{n} \\
&= \sum_{k=1}^{n} (k-1)^2 \frac{1}{n^3} \\
&= \frac{1}{n^3} \sum_{k=1}^{n} (k-1)^2 \\
&= \frac{1}{n^3} [\sum_{k=1}^{n} k^2 - n^2] \\
&= \frac{1}{n^3} [\frac{n(n+1)(2n+1)}{6} - n^2] \\
&= \frac{2n^2 - 3n + 1}{6n^2}.
\end{aligned}$$

Therefore, $L(f,P_n) = \frac{2n^2-3n+1}{6n^2}$ for all $n \in \mathbb{Z}^+$.

Let $A = \{L(f,P_n) : n \in \mathbb{Z}^+\}$.

Then $A = \{\frac{2n^2-3n+1}{6n^2} : n \in \mathbb{Z}^+\}$, so $A$ is the range of the sequence $(a_n)$ defined by $a_n = \frac{2n^2-3n+1}{6n^2}$.

Let $x \in A$.
Then $x = L(f, P_n)$ for some positive integer $n$.
Since $P_n$ is a partition of $[0, 1]$, then $x \in S$.
Hence, $A \subset S$.
Let $n \in \mathbb{Z}^+$ be given.
Then $a_n = \frac{2n^2-3n+1}{6n^2} \in A$.
Since $A \subset S$, then $a_n \in S$.
Since $\sup S$ is an upper bound of $S$, then $a_n \leq \sup S$.
Since $n$ is arbitrary, then $a_n \leq \sup S$ for all $n \in \mathbb{Z}^+$, so $\sup S$ is an upper bound of $A$.
Since $\lim_{n\to\infty} \frac{2n^2-3n+1}{6n^2} = \frac{1}{3}$, then $(a_n)$ is a convergent sequence, so $\frac{1}{3} = \lim_{n\to\infty} \frac{2n^2-3n+1}{6n^2} \leq \sup S$.
Hence, $\frac{1}{3} \leq \underline{\int_0^1} x^2$. □

*Proof.* For every positive integer $n$ the upper Riemann sum is

$$\begin{aligned} U(f, P_n) &= \sum_{k=1}^{n} \sup f(I_k)\Delta_k \\ &= \sum_{k=1}^{n} (\frac{k}{n})^2 \frac{1}{n} \\ &= \sum_{k=1}^{n} k^2 \frac{1}{n^3} \\ &= \frac{1}{n^3} \sum_{k=1}^{n} k^2 \\ &= \frac{1}{n^3} [\frac{n(n+1)(2n+1)}{6}] \\ &= \frac{(n+1)(2n+1)}{6n^2} \\ &= \frac{2n^2+3n+1}{6n^2}. \end{aligned}$$

Therefore, $U(f, P_n) = \frac{2n^2+3n+1}{6n^2}$ for all $n \in \mathbb{Z}^+$.
Let $B = \{U(f, P_n) : n \in \mathbb{Z}^+\}$.
Then $B = \{\frac{2n^2+3n+1}{6n^2} : n \in \mathbb{Z}^+\}$, so $B$ is the range of the sequence $(b_n)$ defined by $b_n = \frac{2n^2+3n+1}{6n^2}$.
Let $y \in B$.
Then $y = U(f, P_n)$ for some positive integer $n$.
Since $P_n$ is a partition of $[0, 1]$, then $y \in T$.
Hence, $B \subset T$.
Let $n \in \mathbb{Z}^+$ be given.
Then $b_n = \frac{2n^2+3n+1}{6n^2} \in B$.
Since $B \subset T$, then $b_n \in T$.

Since $\inf T$ is a lower bound of $T$, then $\inf T \leq b_n$.

Since $n$ is arbitrary, then $\inf T \leq b_n$ for all $n \in \mathbb{Z}^+$, so $\inf T$ is a lower bound of $B$.

Since $\lim_{n\to\infty} \frac{2n^2+3n+1}{6n^2} = \frac{1}{3}$, then $(b_n)$ is a convergent sequence, so $\inf T \leq \lim_{n\to\infty} \frac{2n^2+3n+1}{6n^2} = \frac{1}{3}$.

