Real Analysis Exercises 2

Jason Sass

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Vector space $\mathbb{R}^n$

Exercise 1. Let $x, y \in \mathbb{R}^n$. Then $||x|| - ||y||| \leq ||x - y||$, where $|| \cdot ||$ is the Euclidean norm of a vector.

Proof. Observe that

$$\begin{aligned} ||x|| &= ||x + (y - y)|| \\ &= ||(x - y) + y|| \\ &\leq ||x - y|| + ||y||. \end{aligned}$$

Hence, $||x|| \leq ||x - y|| + ||y||$, so $||x|| - ||y|| \leq ||x - y||$ for any $x, y \in \mathbb{R}^n$.

Let $x, y \in \mathbb{R}^n$. Then $||x||, ||y|| \in \mathbb{R}$, so either $||x|| \geq ||y||$ or $||x|| < ||y||$.

We consider these cases separately.

Case 1: Suppose $||x|| \geq ||y||$.

Then $||x|| - ||y|| \geq 0$. Thus,

$$\begin{aligned} |||x|| - ||y||| &= ||x|| - ||y|| \\ &\leq ||x - y||. \end{aligned}$$

Therefore, $|||x|| - ||y||| \leq ||x - y||$.

Case 2: Suppose $||x|| < ||y||$.

Then $||x|| - ||y|| < 0$. Since $||x|| - ||y|| \leq ||x - y||$ for any $x, y \in \mathbb{R}^n$, if we substitute x for y and y for x we have $||y|| - ||x|| \leq ||y - x||$. Thus,

$$\begin{aligned} |||x|| - ||y||| &= -(||x|| - ||y||) \\ &= ||y|| - ||x|| \\ &\leq ||y - x|| \\ &= ||x - y||. \end{aligned}$$

Therefore, $|||x|| - ||y||| \leq ||x - y||$.

Hence, in all cases, $|||x|| - ||y||| \leq ||x - y||$. □

Exercise 2. Let $x, y \in \mathbb{R}^n$. Then $||x|| + ||y|| \leq ||x + y|| + ||x - y||$, where $|| \cdot ||$ is the Euclidean norm of a vector.

Proof. Observe that

$$\begin{aligned} ||x|| &= ||\frac{x+y}{2} + \frac{x-y}{2}|| \\ &\leq ||\frac{x+y}{2}|| + ||\frac{x-y}{2}|| \\ &= \frac{1}{2}||x+y|| + \frac{1}{2}||x-y||. \end{aligned}$$

and

$$\begin{aligned} ||y|| &= ||\frac{x+y}{2} + \frac{y-x}{2}|| \\ &\leq ||\frac{x+y}{2}|| + ||\frac{y-x}{2}|| \\ &= \frac{1}{2}||x+y|| + \frac{1}{2}||y-x|| \\ &= \frac{1}{2}||x+y|| + \frac{1}{2}||x-y||. \end{aligned}$$

Adding the inequalities $||x|| \leq \frac{1}{2}||x+y|| + \frac{1}{2}||x-y||$ and $||y|| \leq \frac{1}{2}||x+y|| + \frac{1}{2}||x-y||$, we obtain $||x|| + ||y|| \leq ||x+y|| + ||x-y||$, as desired. □

Exercise 3. Let $x, y \in \mathbb{R}^n$. Then $||x+y||^2 + ||x-y||^2 = 2||x||^2 + 2||y||^2$, where $||\cdot||$ is the Euclidean norm of a vector.

Proof. Observe that

$$\begin{aligned} ||x+y||^2 + ||x-y||^2 &= (x+y)\cdot(x+y) + (x-y)\cdot(x-y) \\ &= (x+y)\cdot x + (x+y)\cdot y + (x-y)\cdot x - (x-y)\cdot y \\ &= x\cdot(x+y) + y\cdot(x+y) + x\cdot(x-y) - y\cdot(x-y) \\ &= x\cdot x + x\cdot y + y\cdot x + y\cdot y + x\cdot x - x\cdot y - y\cdot x + y\cdot y \\ &= 2x\cdot x + 2y\cdot y \\ &= 2||x||^2 + 2||y||^2. \end{aligned}$$

□

Boundedness of subsets of $\mathbb{R}^n$

Exercise 4. Let $B = \{(x, y) \in \mathbb{R}^2 : x > 0\}$. Then B is unbounded in $\mathbb{R}^2$.

Solution. Note that B is the right half plane excluding the y axis and $B = (0, \infty) \times (-\infty, \infty) =]0, \infty[\times] - \infty, \infty[=]0, \infty \times \mathbb{R}$, an open 2 cell. It is obvious that B is unbounded. □

Proof. To prove B is unbounded, we must prove $(\forall R > 0)(\exists x \in B)(||x|| > R)$.

Let $R > 0$. Let $x = (R + 1, 0)$.

Since $0 < R < R + 1$, then $R + 1 > 0$, so $(R + 1, 0) \in B$.

Observe that $||(R + 1, 0)|| = R + 1 > R$, so $(R + 1, 0) \notin \overline{B_R}(0)$. Therefore, B is unbounded. □

Exercise 5. Let $C = \{(x, y) \in \mathbb{R}^2 : 0 < x < y < 3\}$. Then C is bounded in $\mathbb{R}^2$.

Solution. Note that C is the upper left triangle formed by the square of 3 units, excluding the line $y = x$ and excluding the line $y = 3$ and excluding the y axis.

It is obvious that C is bounded. We can pick any $R \geq 3\sqrt{2}$. □

Proof. To prove C is bounded, we must prove $(\exists R > 0)(\forall x \in C)(||x|| \leq R)$.

Let $R = 3\sqrt{2}$. We must prove $||(x, y)|| \leq 3\sqrt{2}$ for all $(x, y) \in C$.

Let $(x, y) \in C$. Then $0 < x < y < 3$. Since $0 < x < y$, then $0 < x^2 < y^2$. Since $0 < y < 3$, then $y^2 < 9$. Thus, $x^2 < y^2 < 9$, so $x^2 < 9$. Adding inequalities we obtain $x^2 + y^2 < 18$.

Since $x^2 > 0$ and $y^2 > 0$, then $x^2 + y^2 > 0$.

Thus, $0 < x^2 + y^2 < 18$, so $\sqrt{x^2 + y^2} < \sqrt{18}$. Hence, $\sqrt{x^2 + y^2} \leq \sqrt{18}$. Thus,

$$\begin{aligned} ||(x, y)|| &= \sqrt{x^2 + y^2} \\ &\leq \sqrt{18} \\ &= 3\sqrt{2}. \end{aligned}$$

Hence, $||(x, y)|| \leq 3\sqrt{2}$, so C is bounded. □

Exercise 6. Let $S = \{(x, y, z) \in \mathbb{R}^3 : x^2 + 2y^2 + 4z^2 < 4\}$. Then S is bounded in $\mathbb{R}^3$.

Solution. Consider the equation $x^2 + 2y^2 + 4z^2 = 4$. In standard form we have $\frac{1}{4}x^2 + \frac{1}{2}y^2 + z^2 = 1$, which is the equation of an ellipsoid, a quadric surface. Hence, S consists of the solid ellipsoid excluding its surface. Clearly, a graph of S, as plotted by Maple or other software such as Apple Grapher, shows that S is bounded. It has axes at $2, \sqrt{2}, 1$. □

Proof. To prove S is bounded, we must prove $(\exists R > 0)(\forall x \in S)(||x|| \leq R)$.

Let $R = \sqrt{5}$. We must prove $||(x, y, z)|| \leq \sqrt{5}$ for all $(x, y, z) \in S$.

Let $(x, y, z) \in S$. Then $x^2 + 2y^2 + 4z^2 < 4$.

Since $x^2 \geq 0, y^2 \geq 0, z^2 \geq 0$, then $x^2 + y^2 + z^2 \geq 0$ and $y^2 + 3z^2 \geq 0$. Since $x^2 + 2y^2 + 4z^2 < 4$, then $x^2 + 2y^2 + 4z^2 \leq 4$. Since $y^2 + 3z^2 \geq 0 > -1$, then $y^2 + 3z^2 > -1$, so $y^2 + 3z^2 \geq -1$. Hence, $-y^2 - 3z^2 \leq 1$. We add the inequalities $x^2 + 2y^2 + 4z^2 \leq 4$ and $-y^2 - 3z^2 \leq 1$ to get $x^2 + y^2 + z^2 \leq 5$.

Since $0 \leq x^2 + y^2 + z^2 \leq 5$, then $\sqrt{x^2 + y^2 + z^2} \leq \sqrt{5}$. Therefore, $||(x, y, z)|| = \sqrt{x^2 + y^2 + z^2} \leq \sqrt{5}$.

Hence, $||(x, y, z)|| \leq \sqrt{5}$, so S is bounded. □

Exercise 7. If A and B are bounded subsets of $\mathbb{R}^n$, then $A \cap B$ is bounded.

Proof. Suppose A and B are bounded subsets of $\mathbb{R}^n$. Since $A \cap B \subset A$ and $A \subset \mathbb{R}^n$, then $A \cap B \subset \mathbb{R}^n$.

To prove $A \cap B$ is bounded, we must find a positive real r for which $A \cap B \subset \overline{B_r}(0)$.

Let $x \in A \cap B$. Then $x \in A$ and $x \in B$. Since A is bounded, then there is a positive real a such that $A \subset \overline{B_a}(0)$. Since $x \in A$ and $A \subset \overline{B_a}(0)$, then $x \in \overline{B_a}(0)$, so $||x|| \leq a$. Since B is bounded, then there is a positive real b such that $B \subset \overline{B_b}(0)$. Since $x \in B$ and $B \subset \overline{B_b}(0)$, then $x \in \overline{B_b}(0)$, so $||x|| \leq b$.

Let $r = a + b$. The sum of any two positive real numbers is positive. Since $a > 0$ and $b > 0$, then $r = a + b > 0$.

