Real Analysis Notes hodgepodge need to merge into correct files

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Analysis is the theory of real numbers.

We develop a theory of functions of a real variable.

What are the analytic properties of real-valued functions?

Topics: real functions, convergence and limits of sequences of real numbers,

calculus of real numbers, continuity/uniform continuity, smoothness, sequences of functions

Real analysis studies completeness of real number system,

limits of sequences,

infinite series, derivatives of functions, integrals, uniform convergence, Taylor's theorem,

fundamental theorem of calculus, topology of Euclidean space, metric spaces, compactness, uniform continuity,

mean value theorem, Riemann-Stieltjes integrals, functions of bounded variation,

function algebras, Weierstrass theorem.

Sets of Numbers

$\mathbb{Q} = \{\frac{m}{n} : m, n \in \mathbb{Z} \wedge n \neq 0\} =$ set of all rational numbers
$\mathbb{R} =$ set of all real numbers
$\mathbb{R}^+ = \{x \in \mathbb{R} : x > 0\} =$ set of all positive real numbers
$\mathbb{R}_+ = \{x \in \mathbb{R} : x \geq 0\} = [0, \infty) = [0, \infty[=$ set of all nonnegative real numbers

Limits

Definition 1. Left and Right hand limits

Let $a, r \in \mathbb{R}$.

Let a be fixed.

Let f be a real valued function defined on $[a, a + r)$ for some $r > 0$.

A real number L^+ is a **right-hand limit** of $f(x)$ as x approaches a, denoted $L^+ = \lim_{x\to a^+} f(x)$, iff $(\forall\epsilon > 0)(\exists\delta > 0)(\forall x)(a < x < a + \delta \to |f(x) - L^+| < \epsilon)$.

Let $f : \mathbb{R} \mapsto \mathbb{R}$ be a function defined on $(a-r, a]$ for some $r > 0$.

A real number L^- is a **left-hand limit** of $f(x)$ as x approaches a, denoted $L^- = \lim_{x \to a^-} f(x)$, iff $(\forall \epsilon > 0)(\exists \delta > 0)(\forall x)(a - \delta < x < a \to |f(x) - L^-| < \epsilon)$.

$\lim_{x \to a} f(x) = L$ iff

1. $\lim_{x \to a^+} f(x) = L$
2. $\lim_{x \to a^-} f(x) = L$

Vector space $\mathbb{R}^n$

Let $n \in \mathbb{N}$.

$\mathbb{R} = \mathbb{R}^1$.

Elements of $\mathbb{R}^n$ may be viewed as either a point(geometric) or as a vector (algebraic).

$x = (x_1, x_2, ..., x_n)$

$y = (y_1, y_2, ..., y_n)$

Definition 2. inner product or dot product of vectors

Let $x, y \in \mathbb{R}^n$.

Then $\langle x, y \rangle = x \cdot y = [x_1, ..., x_n] \cdot [y_1, ..., y_n] = x_1y_1 + x_2y_2 + ... + x_ny_n = \sum_{i=1}^n x_iy_i$.

Definition 3. Euclidean norm, length of vector

Let $x \in \mathbb{R}^n$.

The **Euclidean norm of** x is $||x|| = \sqrt{\langle x, x \rangle} = \sqrt{\sum_{i=1}^n |x_i|^2}$. (Pythagorean rule)

It represents the length of the vector x.

Theorem 4. ***Cauchy-Schwarz inequality***

Let $x, y \in \mathbb{R}^n$.

Then $|\langle x, y \rangle| \leq ||x|| \; ||y||$.

Definition 5. normed vector space

A **norm** on the vector space $\mathbb{R}^n$ is a function $f : \mathbb{R}^n \to \mathbb{R}_+$ such that the following axioms hold:

N1. $f(x + y) \leq f(x) + f(y)$

N2. $f(\lambda x) = \lambda f(x)$

N3. $f(x) = 0$ iff $x = 0$ for all $x, y \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$.

Theorem 6. *The function $|| \cdot || : \mathbb{R}^n \to \mathbb{R}_+$ has the following properties:*

1. $||x + y|| \leq ||x|| + ||y||$ (Triangle inequality)

2. $||\lambda x|| = |\lambda| * ||x||$. (homogeneity)

3. $||x|| = 0$ iff $x = 0$.

for $x, y \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$.

