Real Analysis Propositions OLD hodgepodge need to merge into correct notes and theory files

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Topology of Real Numbers

Proposition 1. *Every interval in $\mathbb{R}^1$ is convex.*

Solution. □

Proof. Let I be an arbitrary interval in $\mathbb{R}^1$. To prove I is convex, let v and w be arbitrary elements of I and let t be an arbitrary real number such that $0 \leq t \leq 1$. We must prove $tv + (1-t)w \in I$.

Since $0 \leq t \leq 1$, then either $t = 0$ or $t = 1$ or $0 < t < 1$.

We consider these cases separately.

Case 1: Suppose $t = 0$.

Then $tv + (1-t)w = 0v + (1-0)w = w$. Since $w \in I$ and $w = tv + (1-t)w$, then $tv + (1-t)w \in I$.

Case 2: Suppose $t = 1$.

Then $tv + (1-t)w = 1v + (1-1)w = v$. Since $v \in I$ and $v = tv + (1-t)w$, then $tv + (1-t)w \in I$.

Case 3: Suppose $0 < t < 1$.

Then $0 < t$ and $t < 1$.

By trichotomy, either $v = w$ or $v < w$ or $w < v$.

We consider these cases separately.

Case 3a: Suppose $v = w$.

Then $tv + (1-t)w = tv + w - tw = tv + w - tv = w$. Since $w \in I$, then $tv + (1-t)w \in I$.

Case 3b: Suppose $v < w$.

Since $t < 1$, then $0 < 1 - t$, so $1 - t > 0$.

Let $c = tv + (1-t)w$. Then $c = tv + w - tw$.

Observe that

$$\begin{array}{rcl} (1-t)v & < & (1-t)w \\ v - tv & < & w - tw \\ v & < & tv + w - tw \\ v & < & c \end{array}$$

Since $t > 0$, then $tv < tw$.

Observe that

$$\begin{aligned} tv - tw &< 0 \\ tv - tw + w &< w \\ c &< w \end{aligned}$$

Thus, $v < c$ and $c < w$, so $v < c < w$. Since $v \in I$ and $w \in I$ and $v < c < w$, then by definition of interval, $c \in I$. Hence, $tv + (1-t)w \in I$.

Case 3c: Suppose $w < v$.

Let $c = tv + (1-t)w$. Then $c = tv + w - tw$.

Since $t > 0$, then $tw < tv$.

Observe that

$$\begin{aligned} 0 &< tv - tw \\ w &< tv - tw + w \\ w &< c \end{aligned}$$

Since $t < 1$, then $0 < 1 - t$, so $1 - t > 0$. Thus, $(1-t)w < (1-t)v$.

Observe that

$$\begin{aligned} w - tw &< v - tv \\ tv + w - tw &< v \\ c &< v \end{aligned}$$

Thus, $w < c$ and $c < v$, so $w < c < v$. Since $w \in I$ and $v \in I$ and $w < c < v$, then by definition of interval, $c \in I$. Hence, $tv + (1-t)w \in I$. ☐

Proposition 2. *Let S be a convex subset of $\mathbb{R}$. Then S is an interval.*

Solution. Our hypothesis is:

1. $S \subset \mathbb{R}$.
2. S is convex.

The conclusion is: S is an interval.

To prove S is an interval, let $a, b, c \in \mathbb{R}$ be arbitrary such that $a, b \in S$ and $a < c < b$. We must prove $c \in I$.

Since S is convex, then for every $v, w \in S$ and for every $t \in [0,1], tv + (1-t)w \in S$.

Thus, we'd like $c = ta + (1-t)b$ for some $t \in [0,1]$.

Hence, we must find some $t \in [0,1]$ such that $c = ta + (1-t)b$.

Work backwards. Suppose $c = ta + (1-t)b$. We want t in terms of a, b, c. Thus, if we solve for t, then $t = \frac{b-c}{b-a}$. We must now show that $t \in [0,1]$; that is, prove $0 \leq t \leq 1$. We use the hypotheses to show $0 < t < 1$. ☐

Proof. Let S be a convex subset of $\mathbb{R}$. To prove S is an interval, let a, b, and c be arbitrary real numbers such that $a < c < b$.

We must prove $c \in S$. Let $t = \frac{b-c}{b-a}$. Since $a < c < b$, then $a < c$ and $c < b$. Thus, $a < b$. Since $0 < b - c$ and $0 < b - a$, then $\frac{b-c}{b-a} > 0$. Hence, $t > 0$. Since $a < c$, then $-a > -c$, so $b - a > b - c$. Since $b - a > 0$, we divide to obtain $1 > \frac{b-c}{b-a}$. Thus, $1 > t$. Since $0 < t$ and $t < 1$, then $0 < t < 1$, so $t \in [0, 1]$.

Observe that

$$\begin{aligned} t &= \frac{b-c}{b-a} \\ tb - ta &= b - c \\ c &= b + ta - tb \\ c &= ta + (1-t)b \end{aligned}$$

Since S is convex, then for every $v, w \in S$ and for every $t \in [0, 1]$, $tv + (1 - t)w \in S$. Hence, in particular, since $t \in [0, 1]$, then $ta + (1 - t)b \in S$. Thus, $c \in S$, as desired. □

Proposition 3. *The empty set is open.*

Solution.

