Integration of real valued functions Notes

Jason Sass

June 29, 2021

Sets of Numbers

$\mathbb{R} =$ set of all real numbers
$\mathbb{R}^+ = \{x \in \mathbb{R} : x > 0\} = (0, \infty) =$ set of all positive real numbers
$\mathbb{R}^* = \{x \in \mathbb{R} : x \neq 0\} = (-\infty, 0) \cup (0, \infty) =$ set of all nonzero real numbers

Indefinite Integral of a real valued function

Definition 1. antiderivative of a function
A function F is called an **antiderivative** of the function f if $F'(x) = f(x)$ for all $x \in dom f$.

If F is an antiderivative of f, then $F'(x) = f(x)$ for all $x \in dom f$.
We may write $F' = f$ to denote that F is an antiderivative of f.

Theorem 2. ***representation of antiderivatives***
Let F be an antiderivative of a function f defined on an interval I.
Then G is an antiderivative of f on I iff there exists a constant C such that $G(x) = F(x) + C$ for all $x \in I$.

Let f be a function defined on an interval I.
Suppose F and G are antiderivatives of f.
Then $F'(x) = f(x) = G'(x)$ for all $x \in I$ and there exists a constant C such that $G(x) = F(x) + C$ for all $x \in I$.
The antiderivative of a function f is denoted $\int f$ or $\int f(x)dx$.

Suppose F is an antiderivative of a function f.
Then $F'(x) = f(x)$ for all $x \in dom f$ and there exists a constant C such that $\int f(x)dx = F(x) + C$.

Definite Integral of a real valued function

Definition 3. integrable function

Let $a, b \in \mathbb{R}$ with $a \leq b$.

A function $f : [a, b] \to \mathbb{R}$ is said to be **integrable on** $[a, b]$ iff there exists a unique real number $\int_a^b f$ such that the following axioms hold:

I1. If C is a constant function, then $\int_a^b C = C(b - a)$.
I2. Let $g : [a, b] \to \mathbb{R}$ be a function.
If $f(x) \leq g(x)$ for all $x \in [a, b]$, then $\int_a^b f \leq \int_a^b g$.
I3. Let $c \in \mathbb{R}$.
If $a \leq c \leq b$, then $\int_a^b f = \int_a^c f + \int_c^b f$.

Let $f : [a, b] \to \mathbb{R}$ be an integrable function.

Then the real number $\int_a^b f$ exists.

We say that $\int_a^b f$ is the **definite integral of** f **from** a **to** b.

Definition 4. set of integrable functions $\mathcal{R}[a, b]$

Let $a, b \in \mathbb{R}$ with $a \leq b$.

The set of all integrable functions over the closed interval $[a, b]$, denoted $\mathcal{R}[a, b]$, is the set of all functions defined on $[a, b]$ such that $\int_a^b f$ exists.

Let $f \in \mathcal{R}[a, b]$.

Then $f : [a, b] \to \mathbb{R}$ is an integrable function and the integral of f is the real number $\int_a^b f$.

Therefore, integration is a mapping that assigns a real number, the definite integral, to each function $f \in \mathcal{R}[a, b]$.

Let $\mathcal{C}[a, b]$ be the set of all continuous functions on $[a, b]$.

Let $\mathcal{B}[a, b]$ be the set of all bounded functions on $[a, b]$.

By EVT, every continuous function $f : [a, b] \to \mathbb{R}$ is bounded.

Therefore, $\mathcal{C}[a, b] \subset \mathcal{R}[a, b] \subset \mathcal{B}[a, b]$.

Axiom 5. ***Asumptions for theory of definite integral***

I0. Let $a, b \in \mathbb{R}$ with $a < b$.
a. Every continuous function $f : [a, b] \to \mathbb{R}$ lies in $\mathcal{R}[a, b]$.
b. Every function $f \in \mathcal{R}[a, b]$ is bounded.

Let $a, b \in \mathbb{R}$ with $a \leq b$.
Let $\mathcal{R}[a, b]$ be the set of all integrable functions over $[a, b]$.
I1. If C is a constant function, then $C \in \mathcal{R}[a, b]$ and $\int_a^b C = C(b - a)$.
I2. Let $f, g \in \mathcal{R}[a, b]$.
If $f(x) \leq g(x)$ for all $x \in [a, b]$, then $\int_a^b f \leq \int_a^b g$.
I3. Let $f : [a, b] \to \mathbb{R}$ be a function and let $c \in (a, b)$.
Then $f \in \mathcal{R}[a, b]$ iff $f \in \mathcal{R}[a, c]$ and $f \in \mathcal{R}[c, b]$.
If $f \in \mathcal{R}[a, b]$, then $\int_a^b f = \int_a^c f + \int_c^b f$.