If $x = (x_1, ..., x_n)$ and $y = (y_1, ..., y_n)$, then $x - y = (x_1 - y_1, ..., x_n - y_n)$.

Thus, $||x - y|| = \sqrt{|x_1 - y_1|^2 + ... + |x_n - y_n|^2}$ = distance from x to y = $||y - x||$.

Definition 7. Euclidean distance
Let $x, y \in \mathbb{R}^n$.
We define the distance between points x and y by $d(x,y) = ||x-y|| = ||y-x|| = d(y,x)$.

Thus, $d(x,y) = ||x-y|| = \sqrt{(x_1-y_1)^2 + ... + (x_n-y_n)^2}$.

Proposition 8. ***triangle inequality for Euclidean distance***
Let $x, y \in \mathbb{R}^n$.
Then $d(x,y) \leq d(x,z) + d(z,y)$ for every $z \in \mathbb{R}^n$.

Proof. Let $x, y \in \mathbb{R}^n$.
Then for every $z \in \mathbb{R}^n$,

$$\begin{aligned} d(x,y) &= ||x-y|| \\ &= ||(x-z)+(z-y)|| \\ &\leq ||x-z|| + ||z-y|| \\ &= d(x,z) + d(z,y). \end{aligned}$$

□

Boundedness of subsets of $\mathbb{R}^n$

Definition 9. open ball
Let $r > 0$ (means r is a positive real number).
Let $x \in \mathbb{R}^n$.
The **open ball with radius** r **and center** x is $B_r(x) = \{y \in \mathbb{R}^n : ||y-x|| < r\}$.
excludes boundary

Definition 10. closed ball
Let $r > 0$ and $x \in \mathbb{R}^n$.
The **closed ball with radius** r **and center** x is $\overline{B_r}(x) = \{y \in \mathbb{R}^n : ||y-x|| \leq r\}$.
includes boundary

Definition 11. open unit ball
Let $r > 0$.
The **open unit ball** is the set $B_1(0) = \{x \in \mathbb{R}^n : ||x|| < 1\}$.

Definition 12. closed unit ball
Let $r > 0$.
The **closed unit ball** is the set $\overline{B_1}(0) = \{x \in \mathbb{R}^n : ||x|| \leq 1\}$.

Example 13. For $n = 1$,

$$\begin{aligned} B_r(x) &= \{y \in \mathbb{R} : ||y - x|| < r\} \\ &= \{y \in \mathbb{R} : |y - x| < r\} \\ &= \{y \in \mathbb{R} : -r < y - x < r\} \\ &= \{y \in \mathbb{R} : x - r < y < x + r\} \\ &= (x - r, x + r) \\ &=]x - r, x + r[. \end{aligned}$$

Therefore, $B_r(x)$ is the open interval $(x - r, x + r)$.

$$\begin{aligned} \overline{B_r}(x) &= \{y \in \mathbb{R} : ||y - x|| \leq r\} \\ &= \{y \in \mathbb{R} : |y - x| \leq r\} \\ &= \{y \in \mathbb{R} : -r \leq y - x \leq r\} \\ &= \{y \in \mathbb{R} : x - r \leq y \leq x + r\} \\ &= [x - r, x + r]. \end{aligned}$$

Therefore, $\overline{B_r}(x)$ is the closed interval $[x - r, x + r]$.

Example 14. For $n = 2$,

$$\begin{aligned} B_r(x) &= \{y \in \mathbb{R}^2 : ||y - x|| < r\} \\ &= \{y \in \mathbb{R}^2 : \sqrt{(y - x) \cdot (y - x)} < r\} \\ &= \{(y_1, y_2) \in \mathbb{R}^2 : \sqrt{(y_1 - x_1)^2 + (y_2 - x_2)^2} < r\} \\ &= \{(y_1, y_2) \in \mathbb{R}^2 : (y_1 - x_1)^2 + (y_2 - x_2)^2 < r^2\} \end{aligned}$$

Therefore, $B_r(x)$ is the disc with radius r about $x = (x_1, x_2)$ without its boundary.

$$\begin{aligned} \overline{B_r}(x) &= \{y \in \mathbb{R}^2 : ||y - x|| \leq r\} \\ &= \{y \in \mathbb{R}^2 : \sqrt{(y - x) \cdot (y - x)} \leq r\} \\ &= \{(y_1, y_2) \in \mathbb{R}^2 : \sqrt{(y_1 - x_1)^2 + (y_2 - x_2)^2} \leq r\} \\ &= \{(y_1, y_2) \in \mathbb{R}^2 : (y_1 - x_1)^2 + (y_2 - x_2)^2 \leq r^2\} \end{aligned}$$

Therefore, $\overline{B_r}(x)$ is the disc with radius r about $x = (x_1, x_2)$ including its boundary.