Since $\inf T \leq \frac{1}{3}$, then $\overline{\int_0^1} x^2 \leq \frac{1}{3}$. ☐

*Proof.* Since $\frac{1}{3} \leq \underline{\int_0^1} x^2$ and $\underline{\int_0^1} x^2 \leq \overline{\int_0^1} x^2$ and $\overline{\int_0^1} x^2 \leq \frac{1}{3}$, then $\frac{1}{3} \leq \underline{\int_0^1} x^2 \leq \overline{\int_0^1} x^2 \leq \frac{1}{3}$, so $\underline{\int_0^1} x^2 \leq \frac{1}{3}$ and $\frac{1}{3} \leq \overline{\int_0^1} x^2$.

Since $\underline{\int_0^1} x^2 \leq \frac{1}{3}$ and $\frac{1}{3} \leq \underline{\int_0^1} x^2$, then $\underline{\int_0^1} x^2 = \frac{1}{3}$.

Since $\overline{\int_0^1} x^2 \leq \frac{1}{3}$ and $\frac{1}{3} \leq \overline{\int_0^1} x^2$, then $\overline{\int_0^1} x^2 = \frac{1}{3}$.

Therefore, $f$ is Darboux integrable and the integral of $x^2$ over $[0,1]$ is $\int_0^1 x^2 = \underline{\int_0^1} x^2 = \frac{1}{3} = \overline{\int_0^1} x^2$. ☐

**Exercise 4.** Let $f : [0, 2] \to \mathbb{R}$ be the function defined by $f(x) = 1$ if $x \neq 1$ and $f(1) = 0$.

Then $\int_0^2 f = 2$.

*Proof.* Let $x \in [0, 2]$.

Then $0 \leq x \leq 2$, so either $x = 1$ or $x \neq 1$.

We consider these cases separately.

**Case 1:** Suppose $x = 1$.

Then $f(x) = f(1) = 0$.

**Case 2:** Suppose $x \neq 1$.

Then $f(x) = 1$.

Hence, either $f(x) = 0$ or $f(x) = 1$, so the range of $f$ is the set $\{0, 1\}$.

Since $\{0, 1\}$ is finite, then $f$ is bounded.

Therefore, the upper Darboux integral $\overline{\int_0^2} f$ and the lower Darboux integral $\underline{\int_0^2} f$ exist and the upper Riemann sum $U(f, P)$ and lower Riemann sum $L(f, P)$ exist for every partition $P$ of the interval $[0, 2]$.

Let $S = \{L(f, P) : P \text{ is a partition of } [0, 2]\}$.

Let $T = \{U(f, P) : P \text{ is a partition of } [0, 2]\}$.

Then $\underline{\int_0^2} f = \sup S$ and $\overline{\int_0^2} f = \inf T$. ☐

*Proof.* Let $P_1 = \{0, 2\}$ be a partition of the interval $[0, 2]$.

The upper Riemann sum is $U(f, P_1) = \sup f([0, 2])(2 - 0) = \sup\{0, 1\}(2) = 1(2) = 2$, so $2 \in T$.

Since $\inf T$ is a lower bound of $T$, then $\inf T \leq 2$, so $\overline{\int_0^2} f \leq 2$. ☐

*Proof.* We prove for every $\epsilon$, if $0 < \epsilon < 2$, then $2 - \epsilon \in S$.

Let $\epsilon$ be an arbitrary real number such that $0 < \epsilon < 2$

Then $0 < \epsilon$ and $\epsilon < 2$.

Since $\epsilon < 2$, then $\frac{\epsilon}{2} < 1$, so $0 < 1 - \frac{\epsilon}{2}$ and $1 + \frac{\epsilon}{2} < 2$.
Since $0 < \epsilon$, then $1 - 1 < \frac{\epsilon}{2} + \frac{\epsilon}{2}$, so $1 - \frac{\epsilon}{2} < 1 + \frac{\epsilon}{2}$.
Since $0 < 1 - \frac{\epsilon}{2}$ and $1 - \frac{\epsilon}{2} < 1 + \frac{\epsilon}{2}$ and $1 + \frac{\epsilon}{2} < 2$, then $0 < 1 - \frac{\epsilon}{2} < 1 + \frac{\epsilon}{2} < 2$.
Let $P_\epsilon = \{0, 1 - \frac{\epsilon}{2}, 1 + \frac{\epsilon}{2}, 2\}$ be a partition of $[0, 2]$.
The lower Riemann sum is