Lemma 6. *If $f \in \mathcal{R}[a,b]$, then $\int_a^a f = 0 = \int_b^b f$.*

Proposition 7. *Let $f \in \mathcal{R}[a,b]$.*
For every $c \in [a,b]$, $\int_c^c f = 0$.

Fundamental Theorem of Calculus

Theorem 8. *Let $f : [a,b] \to \mathbb{R}$ be an integrable function.*
Let $F : [a,b] \to \mathbb{R}$ be defined by $F(x) = \int_a^x f$ for $x \in [a,b]$.
The function F is continuous.

Theorem 9. ***Fundamental Theorem of Calculus (derivative of an integral***
Let $f : [a,b] \to \mathbb{R}$ be an integrable function.
Let $F : [a,b] \to \mathbb{R}$ be defined by $F(x) = \int_a^x f$ for all $x \in [a,b]$.
If f is continuous at x, then F is differentiable at x and $F'(x) = f(x)$.

Let $f : [a,b] \to \mathbb{R}$ be an integrable function.
Let $F : [a,b] \to \mathbb{R}$ be defined by $F(x) = \int_a^x f$ for all $x \in [a,b]$.
Suppose f is continuous.
Let $x \in [a,b]$.
Then f is continuous at x, so F is differentiable and $F'(x) = f(x)$.
Hence, $F'(x) = f(x)$ for all $x \in [a,b]$, so F is an antiderivative of f.
Thus, $F' = f$.
Therefore, $\frac{d}{dx} \int_a^x f = \frac{d}{dx} F = F' = f$.
Hence, if f is continuous, then F is an antiderivative of f.

Theorem 10. ***Fundamental Theorem of Calculus (integral of a derivative)***
Let $F : [a,b] \to \mathbb{R}$ be a differentiable function.
If F' is continuous, then F' is integrable and $\int_a^b F' = F(b) - F(a)$.

Let $f : [a,b] \to \mathbb{R}$ be a function.
Suppose f is continuous and F is an antiderivative of f.
Since F is an antiderivative of f, then $F : [a,b] \to \mathbb{R}$ is a function such that $F'(x) = f(x)$ for all $x \in [a,b]$.
Since $F'(x) = f(x)$ for all $x \in [a,b]$, then $F' = f$ and F is differentiable.
Since f is continuous and $f = F'$, then F' is continuous.
Hence, $F' = f$ is integrable and $\int_a^b f = \int_a^b F' = F(b) - F(a)$.

Darboux Integral of a real valued function

A partition of an interval is a finite collection of non-overlapping intervals whose union is the interval.

Definition 11. partition of an interval

Let $a, b \in \mathbb{R}$ with $a < b$.

Let $[a, b]$ be a closed bounded interval.

Let n be a fixed positive integer.

A **partition of [a,b]** is a finite set of points $\{x_0, x_1, ..., x_{n-1}, x_n\}$ such that $a = x_0 < x_1 < ... < x_{n-1} < x_n = b$.

Let P be a partition of an interval $[a, b]$.

Then P is a finite set of points $\{x_0, x_1, ..., x_{n-1}, x_n\}$ and $a = x_0 < x_1 < ... < x_{n-1} < x_n = b$.

Therefore $P = \{a, x_1, ..., x_{n-1}, b\}$ and $x_{k-1} < x_k$ for each $k = 1, 2, ..., n$.

Let $I_k = [x_{k-1}, x_k]$ for each $k = 1, 2, ..., n$.

Then each $[x_{k-1}, x_k]$ is a subinterval of the partition.

Therefore, P consists of n subintervals and $[a, b] = [a, x_1] \cup [x_1, x_2] \cup ... \cup [x_{n-1}, b]$.

Since each subinterval is a subset of $[a, b]$, then $I_k \subset [a, b]$ for each $k = 1, 2, .., n$.

Lemma 12. ***The existence of the supremum and infimum of the direct image of each subinterval of a partition for a bounded function.***

Let $f : [a, b] \to \mathbb{R}$ be a bounded function.

Let n be a fixed positive integer.

Let $P = \{a, x_1, ..., x_{n-1}, b\}$ be a partition of $[a, b]$.