To prove $x \in \overline{B_r}(0)$, we must prove $||x|| \leq r$.

Observe that

$$\begin{aligned} 0 < b &\Rightarrow a < a + b \\ &\Rightarrow a < r \\ &\Rightarrow a \leq r. \end{aligned}$$

Since $||x|| \leq a$ and $a \leq r$, then $||x|| \leq r$. Hence, $x \in \overline{B_r}(0)$, so $A \cap B \subset \overline{B_r}(0)$. ☐

Exercise 8. If A and B are bounded subsets of $\mathbb{R}^n$, then $A \cup B$ is bounded.

Proof. Suppose A and B are bounded subsets of $\mathbb{R}^n$. Since $A \subset \mathbb{R}^n$ and $B \subset \mathbb{R}^n$, then $A \cup B \subset \mathbb{R}^n$.

To prove $A \cup B$ is bounded, we must find a positive real r for which $A \cup B \subset \overline{B_r}(0)$.

Since A is bounded, then there is a positive real a such that $A \subset \overline{B_a}(0)$. Since B is bounded, then there is a positive real b such that $B \subset \overline{B_b}(0)$.

Let $r = \max\{a, b\}$. Since $a > 0$ and $b > 0$, then $r > 0$.

Let $x \in A \cup B$. Then either $x \in A$ or $x \in B$.

We consider these cases separately.

Case 1: Suppose $x \in A$.

Since $A \subset \overline{B_a}(0)$, then $x \in \overline{B_a}(0)$. Hence, $||x|| \leq a$. Since $r = \max\{a, b\}$, then $a \leq r$. Thus, $||x|| \leq a$ and $a \leq r$, so $||x|| \leq r$.

Case 2: Suppose $x \in B$.

Since $B \subset \overline{B_b}(0)$, then $x \in \overline{B_b}(0)$. Hence, $||x|| \leq b$. Since $r = \max\{a, b\}$, then $b \leq r$. Thus, $||x|| \leq b$ and $b \leq r$, so $||x|| \leq r$.

Hence, in all cases, $||x|| \leq r$.

Therefore, $x \in \overline{B_r}(0)$, so $A \cup B \subset \overline{B_r}(0)$. ☐

Exercise 9. Any finite subset of $\mathbb{R}^n$ is bounded.

Proof. Let S be a finite subset of $\mathbb{R}^n$.

Either S is empty or not.

We consider these cases separately.

Case 1: Suppose $S = \emptyset$.

The empty set is bounded, so S is bounded.

Case 2: Suppose $S \neq \emptyset$.

Then there is at least one point of $\mathbb{R}^n$ in S. Hence, there is a positive integer n such that $S = \{p_1, ..., p_n\}$ and each $p_i \in \mathbb{R}^n$.

To prove S is bounded, we must prove there exists $R > 0$ such that $||x|| \leq R$ for all $x \in S$.

Let $p_i \in S$ for some integer i such that $1 \leq i \leq n$.

We must prove there exists $R > 0$ such that $||p_i|| \leq R$.

Let $m = \max\{||p_i|| : p_i \in S\}$. Then $m \geq ||p_i||$ for all $p_i \in S$. In particular, $m \geq ||p_i||$.

Let $R = m + 1$.

We prove $||p_i|| \leq R$.

Clearly,

$$\begin{aligned} ||p_i|| &\leq m \\ &\leq m+1 \\ &= R. \end{aligned}$$

Hence, $||p_i|| \leq R$.

We prove $R > 0$.

Since the norm of any vector is nonnegative, then in particular, $||p_i|| \geq 0$. Thus, $m \geq ||p_i||$ and $||p_i|| \geq 0$, so $m \geq 0$. Hence,

$$\begin{aligned} 0 &\leq m \\ &< m+1 \\ &= R. \end{aligned}$$

Hence, $0 < R$, so $R > 0$.

Therefore, S is bounded.

Hence, in all cases, S is bounded. ☐

Exercise 10. Let $S \subset \mathbb{R}^d$. Then the statement S is unbounded is equivalent to the statement there exists a sequence $p_1, p_2, p_3, ...$ of points in S such that $||p_n|| \geq n$ for all $n \in \mathbb{N}$.

Proof. We first prove if S is unbounded, then there exists a sequence $p_1, p_2, p_3, ...$ of points in S such that $||p_n|| \geq n$ for all $n \in \mathbb{N}$.

Suppose S is unbounded. Then for all real $r > 0$ there exists $x \in S$ such that $||x|| > r$.

Let $n \in \mathbb{N}$. Then $n > 0$ and $n \in \mathbb{R}$. Hence, there exists $x \in S$ such that $||x|| > n$. Let $x = p_n$. Then there exists $p_n \in S$ such that $||p_n|| > n$, so there exists $p_n \in S$ such that $||p_n|| \geq n$. Since n is arbitrary, then there exists $p_n \in S$ such that $||p_n|| \geq n$ for all $n \in \mathbb{N}$.

Define $f : \mathbb{N} \to S$ by $f(n) = p_n$ such that $||p_n|| \geq n$ for all $n \in \mathbb{N}$. Then f is a function with domain $\mathbb{N}$, so f is a sequence of points p_n of S such that $||p_n|| \geq n$ for all $n \in \mathbb{N}$. Therefore there exists a sequence $p_1, p_2, p_3, ...$ of points of S such that $||p_n|| \geq n$ for all $n \in \mathbb{N}$.

Conversely, we prove if there exists a sequence $p_1, p_2, p_3, ...$ of points in S such that $||p_n|| \geq n$ for all $n \in \mathbb{N}$, then S is unbounded.

Let (p_n) be a sequence of points in S such that $||p_n|| \geq n$ for all $n \in \mathbb{N}$.

To prove S is unbounded, we must prove for each $R > 0$ there is at least one point of S whose norm exceeds R. Let $R > 0$. By the Archimedean property of $\mathbb{R}$, to each real number R there corresponds a natural number n greater than R. Hence, choose $n \in \mathbb{N}$ such that $n > R$. Then there exists a point $p_n \in S$ such that $||p_n|| \geq n$. Thus, $||p_n|| \geq n$ and $n > R$, so $||p_n|| > R$.

Therefore, there exists $p_n \in S$ such that $||p_n|| > R$. Hence, S is unbounded. $\square$

Exercise 11. Show that a subset S of $\mathbb{R}$ is bounded iff S is bounded below and bounded above in $\mathbb{R}$.

Proof. Let S be a subset of $\mathbb{R}$.

We prove if S is bounded, then S is bounded below and bounded above in $\mathbb{R}$.

Suppose S is bounded. Then there exists $R > 0$ such that $|x| \leq R$ for each $x \in S$.

Let $x \in S$. Then $|x| \leq R$, so $-R \leq x \leq R$. Thus, $-R \leq x$ and $x \leq R$.

Since x is arbitrary, then $x \leq R$ for all $x \in S$. Hence, R is an upper bound of S in $\mathbb{R}$. Therefore, S is bounded above in $\mathbb{R}$.

Since $R \in \mathbb{R}$, then $-R \in \mathbb{R}$. Since $-R \leq x$, then $-R \leq x$ for all $x \in S$. Hence, $-R$ is a lower bound of S in $\mathbb{R}$. Therefore, S is bounded below in $\mathbb{R}$.

We prove if S is bounded below and bounded above in $\mathbb{R}$, then S is bounded.

Suppose S is bounded below and bounded above in $\mathbb{R}$. To prove S is bounded, let $x \in S$.

We must prove there exists $R > 0$ such that $|x| \leq R$.

Since S is bounded above, then there exists $b \in \mathbb{R}$ such that $x \leq b$ for each $x \in S$. Hence, $x \leq b$. Since S is bounded below, then there exists $a \in \mathbb{R}$ such that $a \leq x$ for each $x \in S$. Hence, $a \leq x$.

Let $M = \max\{|a|, |b|\}$. Choose $R = M + 1$.

We prove $R > 0$.

Since M is the max of $|a|$ and $|b|$, then $M \geq |a|$ and $M \geq |b|$. By definition of absolute value, $|a| \geq 0$ and $|b| \geq 0$. Hence, $M \geq |a| \geq 0$ and $M \geq |b| \geq 0$, so $M \geq 0$. Thus, $0 \leq M < M + 1 = R$, so $0 < R$. Therefore, $R > 0$.

We prove $|x| \leq R$.

Since $|a| \leq M$, then $-|a| \geq -M$.

Observe that

$$\begin{aligned}
-R &= -(M+1) \\
&= -M-1 \\
&\leq -M \\
&\leq -|a| \\
&\leq a \\
&\leq x \\
&\leq b \\
&\leq |b| \\
&\leq M \\
&\leq M+1 \\
&= R.
\end{aligned}$$

Therefore, $-R \leq x$ and $x \leq R$, so $|x| \leq R$. ∎

Exercise 12. If S_1 and S_2 are bounded subsets of $\mathbb{R}$, then $S_1 \times S_2$ is bounded in $\mathbb{R}^2$.

Proof. Suppose S_1 and S_2 are each bounded subsets of $\mathbb{R}$.

Since S_1 is bounded in $\mathbb{R}$, then there exists $M_1 > 0$ such that $|x| \leq M_1$ for all $x \in S_1$. Since S_2 is bounded in $\mathbb{R}$, then there exists $M_2 > 0$ such that $|x| \leq M_2$ for all $x \in S_2$.

To prove $S_1 \times S_2$ is bounded in $\mathbb{R}^2$, we must prove there exists $R > 0$ such that $|(x,y)| \leq R$ for all $(x,y) \in S_1 \times S_2$.

Let $R = \sqrt{M_1^2 + M_2^2}$.

We prove $R > 0$.

Since $M_1 > 0$, then $M_1^2 > 0$. Since $M_2 > 0$, then $M_2^2 > 0$. Hence, $M_1^2 + M_2^2 > 0$. Therefore, $R = \sqrt{M_1^2 + M_2^2} > 0$.