$$\begin{aligned}
L(f, P_\epsilon) &= \sum_{k=1}^{3} \inf f(I_k)\Delta_k \\
&= \inf f(I_1)\Delta_1 + \inf f(I_2)\Delta_2 + \inf f(I_3)\Delta_3 \\
&= \inf f([0, 1 - \frac{\epsilon}{2}])(1 - \frac{\epsilon}{2}) + \inf f([1 - \frac{\epsilon}{2}, 1 + \frac{\epsilon}{2}])\epsilon + \inf f([1 + \frac{\epsilon}{2}, 2])(1 - \frac{\epsilon}{2}) \\
&= \inf\{1\}(1 - \frac{\epsilon}{2}) + \inf\{0, 1\}\epsilon + \inf\{1\}(1 - \frac{\epsilon}{2}) \\
&= 1(1 - \frac{\epsilon}{2}) + 0(\epsilon) + 1(1 - \frac{\epsilon}{2}) \\
&= 2 - \epsilon.
\end{aligned}$$

Since $P_\epsilon$ is a partition of $[0, 2]$ and $L(f, P_\epsilon) = 2 - \epsilon$, then $2 - \epsilon \in S$.
Therefore, if $0 < \epsilon < 2$, then $2 - \epsilon \in S$. □

*Proof.* We prove $\sup S \geq 2$.
Suppose $\sup S < 2$.
Then $2 - \sup S > 0$, so $\frac{2 - \sup S}{2} > 0$.
Let $\epsilon = \frac{2 - \sup S}{2}$.
Then $\epsilon > 0$.
Since $\sup S \geq \inf f * (2 - 0) = \inf\{0, 1\} * (2) = 0 * 2 = 0 > -2$, then $\sup S > -2$, so $\sup S > 2 - 4$.
Hence, $4 > 2 - \sup S$, so $2 > \frac{2 - \sup S}{2}$.
Thus, $2 > \epsilon$.
Since $0 < \epsilon$ and $\epsilon < 2$, then $0 < \epsilon < 2$.
Hence, $2 - \epsilon = 2 - \frac{2 - \sup S}{2} = \frac{2 + \sup S}{2} = \frac{\sup S + 2}{2} \in S$.
Therefore, $\frac{\sup S + 2}{2} \in S$.
Since $\sup S < 2$, then $2 \sup S < \sup S + 2$, so $\sup S < \frac{\sup S + 2}{2}$.
Thus, $\frac{\sup S + 2}{2}$ is an element of $S$ that is greater than the least upper bound of $S$.
This contradicts the fact that $\sup S$ is an upper bound of $S$.
Hence, $\sup S \geq 2$, so $\underline{\int_0^2} f \geq 2$. □

*Proof.* Since $2 \leq \underline{\int_0^2} f \leq \overline{\int_0^2} f \leq 2$, then $2 \leq \underline{\int_0^2} f$ and $\underline{\int_0^2} f \leq 2$ and $2 \leq \overline{\int_0^2} f$ and $\overline{\int_0^2} f \leq 2$.
Since $\underline{\underline{\int_0^2}} f \leq 2$ and $2 \leq \underline{\underline{\int_0^2}} f$, then $\underline{\underline{\int_0^2}} f = 2$.
Since $\overline{\int_0^2} f \leq 2$ and $2 \leq \overline{\int_0^2} f$, then $\overline{\int_0^2} f = 2$.
Therefore, $f$ is Darboux integrable and the integral of $f$ over $[0, 2]$ is $\int_0^2 f = \underline{\int_0^2} f = 2 = \overline{\int_0^2} f$. □

**Exercise 5.** Let $f : [0,1] \to \mathbb{R}$ be the function defined by

$$f(x) = \begin{cases} 0 & \text{if } 0 \leq x \leq \frac{1}{2} \\ 1 & \text{if } \frac{1}{2} < x \leq 1 \end{cases}$$

Then $\underline{\int_0^1} f = \overline{\int_0^1} f = \int_0^1 f = \frac{1}{2}$.