Definition 15. bounded subset of $\mathbb{R}^n$

A subset $S \subset \mathbb{R}^n$ is **bounded** iff there exists a real number $R > 0$ such that $S \subset \overline{B_R}(0)$.

A set which is not bounded is **unbounded**.

$S \subset \mathbb{R}^n$ is bounded iff

$$\begin{aligned}(\exists R > 0)(S \subset \overline{B_R}(0) &\Leftrightarrow (\exists R > 0)(\forall x)(x \in S \to x \in \overline{B_R}(0)) \\ &\Leftrightarrow (\exists R > 0)(\forall x \in S)(x \in \overline{B_R}(0)) \\ &\Leftrightarrow (\exists R > 0)(\forall x \in S)(||x|| \leq R).\end{aligned}$$

Therefore, S is not bounded iff

$$\neg(\exists R > 0)(\forall x \in S)(||x|| \leq R) \quad \Leftrightarrow \quad (\forall R > 0)(\exists x \in S)(||x|| > R).$$

Example 16. Any open or closed ball centered on the origin is bounded.

Proof. Let $r > 0$. We show $B_r(0)$ and $\overline{B_r}(0)$ are bounded. Take $R = r$ (or any value greater than r). Then $B_r(0) \subset \overline{B_r}(0) \subset \overline{B_R}(0)$. Hence, $B_r(0) \subset \overline{B_R}(0)$ and $\overline{B_r}(0) \subset \overline{B_R}(0)$. Therefore, $B_r(0)$ and $\overline{B_r}(0)$ are bounded. □

Example 17. All open/closed balls in $\mathbb{R}^n$ are bounded, whether or not they are centered on the origin.

Proof. Let $r > 0$ and $x \in \mathbb{R}^n$.

We show $B_r(x)$ and $\overline{B_r}(x)$ are bounded.

To prove $\overline{B_r}(x)$ is bounded, we must find $R > 0$ such that $\overline{B_r}(x) \subset \overline{B_R}(0)$. Take $R = ||x|| + r$ (or any value larger than $||x|| + r$). Let $y \in \overline{B_r}(x)$. Then $||y - x|| \leq r$. Thus,

$$\begin{aligned}||y|| &= ||y - x + x|| \\ &\leq ||y - x|| + ||x|| \\ &\leq r + ||x|| \\ &= R\end{aligned}$$

Hence, $||y|| \leq R$, so $y \in \overline{B_R}(0)$. Therefore, $\overline{B_r}(x) \subset \overline{B_R}(0)$.

The proof of the open ball case is similar. □

Example 18. The empty set $\emptyset$ is bounded but $\mathbb{R}^n$ is unbounded.

Proof. We prove $\emptyset$ is bounded.

For all sets S, we have $\emptyset \subset S$. In particular, $\emptyset \subset \overline{B_1}(0)$, so $\emptyset$ is bounded.

We prove $\mathbb{R}^n$ is unbounded. Suppose that $\mathbb{R}^n$ is bounded. Then $\exists R > 0$ such that $\mathbb{R}^n \subset \overline{B_R}(0)$. Thus, for any $x \in \mathbb{R}^n$, $||x|| \leq R$. In particular, $||2R|| \leq R$ for the vector $2R \in \mathbb{R}^n$. Thus, $|2|||R|| \leq R$, so $2R \leq R$. Hence, $R \leq 0$. Therefore, we have $R \leq 0$ and $R > 0$, a violation of trichotomy of $\mathbb{R}$. Therefore, $\mathbb{R}^n$ is unbounded. □

Example 19. Every subset of a bounded set is bounded.

Proof. Let A be a subset of a bounded set B. To prove A is bounded, we must prove $\exists R > 0$ such that $A \subset \overline{B_R}(0)$.