Let $(x,y) \in S_1 \times S_2$. Then $x \in S_1$ and $y \in S_2$. Since $x \in S_1$, then $|x| \leq M_1$. Thus, $0 \leq |x| \leq M_1$, so $0 \leq |x|^2 \leq M_1^2$. Since $y \in S_2$, then $|y| \leq M_2$. Thus, $0 \leq |y| \leq M_2$, so $0 \leq |y|^2 \leq M_2^2$.

Adding the inequalities $0 \leq |x|^2 \leq M_1^2$ and $0 \leq |y|^2 \leq M_2^2$, we obtain $0 \leq |x|^2 + |y|^2 \leq M_1^2 + M_2^2$. Hence, $0 \leq \sqrt{|x|^2 + |y|^2} \leq \sqrt{M_1^2 + M_2^2}$.

Therefore,

$$\begin{aligned}
|(x,y)| &= \sqrt{|x|^2 + |y|^2} \\
&\leq \sqrt{M_1^2 + M_2^2} \\
&= R.
\end{aligned}$$

Thus, $|(x,y)| \leq R$. Hence, there exists $R > 0$ such that $|(x,y)| \leq R$. Therefore, $S_1 \times S_2$ is bounded in $\mathbb{R}^2$. ∎

Open subsets of $\mathbb{R}^n$

Exercise 13. The number 1 is not an interior point of the closed interval $[0, 1]$.

Proof. To prove 1 is not an interior point of $[0, 1]$ in $\mathbb{R}$, we prove $(\forall r > 0)(\exists y)(|y-1| < r \wedge y \notin [0, 1])$. Let $r > 0$. We must find a real number y such that $|y-1| < r$ and $y \notin [0, 1]$. Let y be the real number $1 + \frac{r}{2}$.

Then

$$\begin{aligned} |y-1| &= |(1+\frac{r}{2})-1| \\ &= |\frac{r}{2}| \\ &= \frac{r}{2} \\ &< r. \end{aligned}$$

Observe that

$$\begin{aligned} r > 0 &\Rightarrow \frac{r}{2} > 0 \\ &\Leftrightarrow 1 + \frac{r}{2} > 1 \\ &\Leftrightarrow y > 1. \end{aligned}$$

Hence, $y \notin [0, 1]$. □

Exercise 14. Every point of $\mathbb{R}$ is an interior point of $\mathbb{R}$.

Proof. Let p be an arbitrary point of $\mathbb{R}$. To prove p is an interior point, we must prove there exists $r > 0$ such that $B_r(p) \subset \mathbb{R}$.

Let $r = 1$. Then $B_1(p) = \{y \in \mathbb{R} : |y - p| < 1\} \subset \mathbb{R}$.

Therefore, p is an interior point of $\mathbb{R}$. □

Exercise 15. For every pair of real numbers x and y with $x \neq y$, there are open intervals U and V such that $x \in U$ and $y \in V$ and $U \cap V = \emptyset$.

Proof. Let $x, y \in \mathbb{R}$ such that $x \neq y$. Then either $x < y$ or $x > y$.

We must prove there exist open intervals U and V such that $x \in U$ and $y \in V$ and $U \cap V = \emptyset$.

Without loss of generality, we may assume $x < y$.

Let $U = (\frac{3x-y}{2}, \frac{x+y}{2})$. Let $V = (\frac{x+y}{2}, \frac{3y-x}{2})$.

To prove $x \in U$, we must prove $\frac{3x-y}{2} < x$ and $x < \frac{x+y}{2}$. Observe that

$$\begin{aligned} x < y &\Rightarrow 3x < 2x + y \\ &\Leftrightarrow 3x - y < 2x \\ &\Leftrightarrow \frac{3x-y}{2} < x. \end{aligned}$$

and

$$\begin{aligned} x < y \quad &\Rightarrow \quad 2x < x + y \\ &\Leftrightarrow \quad x < \frac{x+y}{2}. \end{aligned}$$

Therefore, $\frac{3x-y}{2} < x$ and $x < \frac{x+y}{2}$, as desired.

To prove $y \in V$, we must prove $\frac{x+y}{2} < y$ and $y < \frac{3y-x}{2}$. Observe that

$$\begin{aligned} x < y \quad &\Rightarrow \quad x + y < 2y \\ &\Leftrightarrow \quad \frac{x+y}{2} < y. \end{aligned}$$

and

$$\begin{aligned} x < y \quad &\Rightarrow \quad x + 2y < 3y \\ &\Leftrightarrow \quad 2y < 3y - x \\ &\Leftrightarrow \quad y < \frac{3y-x}{2}. \end{aligned}$$

Therefore, $\frac{x+y}{2} < y$ and $y < \frac{3y-x}{2}$, as desired.

To prove $U \cap V = \emptyset$, suppose $U \cap V \neq \emptyset$. Then there is at least one element in $U \cap V$. Let r be an element of $U \cap V$. Then $r \in U$ and $r \in V$. Since $r \in U$, then $\frac{3x-y}{2} < r$ and $r < \frac{x+y}{2}$. Since $r \in V$, then $\frac{x+y}{2} < r$ and $r < \frac{3y-x}{2}$.

Thus, we have $r < \frac{x+y}{2}$ and $r > \frac{x+y}{2}$, a violation of trichotomy of $\mathbb{R}$. Hence, $U \cap V = \emptyset$. □

Exercise 16. Suppose p and q are prime numbers with $p \neq q$. Prove that $\sqrt[3]{pq}$ is irrational.

Solution. To get a contradiction, suppose that $\sqrt[3]{pq}$ is rational. Then we may write $\sqrt[3]{pq} = a/b$ with $a, b \in \mathbb{Z}$ and $b > 0$. Cubing both sides results in $pq = a^3/b^3$ and so

$$a^3 = pqb^3. \tag{1}$$

Now think about the number of times that p appears in the prime factorization of each side of (**??**). On the left hand side, the number of times that p appears in the prime factorization of a^3 is three times the number of times p appears in the prime factorization of a. In particular the number of times that p appears in the prime factorization of (**??**) is a multiple of 3. Similar reasoning shows that the number of times that p appears in the prime factorization of b^3 is a multiple of 3, and so the number of times that p appears in the prime factorization of pqb^3 is one more than a multiple of 3. This shows that the number of times that p appears in the prime factorization of (**??**) is both a multiple of 3 and one more than a multiple of 3, a contradiction. □

Exercise 17. Prove that for every $n \in \mathbb{Z}^+$

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} \leq 2 - \frac{1}{n}.$$

Limits and Continuity

Exercise 18. Show that $\lim_{x\to 0^+} \sin\frac{1}{x} \neq 0$.

Solution. To prove this, we use the epsilon delta definition of right hand limit. We know $L = \lim_{x\to a^+} f(x)$ iff $(\forall\epsilon > 0)(\exists\delta > 0)(\forall x)(a < x < a + \delta \to |f(x) - L| < \epsilon)$.

Thus, $L \neq \lim_{x\to a^+} f(x)$ iff $(\exists\epsilon > 0)(\forall\delta > 0)(\exists x)(a < x < a+\delta \wedge |f(x)-L| \geq \epsilon)$.

Hence, we must prove $(\exists\epsilon > 0)(\forall\delta > 0)(\exists x)(0 < x < \delta \wedge |\sin\frac{1}{x}| \geq \epsilon)$.

We know that max $|\sin\frac{1}{x}|$ is 1, so, if we want $|\sin\frac{1}{x}|$ to be greater than ϵ, then ϵ needs to be no larger than 1. Hence, $0 < \epsilon \leq 1$. To find $1/x$ such that $|\sin\frac{1}{x}| \geq 1/2$,we work backwards. We ask what values for $1/x$ will cause $|\sin\frac{1}{x}|$ to be greater than $1/2$? We can try values such that $|\sin\frac{1}{x}| = 1$, which certainly satisfies being greater than one-half. We know $|\sin\theta| = 1$ iff θ is an odd multiple of $\frac{\pi}{2}$, so $\theta = \frac{\pi}{2} + 2\pi k = \frac{4\pi k+\pi}{2} = \frac{\pi(4k+1)}{2}$. We know $\sin\theta = -1$ iff $\theta = \frac{3\pi}{2} + 2\pi k$. □

Proof. Let ϵ be a positive real number less than or equal to 1, say $\epsilon = \frac{1}{2}$. Let $\delta > 0$ be arbitrary. We must prove there exists $x \in \mathbb{R}$ such that $0 < x < \delta$ and $|\sin\frac{1}{x}| \geq \frac{1}{2}$. To each δ there corresponds $k \in \mathbb{Z}$ such that $\frac{2}{(4k+1)\pi} < \delta$ and $\frac{2}{(4k+3)\pi} < \delta$. We consider these cases separately.

Let $x \in \mathbb{R}$ such that $0 < x < \delta$.

Case 1: Suppose $x = \frac{2}{(4k+1)\pi}$.

Then $|\sin\frac{1}{x}| = |\sin\frac{(4k+1)\pi}{2}| = |\sin 2\pi k + \pi/2| = 1 > 1/2$.

Case 2: Suppose $x = \frac{2}{(4k+3)\pi}$.

Then $|\sin\frac{1}{x}| = |\sin\frac{(4k+3)\pi}{2}| = |\sin 2\pi k + 3\pi/2| = 1 > 1/2$. □

Exercise 19. Prove $\lim_{x\to 0^+} x\sin\frac{1}{x} = 0$.

Solution.

To prove $\lim_{x\to 0^+} x\sin\frac{1}{x} = 0$, we must show that

$(\forall\epsilon > 0)(\exists\delta > 0)(\forall x)[a < x < a + \delta \to |f(x) - L| < \epsilon]$.

So, we let $\epsilon > 0$ be arbitrary.

In this case, $a = 0, L = 0$ and $f(x) = x\sin\frac{1}{x}$.