The conclusion is: $\emptyset$ is open.

A set S is open iff $(\forall p \in S)(p$ is an interior point of $S)$. A set is not open iff $\neg(\forall p \in S)(p$ is not an interior point of $S) \Leftrightarrow (\exists p \in S)(p$ is not an interior point).

We use proof by contradiction. Thus,we assume the negation of the conclusion: Hence, we assume $\emptyset$ is not open; that is, we assume $(\exists p \in S)(p$ is not an interior point).

Suppose $\emptyset$ is not open. Then there is some point $p \in \emptyset$ such that p is not an interior point. But, $\emptyset$ has no elements, so there can be no point $p \in \emptyset$ such that p is not an interior point. Therefore, $\emptyset$ is open. □

Proof. Suppose for the sake of contradiction that the empty set $\emptyset$ is not open. Then there is some point p in $\emptyset$ such that p is not an interior point. But, the empty set has no elements, so there can be no point point p in $\emptyset$ such that p is not an interior point. Therefore, $\emptyset$ cannot be not open, so $\emptyset$ must be open. □

Proposition 4. $\mathbb{R}$ *is an open subset of itself.*

Solution. Intuitively, this seems obvious because no matter how large $\delta > 0$ is, the δ neighborhood of a point $p \in \mathbb{R}$ is always a subset of $\mathbb{R}$.

To prove $\mathbb{R}$ is open, we must prove every point in $\mathbb{R}$ is an interior point of $\mathbb{R}$. Let $p \in \mathbb{R}$ be arbitrary. To prove p is an interior point, we must prove $(\exists \delta > 0)(N(p; \delta) \subset \mathbb{R})$.

To prove $N(p; \delta) \subset \mathbb{R}$, let $x \in N(p; \delta)$ be arbitrary. We must prove $x \in \mathbb{R}$.

Since $x \in N(p; \delta)$, then $x \in \mathbb{R}$ and $|p - x| < \delta$. Therefore, $x \in \mathbb{R}$. This is true regardless of the value of δ chosen. Hence, we may choose any positive δ. □

Proof. To prove $\mathbb{R}$ is an open set, we must prove every point in $\mathbb{R}$ is an interior point of $\mathbb{R}$.

Let p be an arbitrary real number.

To prove p is an interior point of $\mathbb{R}$, we will show there is some positive δ such that the δ neighborhood of p is contained in $\mathbb{R}$.

By definition, the δ neighborhood of p is a subset of $\mathbb{R}$ for every positive δ. Thus, if we let $\delta = 1$, then $N(p; 1) \subset \mathbb{R}$, as desired. □

Proposition 5. *Any interval of the form* (a, b)*, where* $a \neq b$*, is an open subset of* $\mathbb{R}$*.*

Solution. The statement to prove is:

Every interval of the form (a, b) with $a \neq b$ is an open subset of $\mathbb{R}$.

This statement has the form $\forall x.p(x)$.

Thus, let (a, b) be an arbitrary interval such that $a \neq b$.

To prove (a, b) is an open subset of $\mathbb{R}$, we must prove every point in (a, b) is an interior point of (a, b).

Hence, we must prove $(\forall x \in (a, b))(\exists \delta > 0)(N(x; \delta) \subset (a, b))$.

Therefore, let $p \in (a, b)$ be arbitrary.

We must prove $(\exists \delta > 0)(N(p; \delta) \subset (a, b))$.

Since $a \neq b$, then either $a < b$ or $a > b$.

We consider these cases separately.

To prove $N(p; \delta) \subset (a, b)$, assume $x \in N(p; \delta)$. We must prove $x \in (a, b)$; that is, prove $a < x$ and $x < b$.

Case 1: Suppose $a < b$.

Since we want $N(p; \delta) \subset (a, b)$ for some $\delta > 0$, we must find a suitable value for δ that will guarantee this condition. How do we find δ?

We can draw a picture of this scenario(ie, a number line with interval (a, b) with point p between a and b and with some δ such that $p - \delta$ is between a and p and $p + \delta$ is between p and b.

Thus, we must find $\delta > 0$ such that $a < p - \delta < p$ and $p < p + \delta < b$. Any such $\delta > 0$ that satisfies these conditions will work. Thus, we can choose δ such that $p - \delta$ is the midpoint of $(a, p$ and such that $p + \delta$ is the midpoint of (p, b). Thus, $p - \delta = (a + p)/2$ and $p + \delta = (p + b)/2$.

We know from predicate logic that δ must be a function of p, a, b.

Since we want x such that $a < x < b$, let's solve for a and b.

Solving for a in the first equation we have $2p - 2\delta = a + p$, so $a = p - 2\delta$.

Solving for b in the second equation we have $2p + 2\delta = p + b$, so $b = p + 2\delta$.

Now, we can eliminate p in these equations to find a relationship between a, b, δ.