*Proof.* We prove $f$ is a bounded function.
Let $x \in [0,1]$.
Then either $0 \leq x \leq \frac{1}{2}$ or $\frac{1}{2} < x \leq 1$.
We consider these cases separately.
**Case 1:** Suppose $0 \leq x \leq \frac{1}{2}$.
Then $|f(x)| = |0| = 0 < 1$.
**Case 2:** Suppose $\frac{1}{2} < x \leq 1$.
Then $|f(x)| = |1| = 1$.
Thus, in all cases, $|f(x)| \leq 1$.
Hence, $|f(x)| \leq 1$ for all $x \in [0,1]$, so $f$ is bounded.

Therefore, the upper Darboux integral $\overline{\int_0^1} f$ and the lower Darboux integral $\underline{\int_0^1} f$ exist and the upper Riemann sum $U(f,P)$ and lower Riemann sum $L(f,P)$ exist for every partition $P$ of the interval $[0,1]$.
Let $S = \{L(f,P) : P \text{ is a partition of } [0,1]\}$.
Let $T = \{U(f,P) : P \text{ is a partition of } [0,1]\}$.
Then $\underline{\int_0^1} f = \sup S$ and $\overline{\int_0^1} f = \inf T$ and $\underline{\int_0^1} f \leq \overline{\int_0^1} f$. ☐

*Proof.* Let $P_2 = \{0, \frac{1}{2}, 1\}$ be a partition of $[0,1]$.
The upper Riemann sum is

$$\begin{aligned}
U(f,P_2) &= \sum_{k=1}^{2} \sup f(I_k)\Delta_k \\
&= \sup f(I_1)\Delta_1 + \sup f(I_2)\Delta_2 \\
&= \sup f([0,\frac{1}{2}])(\frac{1}{2} - 0) + \sup f([\frac{1}{2},1])(1 - \frac{1}{2}) \\
&= \sup\{0\}(\frac{1}{2}) + \sup\{0,1\}(\frac{1}{2}) \\
&= 0(\frac{1}{2}) + 1(\frac{1}{2}) \\
&= \frac{1}{2}.
\end{aligned}$$

Since $P_2$ is a partition of $[0,1]$ and $U(f,P_2) = \frac{1}{2}$, then $\frac{1}{2} \in T$.
Since $\overline{\int_0^1} f$ is a lower bound of $T$ and $\frac{1}{2} \in T$, then $\overline{\int_0^1} f \leq \frac{1}{2}$. ☐

*Proof.* We prove for every $\epsilon$ if $0 < \epsilon < \frac{1}{2}$, then $\frac{1}{2} - \epsilon \in S$.

Let $\epsilon$ be an arbitrary real number such that $0 < \epsilon < \frac{1}{2}$.

Since $0 < \epsilon < \frac{1}{2}$, then $\frac{1}{2} < \frac{1}{2} + \epsilon < 1$.

Since $0 < \frac{1}{2} < \frac{1}{2} + \epsilon < 1$, then $0 < \frac{1}{2} + \epsilon < 1$, so let $P_\epsilon = \{0, \frac{1}{2} + \epsilon, 1\}$ be a partition of the interval $[0, 1]$.

The lower Riemann sum is

$$\begin{aligned}
L(f, P_\epsilon) &= \sum_{k=1}^{2} \inf f(I_k)\Delta_k \\
&= \inf f(I_1)\Delta_1 + \inf f(I_2)\Delta_2 \\
&= \inf f([0, \frac{1}{2} + \epsilon])(\frac{1}{2} + \epsilon) + \inf f([\frac{1}{2} + \epsilon, 1])[1 - (\frac{1}{2} + \epsilon)] \\
&= \inf\{0, 1\}(\frac{1}{2} + \epsilon) + \inf\{1\}(\frac{1}{2} - \epsilon) \\
&= 0(\frac{1}{2} + \epsilon) + 1(\frac{1}{2} - \epsilon) \\
&= \frac{1}{2} - \epsilon.
\end{aligned}$$

Since $P_\epsilon$ is a partition of $[0, 1]$ and $L(f, P_\epsilon) = \frac{1}{2} - \epsilon$, then $\frac{1}{2} - \epsilon \in S$.

Therefore, $\frac{1}{2} - \epsilon \in S$ for every $\epsilon$ such that $0 < \epsilon < \frac{1}{2}$. $\square$

*Proof.* We prove $\sup S \geq \frac{1}{2}$.