We must find some $\delta > 0$ in terms of ϵ such that, for every x, if $a < x < a+\delta$, then $|f(x) - L| < \epsilon$. Thus, we must find $\delta > 0$ in terms of ϵ such that, for every x, if $0 < x < \delta$, then $|x\sin\frac{1}{x}| < \epsilon$.

We want $|x - a| = |x - 0| = |x| < \delta$ and $|x\sin\frac{1}{x} - 0| = |x\sin\frac{1}{x}| < \epsilon$.

We work backwards. Suppose $0 < x < \delta$ and $|x\sin\frac{1}{x}| < \epsilon$. Then $|x||\sin\frac{1}{x}| < \epsilon$. Since $0 < x < \delta$, then $x = |x|$, so $0 < |x| < \delta$. Hence, $|x| < \delta$.

If we solve for $|x|$, then $|x| < \epsilon/|\sin\frac{1}{x}|$.

Thus, we let $\delta = \epsilon/|\sin\frac{1}{x}|$.

Unfortunately, δ depends on ϵ, not on x. To eliminate this dependency on x we find some $\delta_1 > 0$ such that $x \in (a, a + \delta_1)$ and find some $M > 0$ such that $|\sin\frac{1}{x}| \leq M$.

Note that $(a, a+\delta_1) = (0, 0+\delta_1) = (0, \delta_1)$. Hence, $x \in (0, \delta_1)$ implies $0 < x < \delta_1$. Hence, we must find some $\delta_1 > 0$ such that $0 < x < \delta_1$ and some $M > 0$ such that $|\sin \frac{1}{x}| \leq M$.

If we can find δ_1 and M, then we could argue:

If $|\sin \frac{1}{x}| < M$, then since $x > 0$, then $|x| = x > 0$, so we could multiply to get $|x||\sin \frac{1}{x}| < |x|M$. Thus, $|x \sin \frac{1}{x}| < |x|M$.

To prove $|x \sin \frac{1}{x}| < \epsilon$, we must establish that $|x|M < \epsilon$. This means we need $|x| < \epsilon/M$, so we need $x < \epsilon/M$.

In order to satisfy both $x < \delta_1$ and $x < \epsilon/M$, we choose δ to be the smaller of δ_1 and ϵ/M. Therefore, we let $\delta = \min(\delta_1, \epsilon/M)$.

We first must find δ_1 and determine M.

We know that $|\sin\theta| \leq 1$ for all $\theta \in \mathbb{R}$. Since $x > 0$, then $1/x \neq 0$, so $|\sin 1/x| \leq 1$ for all $x > 0$. Thus, the choice of δ_1 does not matter. We can let δ_1 be any positive value we like. For convenience, let's choose $\delta_1 = \epsilon$. Then $M = 1$.

We now let $\delta = \min(\delta_1, \epsilon/M)$, so $\delta = \min(\epsilon, \epsilon/1) = \min(\epsilon, \epsilon) = \epsilon$. □

Proof. Let $\epsilon > 0$ be given. We must find $\delta > 0$ such that for every x, if $0 < x < \delta$, then $|x \sin \frac{1}{x}| < \epsilon$.

Let $\delta = \epsilon$. Since $\epsilon > 0$, then $\delta > 0$.

Suppose x is an arbitrary real number such that $0 < x < \delta$. We must prove $|x \sin \frac{1}{x}| < \epsilon$.

Since $x > 0$, then $x \neq 0$. Therefore, $|\sin 1/x| \leq 1$ for all $x > 0$.

Observe that

$$\begin{aligned} |x \sin \frac{1}{x}| &= |x||\sin \frac{1}{x}| \\ &\leq |x| \\ &= x \\ &< \delta \\ &= \epsilon \text{ , as desired.} \end{aligned}$$

□

Exercise 20. Let $f_1(x), f_2(x), ..., f_n(x)$ be real valued functions such that $\lim_{x\to a} f_i(x)$ exists for each $i = 1, 2, ..., n$.

Then $\lim_{x\to a} \sum_{i=1}^{n} f_i(x)$ exists and $\lim_{x\to a} \sum_{i=1}^{n} f_i(x) = \sum_{i=1}^{n} \lim_{x\to a} f_i(x)$.

Solution.

There are two statements to prove:

1. $\lim_{x\to a} \sum_{i=1}^{n} f_i(x)$ exists

2. $\lim_{x\to a} \sum_{i=1}^{n} f_i(x) = \sum_{i=1}^{n} \lim_{x\to a} f_i(x)$.

To prove 1:

Define predicate $p(n) : \lim_{x\to a} \sum_{i=1}^{n} f_i(x)$ exists.

Let S be the truth set of $p(n)$.

To prove $(\forall n \in \mathbb{N})[p(n)]$, we must prove $S = \mathbb{N}$.

We prove by induction.

Thus, we must prove:

1a. $1 \in S$, so we must prove $p(1)$ is true.

1b. $(\forall m \in \mathbb{N})(m \in S \to m+1 \in S)$.

Let S be the truth set of $p(n)$.

To prove $(\forall n \in \mathbb{N})[p(n)]$, we must prove $S = \mathbb{N}$.

We prove by induction.

Thus, we must prove:

1. $1 \in S$. Thus,we must prove $p(1)$ is true.

2. $(\forall m \in \mathbb{N})(m \in S \to m+1 \in S)$. To prove, we assume $m \in \mathbb{N}$ is arbitrary such that $m \in S$. To prove $m+1 \in S$, we must prove $p(m+1)$ is true. Thus, we must prove $\lim_{x\to a} \sum_{i=1}^{m} f_i(x)$ exists.

We note that this is the familiar calculus rule that the limit of a sum equals the sum of the limits, but in a generalized form. □

Proof. **Existence:**

We first prove $\lim_{x\to a} \sum_{i=1}^{n} f_i(x)$ exists for every natural number n.

Let S be the truth set of $p(n) : \lim_{x\to a} \sum_{i=1}^{n} f_i(x)$ exists.

We prove $S = \mathbb{N}$ by induction.

Basis:

Clearly, $1 \in S$, since $\lim_{x\to a} \sum_{i=1}^{1} f_i(x) = \lim_{x\to a} f_1(x)$ and $\lim_{x\to a} f_1(x)$ exists.

Induction:

Suppose $m \in S$.

To prove $m+1 \in S$, we must prove $\lim_{x\to a} \sum_{i=1}^{m+1} f_x(x)$ exists.

Since $m \in S$, then $\lim_{x\to a} \sum_{i=1}^{m} f_i(x)$ exists.

Observe that

$$\begin{aligned} \lim_{x\to a} \sum_{i=1}^{m+1} f_i(x) &= \lim_{x\to a} [\sum_{i=1}^{m} f_i(x) + f_{m+1}(x)] \\ &= \lim_{x\to a} \sum_{i=1}^{m} f_i(x) + \lim_{x\to a} f_{m+1}(x) \end{aligned}$$

By hypothesis, $\lim_{x\to a} f_{m+1}(x)$ exists.

By the inductive hypothesis, $\lim_{x\to a} \sum_{i=1}^{m} f_i(x)$ exists.

Therefore, $\lim_{x\to a} \sum_{i=1}^{m+1} f_i(x)$ exists, as desired.

We now prove $\lim_{x\to a} \sum_{i=1}^{n} f_i(x) = \sum_{i=1}^{n} \lim_{x\to a} f_i(x)$ for every $n \in \mathbb{N}$.

Let T be the truth set of $q(n) : \lim_{x\to a} \sum_{i=1}^{n} f_i(x) = \sum_{i=1}^{n} \lim_{x\to a} f_i(x)$.

To prove $T = \mathbb{N}$, we use induction.

Basis:

Clearly, $1 \in T$, since $\lim_{x\to a} \sum_{i=1}^{1} f_i(x) = \lim_{x\to a} f_1(x) = \sum_{i=1}^{1} \lim_{x\to a} f_i(x)$.

Induction:

Suppose $m \in T$.

To prove $m+1 \in T$, we must prove $\lim_{x\to a} \sum_{i=1}^{m+1} f_i(x) = \sum_{i=1}^{m+1} \lim_{x\to a} f_i(x)$.

Since $m \in T$, then $\lim_{x\to a} \sum_{i=1}^{m} f_i(x) = \sum_{i=1}^{m} \lim_{x\to a} f_i(x)$.

Observe that

$$\begin{aligned}
\lim_{x\to a}\sum_{i=1}^{m+1} f_i(x) &= \lim_{x\to a}[\sum_{i=1}^{m} f_i(x) + f_{m+1}(x)] \\
&= \lim_{x\to a}\sum_{i=1}^{m} f_i(x) + \lim_{x\to a} f_{m+1}(x) \\
&= \sum_{i=1}^{m}\lim_{x\to a} f_i(x) + \lim_{x\to a} f_{m+1}(x) \\
&= \sum_{i=1}^{m+1}\lim_{x\to a} f_i(x) \text{ , as desired.}
\end{aligned}$$

□

Exercise 21. Prove $\lim_{x\to 3}(4x+7) = 19$.

Solution. The conclusion is $\lim_{x\to 3}(4x+7) = 19$.

To prove $\lim_{x\to 3}(4x+7) = 19$, we must show that
$(\forall\epsilon > 0)(\exists\delta > 0)(\forall x)[0 < |x-a| < \delta \to |f(x) - L| < \epsilon]$.

So, we let $\epsilon > 0$ be arbitrary.

We must find some $\delta > 0$ in terms of ϵ such that, for every x, if $0 < |x-a| < \delta$, then $|f(x) - L| < \epsilon$.

The key is to find a relationship between $|x-a|$ and $|f(x)-L|$. In this, case $|x-3|$ and $|(4x+7)-19|$. Observe that $|(4x+7)-19| = |4x-12| = |4(x-3)| = 4|x-3|$.