For each $k = 1, 2, ..., n$ the supremum and infimum of the set $\{f(x) : x \in [x_{k-1}, x_k]\}$ exist.

Definition 13. upper and lower Riemann sums

Let $f : [a, b] \to \mathbb{R}$ be a bounded function.

Let n be a fixed positive integer.

Let $P = \{a, x_1, ..., x_{n-1}, b\}$ be a partition of $[a, b]$.

For each $k = 1, 2, ..., n$ let

$M_k = \sup\{f(x) : x \in [x_{k-1}, x_k]\}$ and

$m_k = \inf\{f(x) : x \in [x_{k-1}, x_k]\}$ and

$\Delta_k = x_k - x_{k-1}$.

The **upper Riemann sum of f with respect to** P is $U(f, P) = \sum_{k=1}^{n} M_k \Delta_k$.

The **lower Riemann sum of f with respect to** P is $L(f, P) = \sum_{k=1}^{n} m_k \Delta_k$.

Let $f : [a, b] \to \mathbb{R}$ be a bounded function.

Let n be a fixed positive integer.

Let $P = \{a, x_1, ..., x_{n-1}, b\}$ be a partition of $[a, b]$.

For each $k = 1, 2, ..., n$ let $I_k = [x_{k-1}, x_k]$ be the k^{th} subinterval.

Then the supremum and infimum of the set $\{f(x) : x \in I_k\}$ exist, so $\sup\{f(x) : x \in I_k\}$ and $\inf\{f(x) : x \in I_k\}$ exist.

For each $k = 1, 2, ..., n$ let

$M_k = \sup\{f(x) : x \in [x_{k-1}, x_k]\}$ and

$m_k = \inf\{f(x) : x \in [x_{k-1}, x_k]\}$ and

$\Delta_k = x_k - x_{k-1}$.

Then
$M_k = \sup\{f(x) : x \in [x_{k-1}, x_k]\} = \sup\{f(x) : x \in I_k\} = \sup f(I_k)$.
$m_k = \inf\{f(x) : x \in [x_{k-1}, x_k]\} = \inf\{f(x) : x \in I_k\} = \inf f(I_k)$.
$U(f, P) = \sum_{k=1}^{n} M_k \Delta_k = \sum_{k=1}^{n} \sup f(I_k)\Delta_k$.
$L(f, P) = \sum_{k=1}^{n} M_k \Delta_k = \sum_{k=1}^{n} \inf f(I_k)\Delta_k$.

Intuitively, we observe the following:
The largest value of f on the k^{th} subinterval $[x_{k-1}, x_k]$ is $M_k = \sup f(I_k)$.
The smallest value of f on the k^{th} subinterval $[x_{k-1}, x_k]$ is $m_k = \inf f(I_k)$.
The length of the k^{th} subinterval $[x_{k-1}, x_k]$ is $\Delta_k = x_k - x_{k-1}$.
The upper Riemann sum is $U(f, P) = \sum_{k=1}^{n} \sup f(I_k)\Delta_k$.
The lower Riemann sum is $L(f, P) = \sum_{k=1}^{n} \inf f(I_k)\Delta_k$.

Lemma 14. ***A lower Riemann sum is smaller than an upper Riemann sum for a given partition.***
Let $f : [a, b] \to \mathbb{R}$ be a bounded function.
Let P be a partition of $[a, b]$.
Let $U(f, P)$ be an upper Riemann sum.
Let $L(f, P)$ be a lower Riemann sum.
Then $L(f, P) \leq U(f, P)$.

Definition 15. refinement of a partition of an interval
Let P and Q be partitions of an interval I.
Then Q **is a refinement of** P iff $P \subset Q$.
Equivalently, we say that Q **refines** P.

Lemma 16. ***Refining a partition increases lower Riemann sums and decreases upper Riemann sums.***
Let $f : [a, b] \to \mathbb{R}$ be a bounded function.
If P is a partition of $[a, b]$ and Q is a refinement of P, then $L(f, P) \leq L(f, Q)$ and $U(f, P) \geq U(f, Q)$.

Proposition 17. ***Any lower Riemann sum is smaller than any upper Riemann sum.***
Let $f : [a, b] \to \mathbb{R}$ be a bounded function.
If P and Q are partitions of $[a, b]$, then $L(f, P) \leq U(f, Q)$.