Suppose $\sup S < \frac{1}{2}$.

Then $\frac{1}{2} - \sup S > 0$, so $\frac{\frac{1}{2} - \sup S}{2} > 0$.

Let $\epsilon = \frac{\frac{1}{2} - \sup S}{2}$.

Then $\epsilon > 0$.

Since $\sup S \geq \inf f * (1 - 0) = 0 * (1 - 0) = 0 > \frac{-1}{2}$, then $\sup S > \frac{-1}{2}$, so $\sup S > \frac{1}{2} - 1$.

Hence, $1 > \frac{1}{2} - \sup S$, so $\frac{1}{2} > \frac{\frac{1}{2} - \sup S}{2}$.

Thus, $\frac{1}{2} > \epsilon$.

Since $0 < \epsilon$ and $\epsilon < \frac{1}{2}$, then $0 < \epsilon < \frac{1}{2}$.

Hence, $\frac{1}{2} - \epsilon = \frac{1}{2} - \frac{\frac{1}{2} - \sup S}{2} = \frac{\sup S + \frac{1}{2}}{2} \in S$.

Therefore, $\frac{\sup S + \frac{1}{2}}{2} \in S$.

Since $\sup S < \frac{1}{2}$, then $2 \sup S < \sup S + \frac{1}{2}$, so $\sup S < \frac{\sup S + \frac{1}{2}}{2}$.

Thus, $\frac{\sup S + \frac{1}{2}}{2}$ is an element of $S$ that is greater than the least upper bound of $S$.

This contradicts the fact that $\sup S$ is an upper bound of $S$.

Hence, $\sup S \geq \frac{1}{2}$, so $\underline{\int_0^1} f \geq \frac{1}{2}$. $\square$

*Proof.* Since $\frac{1}{2} \leq \underline{\int_0^1} f \leq \overline{\int_0^1} f \leq \frac{1}{2}$, then $\frac{1}{2} \leq \underline{\int_0^1} f$ and $\underline{\int_0^1} f \leq \frac{1}{2}$ and $\frac{1}{2} \leq \overline{\int_0^1} f$ and $\overline{\int_0^1} f \leq \frac{1}{2}$.

Since $\underline{\int_0^1} f \leq \frac{1}{2}$ and $\frac{1}{2} \leq \underline{\int_0^1} f$, then $\underline{\int_0^1} f = \frac{1}{2}$.

Since $\overline{\int_0^1} f \leq \frac{1}{2}$ and $\frac{1}{2} \leq \overline{\int_0^1} f$, then $\overline{\int_0^1} f = \frac{1}{2}$.

Therefore, $f$ is Darboux integrable and the integral of $f$ over $[0,1]$ is $\int_0^1 f = \underline{\int_0^1} f = \frac{1}{2} = \overline{\int_0^1} f$. □

# Riemann Integral of a real valued function

**Exercise 6.** Let $k \in \mathbb{R}$ be a fixed.

Then $\int_a^b k dx = k(b-a)$.

*Proof.* Let $f : [a,b] \to \mathbb{R}$ be a function defined by $f(x) = k$.

Let $\epsilon > 0$ be given.

Let $\delta = 1$.

Let $\dot{P} = \{([x_{i-1}, x_i], t_i) : i \in \mathbb{Z}^+, 1 \leq i \leq n\}$ be an arbitrary tagged partition of $[a,b]$ with $||\dot{P}|| < 1$.

Since $\dot{P}$ is a partition, then $x_0 = a$ and $x_n = b$.

Observe that

$$\begin{aligned} |S(f,\dot{P}) - k(b-a)| &= |\sum_{i=1}^{n} f(t_i)(x_i - x_{i-1}) - k(b-a)| \\ &= |\sum_{i=1}^{n} k(x_i - x_{i-1}) - k(b-a)| \\ &= |k\sum_{i=1}^{n}(x_i - x_{i-1}) - k(b-a)| \\ &= |k(x_n - x_0) - k(b-a)| \\ &= |k(b-a) - k(b-a)| \\ &= 0 \\ &< \epsilon. \end{aligned}$$

Therefore, $\int_a^b k dx = k(b-a)$. □