Since we want $|f(x) - L| < \epsilon$ and $|f(x) - L| = 4|x-a|$, then $4|x-a| < \epsilon$, so $|x-a| < \epsilon/4$.

Since we're assuming $|x-a| < \delta$, then we simply let $\delta = \epsilon/4$. □

Proof. Let $f(x) = 4x+7$. Let $\epsilon > 0$ be given. Let $\delta = \epsilon/4$.

Suppose $x \in \mathbb{R}$ such that $0 < |x-3| < \delta$. We must show that $|f(x)-19| < \epsilon$.

Observe that

$$\begin{aligned}
|f(x) - 19| &= |4x-12| \\
&= 4|x-3| \\
&< 4\delta \\
&= 4(\epsilon/4) \\
&= \epsilon \text{ , as desired.}
\end{aligned}$$

□

Exercise 22. Prove $\lim_{x\to 9}(x^2+5) = 86$.

Solution.

To prove $\lim_{x\to 9}(x^2+5) = 86$, we must show that
$(\forall\epsilon > 0)(\exists\delta > 0)(\forall x)[0 < |x-a| < \delta \to |f(x) - L| < \epsilon]$.

So, we let $\epsilon > 0$ be arbitrary.

We must find some $\delta > 0$ in terms of ϵ such that, for every x, if $0 < |x-a| < \delta$, then $|f(x) - L| < \epsilon$.

In this case, $a = 9$, $L = 86$ and $f(x) = x^2 + 5$.

We want $|x - a| = |x - 9| < \delta$

and $|f(x) - L| = |x^2 + 5 - 86| = |x^2 - 81| = |x + 9||x - 9| < \epsilon$.

We work backwards. Suppose $|x + 9||x - 9| < \epsilon$. Then $|x - 9| < \epsilon/|x + 9|$. Thus, $\delta = \epsilon/|x + 9|$.

Unfortunately, δ depends on ϵ, not on x. To eliminate this dependency on x we find some deleted δ_1 neighborhood of $a = 9$ and $M > 0$ such that $|x+9| \leq M$.

We do this because if $|x+9| < M$ and $x \neq 9$, then $|x+9||x-9| < M|x-9|$. To prove $|x + 9||x - 9| < \epsilon$, we would need to prove $M|x - 9| < \epsilon$. Thus, we would need to establish that $|x - 9| < \epsilon/M$.

If we can find such an M for some δ_1, then we have $|x - 9| < \delta_1$ and $|x - 9| < \epsilon/M$.

Thus, we choose δ such that both $|x - 9| < \delta_1$ and $|x - 9| < \epsilon/M$.

Hence, we may choose δ to be the smaller of δ_1 and ϵ/M, so let $\delta = \min(\delta_1, \epsilon/M)$.

We first must find a δ_1 neighborhood of $a = 9$ and determine M.

Let's try say $\delta_1 = 1$(because 1 is a nice number to work with, actually any number would work). We will assume $0 < |x - 9| \leq \delta_1 = 1$. We want to find M such that $|x + 9| \leq M$. Let $g(x) = |x + 9|$. We want M to be an upper bound for $g(x)$ for all x such that $0 < |x - 9| \leq 1$. Since $0 < |x - 9| \leq 1$, then $0 < |x - 9|$ and $|x - 9| \leq 1$. Hence, $x \neq 9$ and $-1 \leq x - 9 \leq 1$, so $x \neq 9$ and $8 \leq x \leq 10$. Thus, $x \in [8, 10]$ and $x \neq 9$. On the interval $[8, 10]$, g is increasing, so the absolute maximum of g occurs when $x = 10$. Thus, if $x = 10$, then $g(x) = g(10) = 10 + 9 = 19$. Therefore, let $M = 19$.

We now let $\delta = \min(\delta_1, \epsilon/M)$, so $\delta = \min(1, \epsilon/19)$. □

Proof. Let $\epsilon > 0$. We must find $\delta > 0$ such that if $0 < |x - 9| < \delta$, then $|f(x) - 86| < \epsilon$.

If $0 < |x - 9| \leq 1$, then $x \neq 9$ and $8 \leq x \leq 10$, so $|x + 9| \leq 19$. Thus, if $0 < |x - 9| < 1$, then $|x + 9| < 19$. Let $\delta = \min(1, \epsilon/19)$.

Suppose $0 < |x - 9| < \delta$. Since $\delta = \min(1, \epsilon/19)$, then $\delta \leq 1$ and $\delta \leq \epsilon/19$. Therefore, $0 < |x - 9| < 1$ and $0 < |x - 9| < \epsilon/19$. Since $0 < |x - 9| < 1$, then $|x + 9| < 19$. Since $0 < |x - 9| < \epsilon/19$, then $|x - 9| < \epsilon/19$. Observe that

$$\begin{aligned}
|f(x) - 86| &= |(x^2 + 5) - 86| \\
&= |x^2 - 81| \\
&= |x + 9||x - 9| \\
&< 19|x - 9| \\
&< 19(\epsilon/19) \\
&= \epsilon \text{ , as desired.}
\end{aligned}$$

□

Exercise 23. Prove $\lim_{x \to 0} g(x) = 0$ where $g(x) = x$ if x is rational and $g(x) = 0$ if x is irrational.

Solution.

To prove $\lim_{x \to 0} g(x) = 0$, we must prove

$(\forall \epsilon > 0)(\exists \delta > 0)(\forall x)[0 < |x - a| < \delta \to |g(x) - L| < \epsilon]$.

In this case, $a = 0, L = 0$.

Let $\epsilon > 0$ be arbitrary.

We must find some $\delta > 0$ in terms of ϵ such that, for every x, if $0 < |x| < \delta$, then $|g(x)| < \epsilon$.

We want $|x - a| = |x - 0| = |x| < \delta$ and $|g(x) - L| = |g(x)| < \epsilon$.

We work backwards. Suppose $|g(x)| < \epsilon$. Either $x \in \mathbb{Q}$ or $x \notin \mathbb{Q}$.

If $x \in \mathbb{Q}$, then $|g(x)| = |x|$. Since we want $|x| < \delta$ and $|g(x)| < \epsilon$, then let $\delta = \epsilon$.

If $x \notin \mathbb{Q}$, then $|g(x)| = |0| = 0$. Since we want $|x| < \delta$ and $|g(x)| < \epsilon$, then we can still let $\delta = \epsilon$.

Therefore, we let $\delta = \epsilon$. □

Proof. Let $\epsilon > 0$ be given. We must exhibit $\delta > 0$ such that, if $0 < |x| < \delta$, then $|g(x)| < \epsilon$. Let $\delta = \epsilon$. Suppose x is an arbitrary real number such that $0 < |x| < \delta$. If x is rational, then $|g(x)| = |x| < \delta = \epsilon$, so $|g(x)| < \epsilon$. If x is irrational, then $|g(x)| = |0| = 0 < \epsilon$, so $|g(x)| < \epsilon$. In either case we have the desired conclusion. □

Exercise 24. If $f(x) = K$ is a constant function defined on $\mathbb{R}$, then $\lim_{x \to a} f(x) = K$, for any $a \in \mathbb{R}$.

Solution. Hypothesis: $f(x) = K$ is a function defined on $\mathbb{R}$.

Conclusion: $(\forall a \in \mathbb{R})(\lim_{x \to a} f(x) = K)$.

Let $a \in \mathbb{R}$ be arbitrary. To prove $\lim_{x \to a} f(x) = K$, we must prove $(\forall \epsilon > 0)(\exists \delta > 0)(\forall x)(0 < |x - a| < \delta \to |f(x) - L| < \epsilon)$.

Here, $L = K$.

Let $\epsilon > 0$ be given. We must find $\delta > 0$ such that for every x, if $0 < |x - a| < \delta$, then $|f(x) - K| < \epsilon$.

We work backwards. Suppose there exists $\delta > 0$ such that $0 < |x - a| < \delta$ and $|f(x) - L| < \epsilon$. Since $f(x) = K$, then $|f(x) - L| < \epsilon \Rightarrow |K - K| < \epsilon \Rightarrow 0 < \epsilon$, which is a true statement. So what should δ be? Well, in this case, δ is independent of ϵ, so any positive value for δ will work. For convenience, let's choose $\delta = \epsilon$. □

Proof. Let a be an arbitrary real number. To prove $\lim_{x \to a} f(x) = K$, let $\epsilon > 0$ be given. Let $\delta = \epsilon$.

Suppose x is an arbitrary real number such that $0 < |x - a| < \delta$.

We must prove $|f(x) - K| < \epsilon$.

Observe that

$$\begin{aligned} |f(x) - K| &= |K - K| \\ &= 0 \\ &< \epsilon, \text{ as desired.} \end{aligned}$$

□

Exercise 25. If $\lim_{x\to a} f(x) = L$ and k is any real number, then $\lim_{x\to a} kf(x) = kL$.

Solution.

Hypothesis is: $\lim_{x\to a} f(x) = L$.

Conclusion is: $(\forall k \in \mathbb{R})(\lim_{x\to a} kf(x) = kL)$.

To prove $\lim_{x\to a} kf(x) = kL$, we must prove that for every positive ϵ, there corresponds a positive δ such that, for every $x \in \mathbb{R}$, if $0 < |x - a| < \delta$, then $|kf(x) - kL| < \epsilon$.

Consider the expression $|kf(x) - kL| < \epsilon$. We have $|k||f(x) - L| < \epsilon$, so $|f(x) - L| < \epsilon/|k|$, assuming $k \neq 0$.

By hypothesis, $\lim_{x\to a} f(x) = L$, so let ϵ_1 be an arbitrary positive real number. Then there is some positive δ_1 such that for every $x \in \mathbb{R}$, if $0 < |x - a| < \delta_1$, then $|f(x) - L| < \epsilon_1$.

Since we want $0 < |x - a| < \delta$ and we have $0 < |x - a| < \delta_1$, we let $\delta = \delta_1$.