Since B is bounded, then $\exists R > 0$ such that $B \subset \overline{B_R}(0)$. Since $A \subset B$ and $B \subset \overline{B_R}(0)$, then $A \subset \overline{B_R}(0)$. Therefore, A is bounded. □

Note that for $n = 1, \mathbb{R}^1 = \mathbb{R}$. For $S \subset \mathbb{R}$, $S \neq \emptyset$, S is bounded iff S is bounded above and bounded below.

There is no connection at all between the terms 'bounded' and 'boundary'.

In particular, a bounded set need not include its boundary.

Example 20. The open unit ball $B_1(0)$ is bounded, but its boundary is excluded.

Many equivalent definitions of the term 'bounded'.

A set S is bounded iff the set of all possible distances between pairs of points of S is bounded above in $\mathbb{R}$.

Thus non empty bounded sets can be described as sets which have **finite diameter**.

Intervals and d cells

Let $a, b \in \mathbb{R}$ with $a < b$.

There are 4 types of bounded intervals:

1. **closed interval** $[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\} \subset \mathbb{R}$.

2. **open interval** $]a, b[= (a, b) = \{x \in \mathbb{R} : a < x < b\} \subset \mathbb{R}$.

3. **half-open intervals**

$]a, b] = (a, b] = \{x \in \mathbb{R} : a < x \leq b\}$

$[a, b[= [a, b) = \{x \in \mathbb{R} : a \leq x < b\}$

For $a = b$ the interval $[a, b] = [a, a] = \{a\}$, a **singleton** set. (degenerate interval)

For $a > b$ the interval from a to b is the empty set $\emptyset$. (degenerate interval)

Most subsets of $\mathbb{R}$ are not intervals!

Example 21. $\mathbb{Q}$ is not an interval and neither is the set of irrational numbers $\overline{\mathbb{Q}} = \mathbb{R} - \mathbb{Q}$.

Example 22. $[0, 1] \cup (3, 5)$ is not an interval.

Analogs of intervals in $\mathbb{R}^n$ can be obtained by taking cartesian products of n intervals to form n**-cells**.

This gives us rectangles in $\mathbb{R}^2$, cuboids in $\mathbb{R}^3$ and hyper-cuboids in higher dimensions.

Definition 23. n**-cells in** $\mathbb{R}^n$

Let $a_1 \leq b_1, a_2 \leq b_2, ..., a_n \leq b_n$ be real numbers.

Corresponding to these we have closed intervals $[a_i, b_i]$ and open intervals $]a_i, b_i[$ for $1 \leq i \leq n$.

Let $x = (x_1, ..., x_n) \in \mathbb{R}^n$.

The **closed n-cell in $\mathbb{R}^n$** is defined to be
$[a_1, b_1] \times [a_2, b_2] \times ... \times [a_n, b_n] = \{x \in \mathbb{R}^n : a_1 \leq x_1 \leq b_1, ..., a_n \leq x_n \leq b_n\}$.
The **open n-cell in $\mathbb{R}^n$** is defined to be
$]a_1, b_1[\times]a_2, b_2[\times...\times]a_n, b_n[= \{x \in \mathbb{R}^n : a_1 < x_1 < b_1, ..., a_n < x_n < b_n\}$.

If $n > 1$ then there are many more half-open combinations possible than just 2.

Example 24. For $n = 2$ and $a_1 = a_2 = 1, b_1 = b_2 = 2$ the closed 2-cell is the closed square $[1, 2] \times [1, 2]$ (boundary included) and the open 2-cell is $]1, 2[\times]1, 2[$ (boundary excluded).

There exist 14 other 2 cells.

Ex. $[1, 2]\times]1, 2[= \{(x, y) \in \mathbb{R}^2 : 1 \leq x \leq 2, 1 < y < 2\}$ is a square including vertical edges, excluding horizontal edges.

The bounded types of intervals for a_1, b_1 is either closed, open, or half-open (2), for a total of 4 bounded types. Similarly, there are 4 bounded types of intervals for a_2, b_2. Hence, there are $4 * 4 = 16$ types of bounded 2 cell. Since there are exactly 2 closed or open 2 cell, then there are $16 - 2 = 14$ other types of bounded 2 cell.

In the same way, there are $4^3 = 64$ types of bounded $3-$ cell in $\mathbb{R}^3$. There are $4^n = (2^2)^n = 2^{2n}$ types of bounded $n-$ cell in $\mathbb{R}^n$.