Thus, solving for p in equation $a = p - 2\delta$, we obtain $p = a + 2\delta$. Now, substituting into equation $b = p+2\delta$, we obtain $b = (a+2\delta)+2\delta)$, so $\delta = (b-a)/4$.

Therefore, we let $\delta = (b - a)/4$.

We note that this proposition asserts an open interval (a, b) is an open set whenever a and b are distinct.

We should note that if $a = b$, then $(a, b) = \{x \in \mathbb{R} : a < x < a\} = \emptyset$. We know $\emptyset$ is an open set. Hence, in general, every open interval is an open set. □

Proof. To prove every interval of the form (a, b) with $a \neq b$ is an open subset of $\mathbb{R}$, let (a, b) be an arbitrary interval such that $a \neq b$ for real numbers a and b.

We must prove (a, b) is an open subset of $\mathbb{R}$.

Since $a \neq b$, then either $a < b$ or $a > b$.

We consider these cases separately.

Case 1: Suppose $a < b$.

To prove (a, b) is an open subset of $\mathbb{R}$, we must prove every point in (a, b) is an interior point of (a, b).

Let p be an arbitrary point in (a, b).

To prove p is an interior point of (a, b), we must prove there is some positive δ such that the δ neighborhood is contained in (a, b).

Let $p - \delta$ be the midpoint of a and p.

Let $p + \delta$ be the midpoint of p and b.

Then $p - \delta = \frac{a+p}{2}$ and $p + \delta = \frac{p+b}{2}$. Since $p - \delta = \frac{a+p}{2}$, then $2p - 2\delta = a + p$, so $p - 2\delta = a$. Hence, $p = a + 2\delta$.

Since $p + \delta = \frac{p+b}{2}$, then $2p + 2\delta = p + b$, so $p + 2\delta = b$. Hence, $p = b - 2\delta$.

Thus, $a + 2\delta = b - 2\delta$, so $\delta = (b - a)/4$.

Therefore, we let $\delta = (b - a)/4$.

We prove $\delta > 0$. Since $a < b$, then $b > a$, so $b - a > 0$. Hence, $\frac{b-a}{4} > 0$, so $\delta > 0$.

To prove the δ neighborhood is contained in (a, b), we must prove $N(p; \delta) \subset (a, b)$.

To prove $N(p; \delta) \subset (a, b)$, assume $y \in N(p; \delta)$. To prove $y \in (a, b)$, we must prove $a < y$ and $y < b$.

Since $y \in N(p; \delta) = (p - \delta, p + \delta)$, then $p - \delta < y$ and $y < p + \delta$.

Since $p - \delta < y$ and $p = a + 2\delta$, then $a + \delta < y$. Since $\delta = \frac{b-a}{4}$ and $a + \delta < y$, then $a + (b - a)/4 < y$, so $4a + b - a < 4y$. Hence, $b < 4y - 3a$.

Since $a < b$ and $b < 4y - 3a$, then $a < 4y - 3a$. Therefore, $4a < 4y$, so $a < y$, as desired.

Since $y < p + \delta$ and $p = b - 2\delta$, then $y < (b - 2\delta) + \delta$, so $y < b - \delta$.

Since $y < b - \delta$ and $\delta = \frac{b-a}{4}$, then $y < b + \frac{a-b}{4}$. Hence, $4y < 4b + a - b$, so $4y - 3b < a$.

Since $4y - 3b < a$ and $a < b$, then $4y - 3b < b$. Therefore, $4y < 4b$, so $y < b$, as desired.

Case 2: Suppose $a > b$.

Suppose the interval (a, b) is not empty. Then there exists some element in (a, b). Let x be an arbitrary real number in (a, b). Then $a < x$ and $x < b$. Therefore, $a < b$.

By trichotomy, it is impossible that $a < b$ and $a > b$. Therefore (a, b) must be empty, so $(a, b) = \emptyset$.

Since the empty set is an open set of $\mathbb{R}$, then (a, b) is also an open set of $\mathbb{R}$. $\square$

Proposition 6. *Let S and T be open subsets of $\mathbb{R}$. Then $S \cap T$ is open.*

Solution. Our hypothesis is:

S is an open subset of $\mathbb{R}$.

T is an open subset of $\mathbb{R}$.

Our conclusion is:

$S \cap T$ is an open set.

To prove $S \cap T$ is open, we must prove each point in $S \cap T$ is an interior point of $S \cap T$.

Thus, we must prove:

$(\forall p \in S \cap T)(\exists \delta > 0)(N(p;\delta) \subset S \cap T)$.

Let $p \in S \cap T$ be arbitrary.

Then $p \in S$ and $p \in T$.

To prove $(\exists \delta > 0)(N(p;\delta) \subset S \cap T)$, we let $y \in N(p;\delta)$. We must find some $\delta > 0$ such that $y \in S$ and $y \in T$.

By hypothesis, S is open, so every point in S is an interior point of S.

Hence, in particular, since $p \in S$, then p is an interior point of S.