Since we want $|f(x) - L| < \epsilon/|k|$ and we have $|f(x) - L| < \epsilon_1$, we let $\epsilon_1 = \epsilon/|k|$.

Now, we must consider the case when $k = 0$ separately. Suppose $k = 0$. We must prove $\lim_{x\to a} kf(x) = kL$. We know $kL = 0L = 0$ and $\lim_{x\to a} kf(x) = \lim_{x\to a} 0f(x) = \lim_{x\to a} 0 = 0$. Thus, we have $\lim_{x\to a} kf(x) = \lim_{x\to a} 0f(x) = \lim_{x\to a} 0 = 0 = 0L = kL$, as desired. □

Proof. Let k be an arbitrary real number.

Either $k = 0$ or $k \neq 0$.

We consider these cases separately.

Case 1: Suppose $k = 0$.

We must prove $\lim_{x\to a} kf(x) = kL$.

Observe that $\lim_{x\to a} kf(x) = \lim_{x\to a} 0f(x) = \lim_{x\to a} 0 = 0 = 0L = kL$, as desired.

Case 2: Suppose $k \neq 0$.

To prove $\lim_{x\to a} kf(x) = kL$, we must prove that for every positive ϵ, there corresponds a positive δ such that, for every $x \in \mathbb{R}$, if $0 < |x - a| < \delta$, then $|kf(x) - kL| < \epsilon$.

By hypothesis, $\lim_{x\to a} f(x) = L$, so let ϵ_1 be an arbitrary positive real number. Then there is some positive δ_1 such that for every $x \in \mathbb{R}$, if $0 < |x - a| < \delta_1$, then $|f(x) - L| < \epsilon_1$.

Let $\delta = \delta_1$.

Then $\delta > 0$ and for every $x \in \mathbb{R}$, if $0 < |x - a| < \delta$, then $|f(x) - L| < \epsilon_1$.

Let $\epsilon = |k|\epsilon_1$. Since $k \neq 0$ and $|k| \geq 0$, then $|k| > 0$. Since $|k| > 0$ and $\epsilon_1 > 0$, then $\epsilon > 0$.

Observe that $|f(x) - L| < \epsilon_1$ implies that $|f(x) - L| < \epsilon/|k|$. Since $|k| > 0$, then $|k||f(x) - L| < \epsilon$, so $|kf(x) - kL| < \epsilon$.

Therefore, there is some positive δ such that, for every $x \in \mathbb{R}$, if $0 < |x-a| < \delta$, then $|kf(x) - kL| < \epsilon$.

Since ϵ_1 is arbitrary and $\epsilon = |k|\epsilon_1$, then ϵ is arbitrary.

Hence, for every ϵ there is some positive δ such that, for every $x \in \mathbb{R}$, if $0 < |x - a| < \delta$, then $|kf(x) - kL| < \epsilon$.

Therefore, $\lim_{x \to a} kf(x) = kL$, as desired. □

Exercise 26. Prove that $1/2$ is an interior point of $[0, 1]$.

Solution.

We sketch out a picture of this situation. Intuitively, it appears $1/2$ is an interior point of the closed interval $[0, 1]$. Why? Because one can place a δ disk around $1/2$ and the resulting δ neighborhood stays inside $[0, 1]$.

To prove $1/2$ is an interior point of $[0, 1]$, we must prove $(\exists\delta > 0)(N(p; \delta) \subset S)$.

Thus, we must find some positive δ such that the δ neighborhood of $1/2$ is entirely contained in the closed interval $[0, 1]$. Hence, we must find a $\delta > 0$ such that $N(1/2; \delta) \subset [0, 1]$; that is, we must find some $\delta > 0$ such that $(1/2 - \delta, 1/2 + \delta) \subset [0, 1]$.

If $1/2 - \delta = 0$, then $\delta = 1/2$. If $1/2 + \delta = 1$, then $\delta = 1/2$.

Thus, the maximum δ we could choose is $\delta = 1/2$. So, any δ such that $0 < \delta \leq 1/2$ would work.

So, let $\delta = 1/2$.

This problem suggests some questions: Is every point of a closed interval an interior point? Is every closed interval an open set? □

Proof. We show there is some positive real number δ such that the δ neighborhood of $1/2$ is contained in the interval $[0, 1]$.

Let δ be the positive real number $1/2$.

The δ neighborhood of $1/2$ is $N(1/2; \delta) = (1/2-\delta, 1/2+\delta) = (1/2-1/2, 1/2+1/2) = (0, 1) \subset [0, 1]$. □

Exercise 27. Prove that 1 is not an interior point of $[0, 1]$.

Solution.

Is 1 an interior point of S?

We can draw 1 on a number line to get a visual picture of the situation to help us understand the problem.

Let $\delta > 0$.

Consider the δ neighborhood of 1.

Let $N(1; \delta)$ be the δ neighborhood of 1. Then $N(1; \delta) = (1 - \delta, 1 + \delta)$.

Observe that a portion of $(1-\delta, 1+\delta)$ lies outside the interval $[0, 1]$, namely, the interval $(1, 1 + \delta)$. Hence, any $x \in (1, 1 + \delta)$ will be in the δ neighborhood of 1, but not in $[0, 1]$. Since δ is completely arbitrary, then this means 1 cannot be an interior point.

To prove 1 is not an interior point of the interval $[0, 1]$, we must prove $\neg(\exists\delta > 0)(N(1;\delta) \subset [0,1]) \Leftrightarrow (\forall\delta > 0)(N(1;\delta \not\subset [0,1]) \Leftrightarrow (\forall\delta > 0)(\exists x \in N(1;\delta))(x \notin [0,1])$.

Let $\delta > 0$ be an arbitrary.

Let the δ neighborhood of 1, $N(1;\delta)$, be the open interval $(1-\delta, 1+\delta)$.

Since we observed that any x in the interval $(1, 1+\delta)$ works, we may choose the midpoint of $(1, 1+\delta)$.

Thus, $x = (2+\delta)/2 = 1 + \delta/2$.

We prove $x \in N(1;\delta)$.

How do we prove $x \in N(1;\delta)$?

We work backwards. Suppose $x \in N(1;\delta)$. Then $x \in N(1;\delta) = (1-\delta, 1+\delta)$, so $1-\delta < x < 1+\delta$. Hence, $1-\delta < 1+\delta/2 < 1+\delta$. Subtracting 1 we obtain $-\delta < \delta/2 < \delta$. Since $\delta > 0$, then $-1 < 1/2 < 1$, which is true.

Thus, we reverse these steps in the proof. Observe that

$$\begin{aligned} -1 &< 1/2 < 1 \\ -\delta &< \delta/2 < \delta \\ 1-\delta &< 1+\delta/2 < 1+\delta \\ 1-\delta &< x < 1+\delta. \end{aligned}$$

Therefore, $x \in (1-\delta, 1+\delta) = N(1;\delta)$, as desired.

We now prove $x \notin [0,1]$.

How do we prove $x \notin [0,1]$?

We work backwards. Suppose $x \notin [0,1]$. Then $\neg(x \in [0,1]) \Leftrightarrow \neg(0 \le x \wedge x \le 1) \Leftrightarrow (0 > x \vee x > 1)$. Since x is positive, we prove $x > 1$.

How do we prove $x > 1$ We work backwards.

Suppose $x > 1$. Then $1 + \delta/2 > 1$, so $\delta/2 > 0$. Hence, $\delta > 0$, which is true by assumption.

Thus we reverse these steps in the proof. Observe that

$$\begin{aligned} \delta &> 0 \\ \delta/2 &> 0 \\ 1+\delta/2 &> 1 \\ x &> 1. \end{aligned}$$

□

Proof. To prove 1 is not an interior point of the interval $[0, 1]$, we must prove the δ neighborhood of 1 is not contained in $[0, 1]$ for every positive real number δ.

Let δ be an arbitrary positive real number.

Let the δ neighborhood of 1, $N(1;\delta)$, be the open interval $(1-\delta, 1+\delta)$.

To prove $N(1;\delta)$ is not contained in $[0, 1]$, we must show there is some real number x that is in $N(1;\delta)$ and not in $[0, 1]$.

Let x be the real number $1 + \delta/2$.

Observe that

$$\begin{aligned} -1 &< 1/2 < 1 \\ -\delta &< \delta/2 < \delta \\ 1-\delta &< 1+\delta/2 < 1+\delta \\ 1-\delta &< x < 1+\delta. \end{aligned}$$

Therefore, $x \in (1-\delta, 1+\delta) = N(1;\delta)$, as desired.

Observe that

$$\begin{aligned} \delta &> 0 \\ \delta/2 &> 0 \\ 1+\delta/2 &> 1 \\ x &> 1. \end{aligned}$$

Since $x > 1$, then $x \notin [0,1]$, as desired. $\square$

Exercise 28. Does $\mathbb{N}$ have any interior points? Is $\mathbb{N}$ an open set?

Solution.

Since $\mathbb{N} = \{1, 2, 3, ...\} = \{n : n \in \mathbb{N}\}$, choose an arbitrary natural number, say $n \in \mathbb{N}$.

Is n an interior point of $\mathbb{N}$?

We can draw n on a number line to get a visual picture of the situation to help us understand the problem.

Consider the δ neighborhood of n with various values for $\delta > 0$, say, in decreasing order. For example, let's choose δ from the set $\{5, 3, 1, 1/2, 1/4, 1/8, 2/19\}$.

Let the δ neighborhood of n be $N(n;\delta) = (n-\delta, n+\delta)$.

We can devise a table with columns for δ, $N(n;\delta)$, and decide whether $N(n;\delta) \subset \mathbb{N}$ and state a reason why.

Set $\delta = 5$. Then $N(n;\delta) = (n-5, n+5)$. We observe that $(n-5, n+5)$ can be decomposed into 10 open intervals $(n+k, n+k+1)$, where $k = -5$ to 4. We arbitrarily choose one of these open intervals $(n+k, n+k+1)$ for some $k \in \mathbb{Z}$ such that $-5 \leq k < 5$. We can choose any number in this chosen interval, say $x \in (n+k, n+k+1)$, and know that x is not a natural number. Therefore, $N(n;5) \not\subset \mathbb{N}$.