Lemma 18. *For every $n \in \mathbb{Z}^+$, $\sum_{i=1}^{n}(x_i - x_{i-1}) = x_n - x_0$.*

Proposition 19. *Let $f : [a, b] \to \mathbb{R}$ be a bounded function.*
Let $M = \sup f([a, b])$.
Let $m = \inf f([a, b])$.
If P is a partition of $[a, b]$, then $m(b - a) \leq L(f, P) \leq U(f, P) \leq M(b - a)$.

Definition 20. upper and lower integrals
Let $f : [a, b] \to \mathbb{R}$ be a bounded function.

The **upper Darboux integral of** f, denoted $\overline{\int_a^b} f$, is $U_f = \inf\{U(f, P) : P$ is a partition of $[a, b]\}$.

The **lower Darboux integral of** f, denoted $\underline{\int_a^b} f$, is $L_f = \sup\{L(f,P) : P \text{ is a partition of } [a,b]\}$.

Theorem 21. ***The upper and lower Darboux integrals exist for a bounded function.***
Let $f : [a,b] \to \mathbb{R}$ be a bounded function.
Then the lower integral $\underline{\int_a^b} f$ and upper integral $\overline{\int_a^b} f$ exist and $\underline{\int_a^b} f \leq \overline{\int_a^b} f$.

Proposition 22. *Let $f : [a,b] \to \mathbb{R}$ be a bounded function.*
Let $M = \sup f([a,b])$.
Let $m = \inf f([a,b])$.
Then $m(b-a) \leq \underline{\int_a^b} f$ and $\overline{\int_a^b} f \leq M(b-a)$.

Let $f : [a,b] \to \mathbb{R}$ be a bounded function.
Let $M = \sup f([a,b])$.
Let $m = \inf f([a,b])$.
Then $m(b-a) \leq \underline{\int_a^b} f$ and $\overline{\int_a^b} f \leq M(b-a)$ and $\underline{\int_a^b} f \leq \overline{\int_a^b} f$.
Therefore, $m(b-a) \leq \underline{\int_a^b} f \leq \overline{\int_a^b} f \leq M(b-a)$.

Definition 23. Darboux integral
Let $f : [a,b] \to \mathbb{R}$ be a bounded function.
Let $\overline{\int_a^b} f$ be the upper Darboux integral of f.
Let $\underline{\int_a^b} f$ be the lower Darboux integral of f.
Then f is **Darboux integrable on** $[a,b]$ iff $\underline{\int_a^b} f = \overline{\int_a^b} f$.
The **Darboux integral of** f **over** $[a,b]$ is denoted by $\int_a^b f$.
When f is Darboux integrable we say that the Darboux integral of f exists and we write $\int_a^b f = \underline{\int_a^b} f = \overline{\int_a^b} f$.

Example 24. The constant function is Darboux integrable.
Let $k \in \mathbb{R}$ be fixed.
Then $\int_a^b k = k(b-a)$.

Example 25. Dirichlet function is not Darboux integrable
Let $f : [0,1] \to \mathbb{R}$ be a function defined by

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$$

Then f is not integrable on $[0,1]$.

Theorem 26. ***Darboux integrability criterion***
Let $f : [a,b] \to \mathbb{R}$ be a bounded function.
Then f is Darboux integrable on $[a,b]$ iff for every $\epsilon > 0$ there exists a partition P of $[a,b]$ such that $U(f,P) - L(f,P) < \epsilon$.

Definition 27. calculus integral of a function

Let $f : [a, b] \to \mathbb{R}$ be a function.

Let $F : [a, b] \to \mathbb{R}$ be any continuous function such that $F'(x) = f(x)$ for all $x \in [a, b]$.

The calculus integral is defined to be $\int_a^b f(x)dx = F(b) - F(a)$.

Definition 28. gauge integral of a function

Let $f : [a, b] \to \mathbb{R}$ be a function.

Let V be some number.

We say that V is the **gauge integral** of f, written $V = \int_a^b f(t)dt$, iff for every $\epsilon > 0$ there exists a corresponding function $\delta : [a, b] \to (0, +\infty)$ such that whenever n is a positive integer, and $t_0, t_1, t_2, ..., t_n$ and $s_1, s_2, ..., s_n$ are some numbers satisfying $a = t_0 \leq s_1 \leq t_1 \leq s_2 \leq t_2 \leq ... \leq t_{n-1} \leq s_n \leq t_n = b$ and $t_i - t_{i-1} < \delta(s_i)$ for all i, then $|V - \sum_{i=1}^n f(s_i)(t_i - t_{i-1})| < \epsilon$.