Therefore, $(\exists \delta_1 > 0)(N(p;\delta_1) \subset S)$.

How can we prove $y \in S$?

We know $N(p;\delta_1) \subset S$ for some $\delta_1 > 0$. So, if we could show $y \in N(p;\delta_1)$, then we could argue that $y \in N(p;\delta_1)$ and $N(p;\delta_1) \subset S$ implies $y \in S$.

Thus, we must show that $y \in N(p;\delta_1)$.

We can show this if we let $\delta = \delta_1$.

Therefore, choose $\delta = \delta_1$.

Since $\delta_1 > 0$, then $\delta > 0$.

Since $y \in N(p;\delta)$, then $y \in N(p;\delta_1)$.

Since $y \in N(p;\delta_1)$ and $N(p;\delta_1) \subset S$, then $y \in S$, as desired.

By hypothesis, T is open, so every point in T is an interior point of T.

Hence, in particular, since $p \in T$, then p is an interior point of T.

Therefore, $(\exists \delta_2 > 0)(N(p;\delta_2) \subset T)$.

How can we prove $y \in T$?

We know $N(p;\delta_2) \subset T$ for some $\delta_2 > 0$. So, if we could show $y \in N(p;\delta_2)$, then we could argue that $y \in N(p;\delta_2)$ and $N(p;\delta_2) \subset T$ implies $y \in T$.

Thus, we must show that $y \in N(p;\delta_2)$.

Now, we already have $\delta = \delta_1$, so we need to re-evaluate what δ value we should choose.

We must choose δ such that $y \in N(p;\delta_1)$ in S and $y \in N(p;\delta_2)$ in T.

Since y must be in $N(p;\delta_1)$ in set S, we need to find some δ neighborhood of p in S that is contained inside of $N(p;\delta_1)$ in S.

How do we find such a neighborhood in S?

We draw a picture of the situation: sets S, T with point p in $N(p;\delta_1)$ in set S and p in $N(p;\delta_2)$ in set T.

We see that every δ neighborhood of p in S is contained entirely in $N(p;\delta_1)$ if only if $0 < \delta \leq \delta_1$.

Therefore, $y \in N(p;\delta_1)$ in S iff $0 < \delta \leq \delta_1$.

Since y must be in $N(p;\delta_2)$ in set T, then we need to find some δ neighborhood of p in T that is contained inside of $N(p;\delta_2)$ in T.

How do we find such a neighborhood in T?

We see that every δ neighborhood of p in T is contained entirely in $N(p;\delta_2)$ if only if $0 < \delta \leq \delta_2$.

Therefore, $y \in N(p;\delta_2)$ in T iff $0 < \delta \leq \delta_2$.

Thus, we want δ such that $0 < \delta \leq \delta_1$ and $0 < \delta \leq \delta_2$.

Hence, we can choose δ to be the smaller of δ_1 and δ_2.

Therefore, let $\delta = \min(\delta_1, \delta_2)$.

□

Proof. To prove $S \cap T$ is open, we must prove each point in $S \cap T$ is an interior point of $S \cap T$.

Let p be an arbitrary point in $S \cap T$.

Then $p \in S$ and $p \in T$.

To prove p is an interior point of $S \cap T$, we must find a positive δ such that the δ neighborhood of p, $N(p;\delta)$, is entirely contained in $S \cap T$.

Since S is open, then every point in S is an interior point of S. Therefore, since $p \in S$, then p is an interior point of S. Hence, there is some positive δ_1 such that $N(p;\delta_1) \subset S$.

Since T is open, then every point in T is an interior point of T. Therefore, since $p \in T$, then p is an interior point of T. Hence, there is some positive δ_2 such that $N(p;\delta_2) \subset T$.

Let $\delta = \min(\delta_1, \delta_2)$.

Since δ_1 and δ_2 are positive, then δ is positive.

To prove $N(p;\delta) \subset S \cap T$, let y be an arbitrary element of $N(p;\delta)$. We must show $y \in S$ and $y \in T$.

Since $y \in N(p;\delta)$, then $|p - y| < \delta$.

By trichotomy, either $\delta_1 \leq \delta_2$ or $\delta_1 > \delta_2$.

We consider these cases separately.

Case 1: Suppose $\delta_1 \leq \delta_2$.

Then $\min(\delta_1, \delta_2) = \delta_1$, so $\delta = \delta_1$. Thus, $|p - y| < \delta_1$. Hence, $y \in N(p;\delta_1) \subset S$, so $y \in S$. Since $|p - y| < \delta_1$ and $\delta_1 \leq \delta_2$, then $|p - y| < \delta_2$. Thus, $y \in N(p;\delta_2) \subset T$, so $y \in T$. Hence, $y \in S$ and $y \in T$.

Case 2: Suppose $\delta_1 > \delta_2$.

Then $\min(\delta_1, \delta_2) = \delta_2$, so $\delta = \delta_2$. Thus, $|p - y| < \delta_2$. Hence, $y \in N(p;\delta_2) \subset T$, so $y \in T$. Since $|p - y| < \delta_2$ and $\delta_2 < \delta_1$, then $|p - y| < \delta_1$. Thus, $y \in N(p;\delta_1) \subset S$, so $y \in S$. Hence, $y \in S$ and $y \in T$.