Ex. Prove that the above n cells are bounded.

We may also form **unbounded** n cells by replacing some of the a_k by $-\infty$ and/or some of the b_k by $\infty = +\infty$.

Note that $\pm\infty$ are not real numbers, so that these must be excluded from the intervals.

The set of real numbers is $\mathbb{R} = (-\infty, \infty) = (-\infty, +\infty) =] - \infty, +\infty[$.

The nonnegative real numbers is $\mathbb{R}_+ = [0, \infty) = [0, \infty[$.

Example 25. The right hand half plane, including the y axis is the unbounded 2 cell $[0, \infty) \times (-\infty, \infty) = [0, [\infty\times] - \infty, \infty[= \mathbb{R}_+ \times \mathbb{R}$.

Open subsets of $\mathbb{R}^n$

Definition 26. interior point of a set

Let S be a subset of $\mathbb{R}^d$.

We classify the points of S as either **interior points of** S or non-interior points in S as follows.

Let $x \in S$. Then x is an **interior point of** S iff there is $r > 0$ such that $B_r(x) \subset S$.

Otherwise x is a non-interior point in S.

The set of all interior points of S is called the **interior of** S, and is denoted by int S or $\mathring{\text{S}}$.

The set of non-interior points in S is denoted by nint S, so that nint $S = S - intS$.

Let $a \notin S$.

Then $a \notin int(S)$ and therefore $a \notin nint(S)$.

Let x be an interior point of S. Then $x \in S$ and $x \in int(S)$.

Let y be an element of S that is on the edge of S. Then $y \in S$. Since we can choose any open ball with $r > 0$, then there will always be at least one point of the open ball that is not in S. Hence, $y \notin int(S)$, so $y \in nint(S)$.

The terms interior point of S and interior are standard.

The term non-interior point in S and the notation nint S are not standard, but they are helpful in understanding this material.

Note that a non-interior point in S must be an element of S.

Although points of $\overline{S} = \mathbb{R}^d - S$ are certainly not interior points of S, such points do not qualify as non-interior points in S and therefore they are not in nint (S).

Example 27. Let $x \in S$. Then x is a non-interior point in S iff x is not an interior point of S iff

$$\begin{aligned}
\neg(\exists r > 0)(B_r(x) \subset S) &\Leftrightarrow (\forall r > 0)\neg(B_r(x) \subset S) \\
&\Leftrightarrow (\forall r > 0)\neg(\forall y)(y \in B_r(x) \to y \in S) \\
&\Leftrightarrow (\forall r > 0)(\exists y)\neg(y \in B_r(x) \to y \in S) \\
&\Leftrightarrow (\forall r > 0)(\exists y)(y \in B_r(x) \wedge y \notin S) \\
&\Leftrightarrow (\forall r > 0)(\exists y)(||y - x|| < r \wedge y \notin S).
\end{aligned}$$

Definition 28. open set

Let $S \subset \mathbb{R}^n$.

Then S is an **open set(or S is open in $\mathbb{R}^n$, or S is an open subset of $\mathbb{R}^n$)** iff every point of S is an interior point of S.

Therefore, $S \subset \mathbb{R}^n$ is open iff for each $x \in S$, there is $r > 0$ such that $B_r(x) \subset S$.

Since $intS \subset S$ then S is open if $S \subset intS$. Hence, S is open iff $intS \subset S$ and $S \subset intS$. Therefore, S is open iff $intS = S$.

S is not open iff S has at least one non-interior point.

Definition 29. limit of a sequence in $\mathbb{R}^n$

Let (s_n) be a sequence of points in $\mathbb{R}^n$.

A sequence (s_n) of points in $\mathbb{R}^n$ **converges to a point** $L \in \mathbb{R}^n$, denoted $s_n \to L$ or $\lim_{n\to\infty} s_n = L$, iff the sequence of real numbers $||s_n - L||$ converges to 0 in the usual sense in $\mathbb{R}$.

Equivalently, for every $\epsilon > 0$ there corresponds a natural number N such that $||s_n - L|| < \epsilon$ whenever $n > N$.

In symbols, $\lim_{n\to\infty} s_n = L$ iff $(\forall \epsilon > 0)(\exists N \in \mathbb{N})(\forall n \in \mathbb{N})(n > N \to ||s_n - L|| < \epsilon)$.