Set $\delta = 3$. Then $N(n;\delta) = (n-3, n+3)$. We observe that $(n-3, n+3)$ can be decomposed into 6 open intervals $(n+k, n+k+1)$, where $k = -3$ to 2. We arbitrarily choose one of these open intervals $(n+k, n+k+1)$ for some $k \in \mathbb{Z}$ such that $-3 \leq k < 3$. We can choose any number in this chosen interval, say $x \in (n+k, n+k+1)$, and know that x is not a natural number. Therefore, $N(n;3) \not\subset \mathbb{N}$.

Set $\delta = 1$. Then $N(n;\delta) = (n-1, n+1)$. We observe that $(n-1, n+1)$ can be decomposed into 2 open intervals $(n+k, n+k+1)$, where $k = -1$ to 0. We arbitrarily choose one of these open intervals $(n+k, n+k+1)$ for some $k \in \mathbb{Z}$

such that $-1 \leq k < 1$. We can choose any number in this chosen interval, say $x \in (n+k, n+k+1)$, and know that x is not a natural number. Therefore, $N(n;1) \not\subset \mathbb{N}$.

Set $\delta = 1/2$. Then $N(n;\delta) = (n-1/2, n+1/2)$. We observe that $(n-1/2, n+1/2)$ is an open interval. Observe that $(n-1/2, n+1/2) - \{n\}$ contains no natural numbers. We can choose any number in this set, say $x \in (n-1/2, n+1/2) - \{n\}$, and know that x is not a natural number. Therefore, $N(n;1/2) \not\subset \mathbb{N}$.

Set $\delta = 1/4$. Then $N(n;\delta) = (n-1/4, n+1/4)$. We observe that $(n-1/4, n+1/4)$ is an open interval. Observe that $(n-1/4, n+1/4) - \{n\}$ contains no natural numbers. We can choose any number in this set, say $x \in (n-1/4, n+1/4) - \{n\}$, and know that x is not a natural number. Therefore, $N(n;1/4) \not\subset \mathbb{N}$.

Set $\delta = 1/8$. Then $N(n;\delta) = (n-1/8, n+1/8)$. We observe that $(n-1/8, n+1/8)$ is an open interval. Observe that $(n-1/8, n+1/8) - \{n\}$ contains no natural numbers. We can choose any number in this set, say $x \in (n-1/8, n+1/8) - \{n\}$, and know that x is not a natural number. Therefore, $N(n;1/8) \not\subset \mathbb{N}$.

Set $\delta = 2/19$. Then $N(n;\delta) = (n-2/19, n+2/19)$. We observe that $(n-2/19, n+2/19)$ is an open interval. Observe that $(n-2/19, n+2/19) - \{n\}$ contains no natural numbers. We can choose any number in this set, say $x \in (n-2/19, n+2/19) - \{n\}$, and know that x is not a natural number. Therefore, $N(n;2/19) \not\subset \mathbb{N}$. We notice some patterns and state some observations.

1. It appears that for every $\delta > 0$, no matter how small, $N(n;\delta) \not\subset \mathbb{N}$.

2. Furthermore, for any $\delta > 1$, there exists an interval $(n-1, n+1) - \{n\}$ that is contained in $N(n;\delta)$, but which contains no natural numbers.

3. Moreover, for any $\delta \leq 1$, there exists an interval $(n-\delta, n+\delta) - \{n\}$ that is contained in $N(n;\delta)$, but contains no natural numbers.

Hence, there do not appear to be any interior points in $\mathbb{N}$.

We prove these observations and results.

We prove: There are no interior points in $\mathbb{N}$.

Define predicate over $\mathbb{N}$ by

$p(n)$: n is an interior point of $\mathbb{N}$.

Thus, the statement to prove is: There does not exist a natural number which is an interior point of $\mathbb{N}$.

In logic symbols we prove: $\neg(\exists n \in \mathbb{N})(q(n)) \Leftrightarrow (\forall n \in \mathbb{N})(\neg q(n))$.

Hence, we must prove:

Every natural number is not an interior point of $\mathbb{N}$.

So, let $n \in \mathbb{N}$ be arbitrary.

To prove n is not an interior point of $\mathbb{N}$, we must prove:

$(\forall \delta > 0)(\exists x \in N(n;\delta))(x \notin \mathbb{N})$.

So, let $\delta > 0$ be arbitrary.

To prove $(\exists x \in N(n;\delta))(x \notin \mathbb{N})$, we must find a real number x (in terms of δ) in the δ neighborhood of n such that x is not a natural number.

Let $N(n;\delta)$ be the δ neighborhood of n.

How shall we find x? We incorporate our observations/results from above.

Whenever $\delta > 1$, then we observed that any x in $(n-1, n+1) - \{n\}$ will work. Hence, since $(n-1, n) \cup (n, n+1) \subset (n-1, n+1) - \{n\}$, we may choose any x in either the interval $(n-1, n)$ or $(n, n+1)$.

Hence, in particular, let's choose x to be say, the midpoint of $(n, n+1)$. Then $x = (n+n+1)/2 = (2n+1)/2 = n+1/2$.

Thus, we must prove $x \in N(n;\delta)$ and $x \notin \mathbb{N}$.

Whenever $\delta \leq 1$, then we observed that any x in $(n-\delta, n+\delta) - \{n\}$ will work. Hence, since $(n-\delta, n) \cup (n, n+\delta) \subset (n-\delta, n+\delta) - \{n\}$, then we may choose any x in either the interval $(n-\delta, n)$ or $(n, n+\delta)$.

Hence, in particular, let's choose x to be the midpoint of $(n, n+\delta)$. Then $x = (n+n+\delta)/2 = (2n+\delta)/2 = n+\delta/2$.

Thus, we must prove $x \in N(n;\delta)$ and $x \notin \mathbb{N}$.

We consider these cases separately: $\delta > 1$ or $\delta \leq 1$. □

Proof. To prove the set of natural numbers $\mathbb{N}$ has no interior points, we prove every natural number is not an interior point.

Let n be an arbitrary natural number.

To prove n is not an interior point of $\mathbb{N}$, let δ be an arbitrary positive real number.

We must prove there is some real number x in the δ neighborhood of n, $N(n;\delta)$ such that x is not a natural number; that is, we will show there exists a real number x such that $x \in N(n;\delta)$ and $x \notin \mathbb{N}$.

Either $\delta \leq 1$ or $\delta > 1$.

We consider these cases separately.

Case 1: Suppose $\delta > 1$.

Let x be the real number $n + 1/2$.

We prove $x \in N(n;\delta)$.

Since $\delta > 1$ and $1 > -1/2$, then $\delta > -1/2$, so $-\delta < 1/2$. Hence, $n - \delta < n + 1/2$, so $n - \delta < x$.

Since $1/2 < 1$ and $1 < \delta$, then $1/2 < \delta$. Hence, $n+1/2 < n+\delta$, so $x < n+\delta$.

Since $n-\delta < x$ and $x < n+\delta$, then $x \in (n-\delta, n+\delta) = N(n;\delta)$, as desired.

We now prove $x \notin \mathbb{N}$.

Since $0 < 1/2 < 1$, then $n < n+1/2 < n+1$. Hence, $n < x < n+1$.

Since n and $n+1$ are natural numbers and there are no natural numbers greater than n and less than $n+1$, then x cannot be a natural number. Hence, $x \notin \mathbb{N}$, as desired.

Case 2: Suppose $\delta \leq 1$.

Let x be the real number $n + \delta/2$.

We prove $x \in N(n;\delta)$.

Observe that

$$\begin{aligned}
-1 &< 1/2 < 1 \\
-\delta &< \delta/2 < \delta \\
n-\delta &< n+\delta/2 < n+\delta \\
n-\delta &< x < n+\delta.
\end{aligned}$$

Therefore, $x \in (n-\delta, n+\delta) = N(n;\delta)$, as desired.

We now prove $x \notin \mathbb{N}$.

Since $\delta > 0$, then $0 < \delta$, so $0 < \delta/2$. Hence, $n < n+\delta/2$, so $n < x$.

Since $\delta \leq 1$, then $\delta < 2$, so $\delta/2 < 1$. Hence, $n + \delta/2 < n + 1$, so $x < n + 1$.

Therefore, $n < x < n + 1$. Since n and $n + 1$ are natural numbers and there are no natural numbers greater than n and less than $n + 1$, then x cannot be a natural number. Hence, $x \notin \mathbb{N}$, as desired.

Thus, in all cases, there is some real number x in the δ neighborhood of n such that x is not a natural number. Therefore, n is not an interior point of $\mathbb{N}$, so $\mathbb{N}$ has no interior points. $\square$

Solution. We consider whether $\mathbb{N}$ is an open set or not.

The statement $\mathbb{N}$ is an open set means every point in $\mathbb{N}$ is an interior point of $\mathbb{N}$.

This means $\mathbb{N}$ is an open set iff every natural number is an interior point of $\mathbb{N}$.

Define predicate $p(n) : n$ is an interior point of $\mathbb{N}$.

Then $\mathbb{N}$ is open iff $(\forall n \in \mathbb{N})(p(n))$.

Hence $\mathbb{N}$ is not open iff $\neg(\forall n \in \mathbb{N})(p(n)) \Leftrightarrow (\exists n \in \mathbb{N})(\neg p(n))$.

Let P be the truth set of p. Then $P = \{n \in \mathbb{N} : p(n)\}$ and $\overline{P} = \mathbb{N} - P$. Then $\mathbb{N}$ is open iff $P = \mathbb{N}$ and $\mathbb{N}$ is not open iff $\overline{P} \neq \emptyset$.