Therefore, in all cases, $y \in S$ and $y \in T$, so that $y \in S \cap T$. Hence, $N(p;\delta) \subset S \cap T$, as desired. □

Exercise 7. If S_1, S_2 and S_3 are open subsets of $\mathbb{R}$, then $S_1 \cap S_2 \cap S_3$ is an open subset of $\mathbb{R}$.

Solution. Let S_1, S_2, S_3 be arbitrary open subsets of $\mathbb{R}$.

To prove $S_1 \cap S_2 \cap S_3$ is open, we must prove every point in $S_1 \cap S_2 \cap S_3$ is an interior point of $S_1 \cap S_2 \cap S_3$. Let p be an arbitrary point of $S_1 \cap S_2 \cap S_3$.

To prove p is an interior point of $S_1 \cap S_2 \cap S_3$, we must prove $(\exists \delta > 0)(N(p;\delta) \subset S_1 \cap S_2 \cap S_3$.

To prove $(N(p;\delta) \subset S_1 \cap S_2 \cap S_3$, let $y \in N(p;\delta)$. To prove $y \in S_1 \cap S_2 \cap S_3$, we must prove $y \in S_1$ and $y \in S_2$ and $y \in S_3$.

Since each of the sets $S_1, S_,2, S_3$ is open, then every point in $S_1, S_,2, S_3$ is an interior point of S_1, S_2, S_3, respectively. Since $p \in S_1 \cap S_2 \cap S_3$, then $p \in S_1$ and $p \in S_2$ and $p \in S_3$. Therefore, p is an interior point of S_1 and p is an interior point of S_2 and p is an interior point of S_3. Hence, $(\exists \delta_1 > 0)(N(p;\delta_1) \subset S_1)$ and $(\exists \delta_2 > 0)(N(p;\delta_2) \subset S_2)$ and $(\exists \delta_3 > 0)(N(p;\delta_3) \subset S_3)$.

How can we show $y \in S_1$? Well, since we have $N(p;\delta_1) \subset S_1)$, if we could show $y \in N(p;\delta_1)$, then we could argue $y \in N(p;\delta_1) \subset S_1 \Rightarrow y \in S_1$. Thus, we must prove $y \in N(p;\delta_1)$. How can we show $y \in N(p;\delta_1)$? Well, since we want $\delta > 0$ such that $y \in N(p;\delta)$, then we can choose $\delta = \delta_1$. Thus, let $\delta = \delta_1$. Then $y \in N(p;\delta) = N(p;\delta_1) \subset S_1$, so $y \in N(p;\delta_1)$.

How can we show $y \in S_2$? Well, since we have $N(p;\delta_2) \subset S_2)$, if we could show $y \in N(p;\delta_2)$, then we could argue $y \in N(p;\delta_2) \subset S_2 \Rightarrow y \in S_2$. Thus, we must prove $y \in N(p;\delta_2)$. How can we show $y \in N(p;\delta_2)$?

Since each of S_1, S_2, S_3 is open, then there exist $\delta_1, \delta_2, \delta_3 > 0$ such that p is in the $\delta_1, \delta_2, \delta_3$ neighborhoods of S_1, S_2, S_3 sets, respectively. Hence $N(p,\delta_1) \subset S_1$ and $N(p,\delta_2) \subset S_2$ and $N(p,\delta_3) \subset S_3$. To prove y is in S_1, if we can show that y is in $N(p,\delta_1)$, then since $N(p,\delta_1) \subset S_1$, then we could conclude $y \in S_1$. Since we want y to be in all 3 sets, then we can take the smallest of all 3 δ_i. This will guarantee that δ would be in each of the δ_i neighborhoods of p for each set S_i. So, we should let δ the smallest of $\delta_1, \delta_2, \delta_3$. □

Proof. To prove the set $S_1 \cap S_2 \cap S_3$ is an open subset of the set of all real numbers $\mathbb{R}$, we must prove every point in $S_1 \cap S_2 \cap S_3$ is an interior point of $S_1 \cap S_2 \cap S_3$.

Let p be an arbitrary point in $S_1 \cap S_2 \cap S_3$.

To prove p is an interior point of $S_1 \cap S_2 \cap S_3$, we must find some positive δ such that the δ neighborhood of p is contained entirely in $S_1 \cap S_2 \cap S_3$; that is, we will show there is some positive real number δ such that $N(p;\delta) \subset S_1 \cap S_3 \cap S_3$.

Since each of the sets S_1, S_2, and S_3 is open, then there exist positive real numbers δ_1, δ_2, and δ_3 such that p is in the δ_1, δ_2, and δ_3 neighborhoods of S_1, S_2, and S_3, respectively.

Therefore, $N(p;\delta_1) \subset S_1$ and $N(p;\delta_2) \subset S_2$ and $N(p;\delta_3) \subset S_3$.

Let δ be the smallest of δ_1, δ_2, and δ_3.