The function δ is called a gauge.

The collection of numbers n, s_i, t_i is called a tagged division(partition) and the numbers s_i are called the tags.

A tagged division is called δ-fine if $t_i - t_{i-1} < \delta(s_i)$ for all i.

Theorem 29. ***FTC(derivatives of integrals)***

Let f be a real-valued, gauge integrable function on $[a, b]$.

Let $F(x) = \int_a^x f$.

Then F is differentiable and $F'(x) = f(x)$, at each x where f is continuous.

Theorem 30. ***FTC(integrals of derivatives)***

Let F be a real-valued, differentiable function on $[a, b]$.

Then F' is gauge integrable and $\int_a^b F' = F(b) - F(a)$.

Riemann Integral of a real valued function

The definite integral of a function can be interpreted as the area under the curve.

The norm of a partition is the length of the largest subinterval.

Definition 31. norm of a partition

Let P be a partition of size n with partition points $x_0, x_1, ..., x_{n-1}, x_n$.

The **norm of P**, denoted $||P||$, is the maximum of the set $\{x_i - x_{i-1} : i \in \mathbb{Z}^+, 1 \leq i \leq n\}$.

Let P be a partition of size n with partition points $x_0, x_1, ..., x_{n-1}, x_n$.

The norm of P is $||P|| = \max\{x_i - x_{i-1} : i \in \mathbb{Z}^+, 1 \leq i \leq n\}$.

Since the set is finite, then the maximum of the set exists.

A tagged partition is a partition together with a distinguished point in each of its subintervals.

Definition 32. tagged partition
Let P be a partition of size n with subintervals $I_i = [x_{i-1}, x_i]$.
Let $t_i \in I_i$ for some $i \in \mathbb{Z}^+$.
A tagged partition of P, denoted $\dot{P}$, is the set $\{(I_i, t_i) : i \in \mathbb{Z}^+, 1 \leq i \leq n\}$.
The points t_i are called tags.

Let P be a partition of size n with subintervals $I_i = [x_{i-1}, x_i]$.
A tagged partition of P is $\dot{P} = \{(I_i, t_i) : 1 \leq i \leq n\}$, where $t_i \in I_i$.

Definition 33. Riemann sum
Let $f : [a, b] \to \mathbb{R}$ be a function.
Let $\dot{P}$ be a tagged partition of $[a, b]$.
The **Riemann sum of** f with respect to $\dot{P}$ is $S(f; \dot{P}) = \sum_{i=1}^{n} f(t_i)(x_i - x_{i-1})$.

A function is Riemann integrable if the limit of the Riemann sums exists as the norm of the partitions approaches zero.

Definition 34. Riemann integrable function
Let $a, b \in \mathbb{R}$ with $a < b$.
Let $f : [a, b] \to \mathbb{R}$ be a function.
Then f is said to be **Riemann integrable on** $[a, b]$ iff there exists $L \in \mathbb{R}$ such that for every $\epsilon > 0$, there exists $\delta > 0$ such that if $\dot{P}$ is any tagged partition of $[a, b]$ with $||\dot{P}|| < \delta$, then $|S(f; \dot{P}) - L| < \epsilon$.
We call L the **Riemann integral of** f **over** $[a, b]$ and we write $\int_a^b f(x)dx = L$.

Theorem 35. ***Integral of a Riemann integrable function is unique.***
If $f : [a, b] \to \mathbb{R}$ is a Riemann integrable function, then the value of the integral is unique.

Let $f : [a, b] \to \mathbb{R}$ be a Riemann integrable function.
Then there exists a unique $L \in \mathbb{R}$ such that for every $\epsilon > 0$, there exists $\delta > 0$ such that if $\dot{P}$ is any tagged partition of $[a, b]$ with $||\dot{P}|| < \delta$, then $|S(f; \dot{P}) - L| < \epsilon$.
Thus, L is the **Riemann integral of** f **over** $[a, b]$.
Therefore, $\int_a^b f(x)dx = L$.
Equivalently, $\int_a^b f = L$.
The set of all Riemann integrable functions on an interval $[a, b]$ is denoted $\mathcal{R}[a, b]$.

Lemma 36. *For every $n \in \mathbb{Z}^+$, $\sum_{i=1}^{n}(x_i - x_{i-1}) = x_n - x_0$.*

Example 37. Let $k \in \mathbb{R}$ be fixed.
Then $\int_a^b k dx = k(b - a)$.