Since we just proved $\mathbb{N}$ has no interior points, then there does not exist a natural number that is an interior point. Hence, $\neg(\exists n \in \mathbb{N})(p(n)) \Leftrightarrow (\forall n \in \mathbb{N})(\neg p(n))$.

Since $\mathbb{N} \neq \emptyset$, then $(\forall n)(\neg p(n)) \Rightarrow (\exists n)(\neg p(n))$. Hence, we can conclude $\mathbb{N}$ is not open. $\square$

Proof. We prove the set of natural numbers $\mathbb{N}$ is not an open subset of $\mathbb{R}$.

Since $\mathbb{N}$ does not have any interior points, then every natural number is not an interior point of $\mathbb{N}$. Hence, there is some natural number that is not an interior point of $\mathbb{N}$. Therefore, not every natural number is an interior point, so $\mathbb{N}$ is not open. $\square$

Exercise 29. Prove that 0 is a point of accumulation of the set $\{1, 1/2, 1/3, ...\}$.

Solution. Let $S = \{1, 1/2, 1/3, ...\} = \{1/n : n \in \mathbb{N}\}$. Let $a = 0$.

We can draw a picture of this on a number line and intuitively see that no matter how small $\delta > 0$ is, we can always find a point $1/n$ contained in $(-\delta, \delta)$.

To prove 0 is a point of accumulation of S, we must prove $(\forall \delta > 0)(N'(0; \delta) \cap S \neq \emptyset)$. $\square$

Proof. Let $S = \{1, 1/2, 1/3, ...\} = \{1/n : n \in \mathbb{N}\}$. To prove 0 is a point of accumulation of S, we must prove every deleted δ neighborhood of 0 contains some point of S. Thus, we must prove $(\forall \delta > 0)(N'(0; \delta) \cap S \neq \emptyset)$.

Let δ be an arbitrary positive real number. Suppose for the sake of contradiction that $N'(0; \delta) \cap S = \emptyset$. Then there is no real number x such that $x \in N'(0; \delta)$ and $x \in S$. Hence, there is no x such that $x \in (-\delta, \delta)$ and $x \neq 0$ and $x = 1/n$ for some natural number n. Therefore, there is no nonzero real x such that $|x| < \delta$ and $x = 1/n$. Hence, there is no nonzero real x such that $|1/n| < \delta$, so no nonzero real x exists such that $1/n < \delta$. Thus, there is no nonzero real number x such that $1/\delta < n$ for some natural number n. But, according to the Archimedean property of $\mathbb{R}$, we know every real number is less than some natural number. Hence, in particular, the real number $1/\delta$ is less than the natural number n. Therefore, it cannot be true that $N'(0;\delta) \cap S = \emptyset$. Thus, it must be that $N'(0;\delta) \cap S \neq \emptyset$, as desired.

Therefore, 0 is a point of accumulation of S. Further, since 0 is a point of accumulation of S, but $0 \notin S$, then S is not a closed set. ☐

Exercise 30. Prove the set of natural numbers has no accumulation points.

Solution. This means we must prove there is no natural number which is an accumulation point of $\mathbb{N}$.

Define predicate $p(n)$: n is an accumulation point of $\mathbb{N}$

We must prove $\neg\exists(n)(n \in \mathbb{N} \wedge p(n)) \Leftrightarrow \neg(\exists n \in \mathbb{N})(p(n))$.

We use proof by contradiction.

Thus, we assume n is an arbitrary natural number that is a point of accumulation of $\mathbb{N}$. Hence, it is true that $(\forall\delta > 0)(N'(n;\delta) \cap S \neq \emptyset)$.

We show that this is false. Let $\delta = 1/2$. Then $N'(n;\delta \cap \mathbb{N} = ((n-1/2, n+1/2) - \{n\}) \cap \mathbb{N} = (((2n-1)/2, (2n+1)/2) - \{n\}) \cap \mathbb{N} = \emptyset$, since there are no natural numbers between $((2n-1)/2, (2n+1)/2,)$, excluding n itself. ☐

Proof. Suppose for the sake of contradiction that there is some natural number that is a point of accumulation of $\mathbb{N}$, the set of all natural numbers. Let n be an arbitrary natural number that is a point of accumulation of $\mathbb{N}$. Then every deleted δ neighborhood of n contains some natural number. Let $\delta = 1/2$. Since there are no natural numbers between $((2n-1)/2, (2n+1)/2,)$, excluding n itself, then $N'(n;\delta \cap \mathbb{N} = ((n-1/2, n+1/2) - \{n\}) \cap \mathbb{N} = (((2n-1)/2, (2n+1)/2) - \{n\}) \cap \mathbb{N} = \emptyset$. Thus, there is some deleted δ neighborhood of n that contains no natural numbers, which contradicts the assumption that every deleted δ neighborhood of n contains some natural number. Therefore, n cannot be a point of accumulation of $\mathbb{N}$. Hence, no natural number is a point of accumulation of $\mathbb{N}$.

A set S is not closed in $\mathbb{R}$ iff there is a point of accumulation of S that is not in S. Since $\mathbb{N}$ contains no points of accumulation, then there is no point of accumulation of $\mathbb{N}$ that is not in $\mathbb{N}$. Hence, $\mathbb{N}$ cannot be not closed in $\mathbb{R}$. Therefore, $\mathbb{N}$ is a closed subset of $\mathbb{R}$. ☐

Exercise 31. Let f be a real valued function. Prove if a is not a point of accumulation of the domain of f, then $\lim_{x\to a} f(x) = L$ is true for any value of L.

Solution.

We must prove :

if a is not a point of accumulation of dom(f), then $(\forall L)(\lim_{x\to a} f(x) = L)$.

Hypothesis is: a is not a point of accumulation of dom(f).

Conclusion: $(\forall L)(\lim_{x\to a} f(x) = L)$.

To prove $(\forall L)(\lim_{x\to a} f(x) = L)$, let L be an arbitrary real number.

We must prove $(\forall \epsilon > 0)(\exists \delta > 0)(\forall x)(0 < |x-a| < \delta \rightarrow |f(x) - L| < \epsilon)$.

Let $\epsilon > 0$ be arbitrary.

We must prove $(\exists \delta > 0)(\forall x)(0 < |x-a| < \delta \rightarrow |f(x) - L| < \epsilon)$.

How can we find $\delta > 0$? We need to use the hypothesis somehow.

Since a is not a point of accumulation of the domain of f, then $\neg(\forall \delta > 0)(N'(a;\delta) \cap S \neq \emptyset)$, so $(\exists \delta > 0)(N'(a;\delta) \cap S = \emptyset)$, where S is the domain of f.

By existential elimination, let δ_1 be some positive real number such that $N'(a;\delta_1) \cap S = \emptyset$.

We must find $\delta > 0$ such that for every x, if $0 < |x-a| < \delta$, then $|f(x)-L| < \epsilon$.

Since $N'(a;\delta_1) \cap S = \emptyset$, then there is no element in both $N'(a;\delta_1)$ and S.

We know $N'(a;\delta_1) = \{x \in \mathbb{R} : |x-a| < \delta_1\} - \{a\}$. Thus, if $x \in N'(a;\delta_1$, then $|x-a| < \delta_1$ and $x \neq a$. Since $|x-a| \geq 0$, then either $|x-a| > 0$ or $|x-a| = 0$. Since $|x-a| = 0$ iff $x = a$, and $x \neq a$, then $|x-a| \neq 0$. Hence, $|x-a| > 0$.

Thus, let $\delta = \delta_1$.

Then we must prove $(\forall x)(0 < |x-a| < \delta \rightarrow |f(x) - L| < \epsilon$.

Thus, we assume arbitrary $x \in \mathbb{R}$ such that $0 < |x-a| < \delta$. We must prove $|f(x) - L| < \epsilon$.

Since $0 < |x-a| < \delta$ and $\delta = \delta_1$, then $0 < |x-a| < \delta_1$. Since $0 < |x-a|$, then $x \neq a$. Since $x \in \mathbb{R}$ and $|x-a| < \delta_1$ and $x \neq a$, then $x \in N'(a;\delta_1)$. Since $N'(a;\delta_1) \cap S$ is empty, then $x \notin S$. Hence, x is not in the domain of f, so $f(x)$ does not exist.

Let $x \in \mathbb{R}$ be arbitrary. Then either $x \in N'(a;\delta_1)$ or $x \notin N'(a;\delta_1)$.

We consider these cases separately.

Suppose $x \in N'(a;\delta_1)$. Then $x \in \mathbb{R}$ and $|x-a| < \delta_1$ and $x \neq a$. Since $|x-a| \geq 0$, then either $|x-a| > 0$ or $|x-a| = 0$. We know $|x-a| = 0$ iff $x = a$. Since $x \neq a$, then $|x-a| \neq 0$. Hence, $|x-a| > 0$. Thus, $0 < |x-a| < \delta_1$.

Since $x \in N'(a;\delta_1$ and $N'(a;\delta_1) \cap S$ is empty, then $x \notin S$. Hence, $f(x)$ does not exist. What does this imply?

Suppose $x \notin N'(a;\delta_1)$. Then since $N'(a;\delta_1) \cap S$ is empty

then there exists some deleted δ_1 neighborhood of a that contains no points of the domain of f. Thus, there exists $\delta_1 > 0$ such that $N'(a;\delta_1) \cap S = \emptyset$, where S is the domain of f. Let $\delta = \delta_1$. Then $\delta > 0$.

Suppose x is an arbitrary real number such that $0 < |x-a| < \delta$. Then $0 < |x-a| < \delta_1$, so $x \in N'(a;\delta_1)$. Since $N'(a;\delta_1) \cap S = \emptyset$, then every element of $N'(a;\delta_1)$ is not in the domain of f. Hence, in particular, x is not in the domain of f. Therefore, $f(x)$ does not exist.

To prove $|f(x) - L| < \epsilon$, we must prove $f(x) \in N(L;\epsilon)$.

How can we do this? We're stuck.

□