Since each of δ_1, δ_2, and δ_3 is a positive real number, then δ is a positive real number.

To prove $N(p;\delta) \subset S_1 \cap S_2 \cap S_3$, let y be an arbitrary element of $N(p;\delta)$.

Then $y \in \mathbb{R}$ and $|p - y| < \delta$.

We must show y is in S_1 and S_2 and S_3.

Since δ is the smallest of δ_1, δ_2, and δ_3, then $\delta \leq \delta_1$ and $\delta \leq \delta_2$ and $\delta \leq \delta_3$.

Since $|p - y| < \delta$ and $\delta \leq \delta_1$, then $|p - y| < \delta_1$. Since $|p - y| < \delta$ and $\delta \leq \delta_2$, then $|p - y| < \delta_2$. Since $|p - y| < \delta$ and $\delta \leq \delta_3$, then $|p - y| < \delta_3$.

Since $y \in \mathbb{R}$ and $|p - y| < \delta_1$, then $y \in N(p;\delta_1)$. Since $y \in \mathbb{R}$ and $|p - y| < \delta_2$, then $y \in N(p;\delta_2)$. Since $y \in \mathbb{R}$ and $|p - y| < \delta_3$, then $y \in N(p;\delta_3)$.

Since $y \in N(p;\delta_1)$ and $N(p;\delta_1) \subset S_1$, then $y \in S_1$. Since $y \in N(p;\delta_2)$ and $N(p;\delta_2) \subset S_2$, then $y \in S_2$. Since $y \in N(p;\delta_3)$ and $N(p;\delta_3) \subset S_3$, then $y \in S_3$.

Therefore, $y \in S_1$ and $y \in S_2$ and $y \in S_3$, so $y \in S_1 \cap S_2 \cap S_3$, as desired. □

Proposition 8. *The intersection of any finite collection of open subsets of* $\mathbb{R}$ *is open.*

Solution.

The statement means:

Let $n \in \mathbb{Z}, n \geq 0$.

Let $\{S_1, S_2, S_3, ..., S_n\}$ be a finite collection of n open subsets of $\mathbb{R}$.

Then $\{S_i : i \in \mathbb{N}\}$ and each S_i is an open subset of $\mathbb{R}$, for $i = 1, 2, 3, ..., n$.

We must prove $\cap_{i=1}^{n} S_i$ is an open subset of $\mathbb{R}$.

Define predicate $p(n) : \cap_{i=1}^{n} S_i$ is an open subset of $\mathbb{R}$ over $\mathbb{N}$.

To prove $\cap_{i=1}^{n} S_i$ is an open subset of $\mathbb{R}$, we must prove $(\forall n \in \mathbb{N})(p(n))$.

Let T be the truth set of $p(n)$.

Then $T = \{n \in \mathbb{N} : p(n)\}$, so $T \subset \mathbb{N}$.

To prove $(\forall n \in \mathbb{N})(p(n))$, we must prove $T = \mathbb{N}$.

To prove $T = \mathbb{N}$, we use induction.

Basis:

To prove $1 \in T$, we must prove $p(1)$ is true. Thus, we must prove $\cap_{i=1}^{1} S_i$ is an open subset of $\mathbb{R}$.

Since $\cap_{i=1}^{1} S_i = S_1$ and S_1 is an open subset of $\mathbb{R}$, then $\cap_{i=1}^{1} S_i$ is an open subset of $\mathbb{R}$, as desired.

Induction:

To prove T is inductive, we must prove $(\forall n \in \mathbb{N})(n \in T \rightarrow n+1 \in T)$.

To prove $(\forall n \in \mathbb{N})(n \in T \rightarrow n+1 \in T)$, we let $k \in \mathbb{N}$ be arbitrary such that $k \in T$. To prove $k+1 \in T$, we must prove $p(k+1)$ is true; that is, we must prove $\cap_{i=1}^{k+1} S_i$ is an open subset of $\mathbb{R}$.

Since $k \in T$, then $p(k)$ is true, so $\cap_{i=1}^{k} S_i$ is an open subset of $\mathbb{R}$. Hence, the intersection of a collection of k open subsets of $\mathbb{R}$ is open.

Observe that $\cap_{i=1}^{k+1} S_i = S_1 \cap (\cap_{i=2}^{k+1} S_i)$.

We know S_1 is an open subset of $\mathbb{R}$. The set $\cap_{i=2}^{k+1} S_i$ is a collection of $k+1-2+1 = k$ open subsets of $\mathbb{R}$. Hence, $\cap_{i=2}^{k+1} S_i$ is open.

Since the intersection of two open subsets of $\mathbb{R}$ is open, then $S_1 \cap (\cap_{i=2}^{k+1} S_i)$ is open. Hence, the set $\cap_{i=1}^{k+1} S_i$ is open, as desired. □

Proof. Let $n \in \mathbb{Z}, n \geq 0$.

Let $\{S_1, S_2, S_3, ..., S_n\}$ be a finite collection of n open subsets of $\mathbb{R}$.

Then $\{S_1, S_2, S_3, ..., S_n\}$ is a collection of either zero open subsets of $\mathbb{R}$ or of 1 or more open subsets of $\mathbb{R}$.

Since a collection of zero open subsets of $\mathbb{R}$ is empty, then we must prove two statements:

1. The empty set is an open subset of $\mathbb{R}$.

2. The set $\cap_{i=1}^{n} S_i$ is an open subset of $\mathbb{R}$, where each S_i is an open subset of $\mathbb{R}$, for $i = 1, 2, 3, ..., n$.

We prove the empty set is open.

Since the empty set is open, then we are done.

To prove the set $\cap_{i=1}^{n} S_i$ is an open subset of $\mathbb{R}$, we define predicate $p(n)$: $\cap_{i=1}^{n} S_i$ is an open subset of $\mathbb{R}$.

To prove $\cap_{i=1}^{n} S_i$ is an open subset of $\mathbb{R}$, we must prove $(\forall n \in \mathbb{N})(p(n))$.

Let T be the truth set of $p(n)$.

Then $T = \{n \in \mathbb{N} : p(n)\}$, so $T \subset \mathbb{N}$.

To prove $(\forall n \in \mathbb{N})(p(n))$, we must prove $T = \mathbb{N}$.

To prove $T = \mathbb{N}$, we use induction.

Basis:

To prove $1 \in T$, we must prove $p(1)$ is true. Thus, we must prove $\cap_{i=1}^{1} S_i$ is an open subset of $\mathbb{R}$.

Since $\cap_{i=1}^{1} S_i = S_1$ and S_1 is an open subset of $\mathbb{R}$, then $\cap_{i=1}^{1} S_i$ is an open subset of $\mathbb{R}$, as desired.

Induction:

To prove T is inductive, we must prove $(\forall n \in \mathbb{N})(n \in T \rightarrow n + 1 \in T)$.

To prove $(\forall n \in \mathbb{N})(n \in T \rightarrow n + 1 \in T)$, we let k be an arbitrary natural number such that $k \in T$. To prove $k + 1 \in T$, we must prove $p(k + 1)$ is true, so we must prove $\cap_{i=1}^{k+1} S_i$ is an open subset of $\mathbb{R}$.

Since $k \in T$, then $p(k)$ is true, so the set $\cap_{i=1}^{k} S_i$ is an open subset of $\mathbb{R}$. Hence, the intersection of a collection of k open subsets of $\mathbb{R}$ is open.

Observe that the intersection of a collection of $k + 1$ open subsets of $\mathbb{R}$ is $\cap_{i=1}^{k+1} S_i = S_1 \cap (\cap_{i=2}^{k+1} S_i)$.

We know the set S_1 is an open subset of $\mathbb{R}$. The set $\cap_{i=2}^{k+1} S_i$ is a collection of $k + 1 - 2 + 1 = k$ open subsets of $\mathbb{R}$. Hence, the set $\cap_{i=2}^{k+1} S_i$ is open.

Since the intersection of two open subsets of $\mathbb{R}$ is open, then $S_1 \cap (\cap_{i=2}^{k+1} S_i)$ is open. Hence, the set $\cap_{i=1}^{k+1} S_i$ is open, as desired. $\square$

Proposition 9. *Let $\{S_i : i \in \mathbb{N}\}$ be a family of open subsets of $\mathbb{R}$ indexed by $\mathbb{N}$. Then is $\cap_{i=1}^{\infty} S_i$ is open?*

Solution.

Let $\{S_i : i \in \mathbb{N}\} = \{S_1, S_2, S_3, ...\}$ be a family of open subsets of $\mathbb{R}$.

Then each S_i is an open subset of $\mathbb{R}$, for $i \in \mathbb{N}$.

We must prove the set $\cap_{i=1}^{\infty} S_i$ is an open subset of $\mathbb{R}$.

We can't use an induction argument because we can't define a predicate in terms of a natural number.

We need to use a different approach, such as the one used to prove $S \cap T$ is open.

To prove $\cap_{i=1}^{\infty} S_i$ is an open set, we must prove every point in $\cap_{i=1}^{\infty} S_i$ is an interior point of $\cap_{i=1}^{\infty} S_i$. Thus, let $p \in \cap_{i=1}^{\infty} S_i$ be arbitrary. Then $(\forall i \in \mathbb{N})(p \in S_i)$. Hence, let $i \in \mathbb{N}$ be particular such that $p \in S_i$ for some particular set S_i.

Then set S_i is a particular open subset of $\mathbb{R}$ such that $p \in S_i$ for some particular $i \in \mathbb{N}$.

To prove p is an interior point of $\cap_{i=1}^{\infty} S_i$, we must prove $(\exists \delta > 0)(N(p; \delta) \subset \cap_{i=1}^{\infty} S_i)$.

How do we prove $N(p; \delta) \subset \cap_{i=1}^{\infty} S_i$?

We assume $x \in N(p;\delta)$. To prove $x \in \cap_{i=1}^{\infty} S_i$, we must prove $(\forall i \in \mathbb{N})(x \in S_i)$.

Since S_i is an open subset of $\mathbb{R}$, then every point in S_i is an interior point of S_i.

Hence, in particular, since $p \in S_i$, then p must be an interior point of S_i. Therefore, $(\exists \delta_1 > 0)(N(p;\delta_1) \subset S_i)$.

If we could show $x \in N(p;\delta_1)$, then we could argue $x \in N(p;\delta_1) \subset S_i \Rightarrow x \in S_i$. Since $x \in N(p;\delta)$ and we want $x \in N(p;\delta_1)$, we choose $\delta = \delta_1$.

Thus, let $\delta = \delta_1$.

Then $\delta = \delta_1 > 0$, so $\delta > 0$, as desired.

To prove $N(p;\delta) \subset \cap_{i=1}^{\infty} S_i$, we assume $x \in N(p;\delta)$. To prove $x \in \cap_{i=1}^{\infty} S_i$, we must prove $(\forall i \in \mathbb{N})(x \in S_i)$.

Since $x \in N(p;\delta) = N(p;\delta_1) \subset S_i$, then $x \in S_i$.

But, this only proves x is in a particular open subset S_i. It does not prove x is in every open subset S_i. Therefore, we cannot conclude $\cap_{i=1}^{\infty} S_i$ is necessarily open. It's possible it might, but not solely based on the given information. □

Proposition 10. *Let S and T be open subsets of $\mathbb{R}$. Then $S \cup T$ is open.*

Solution. To prove $S \cup T$ is open, we must prove every point in $S \cup T$ is an interior point.

Thus, let $p \in S \cup T$ be arbitrary.

To prove p is an interior point of $S \cup T$, we must find $\delta > 0$ such that $N(p;\delta) \subset S \cup T$.

Since $p \in S \cup T$, then either $p \in S$ or $p \in T$.

We consider these cases separately.

Case 1: Suppose $p \in S$.

Since S is open, then every point in S is an interior point of S. Hence, in particular, p is an interior point of S. Therefore, $N(p;\delta_1) \subset S$ for some $\delta_1 > 0$.

Let $\delta = \delta_1$. Then $\delta > 0$.

We must prove $N(p;\delta) \subset S \cup T$.

Since $\delta = \delta_1$, then $N(p;\delta) = N(p;\delta_1)$.

Since $N(p;\delta_1) \subset S$, then $N(p;\delta) \subset S$.

Since $N(p;\delta) \subset S$ and $S \subset S \cup T$, then by transitivity of $\subset$, $N(p;\delta) \subset S \cup T$.

Therefore, $N(p;\delta) \subset S \cup T$ for $\delta = \delta_1$.

Case 2: Suppose $p \in T$.

Since T is open, then every point in T is an interior point of T. Hence, in particular, p is an interior point of T. Therefore, $N(p;\delta_2) \subset T$ for some $\delta_2 > 0$.

Let $\delta = \delta_2$. Then $\delta > 0$.

We must prove $N(p;\delta) \subset S \cup T$.

Since $\delta = \delta_2$, then $N(p;\delta) = N(p;\delta_2)$.

Since $N(p;\delta_2) \subset T$, then $N(p;\delta) \subset T$.

Since $N(p;\delta) \subset T$ and $T \subset S \cup T$, then by transitivity of $\subset$, $N(p;\delta) \subset S \cup T$.

Therefore, $N(p;\delta) \subset S \cup T$ for $\delta = \delta_2$.

Therefore, in all cases, for some $\delta > 0$, $N(p;\delta) \subset S \cup T$, as desired. □

Proof. To prove set $S \cup T$ is open, we must prove every point in $S \cup T$ is an interior point of $S \cup T$.

Let p be an arbitrary point in $S \cup T$.

To prove p is an interior point of $S \cup T$, we must find a positive δ such that the δ neighborhood of p, $N(p;\delta)$, is contained in $S \cup T$; that is, we will show there is some positive δ such that $N(p;\delta) \subset S \cup T$.

Since $p \in S \cup T$, then either $p \in S$ or $p \in T$.

We consider these cases separately.

Case 1: Suppose $p \in S$.

Since S is open, then every point in S is an interior point of S. Hence, in particular, p is an interior point of S. Therefore, $N(p;\delta_1) \subset S$ for some $\delta_1 > 0$.

Let $\delta = \delta_1$. Then $\delta > 0$.

Observe that $N(p;\delta) = N(p;\delta_1) \subset S \subset S \cup T$. Hence, $N(p;\delta) \subset S \cup T$.

Therefore, $N(p;\delta) \subset S \cup T$ for some positive δ.

Case 2: Suppose $p \in T$.

Since T is open, then every point in T is an interior point of T. Hence, in particular, p is an interior point of T. Therefore, $N(p;\delta_2) \subset T$ for some $\delta_2 > 0$.

Let $\delta = \delta_2$. Then $\delta > 0$.

Observe that $N(p;\delta) = N(p;\delta_2) \subset T \subset S \cup T$. Hence, $N(p;\delta) \subset S \cup T$.

Therefore, $N(p;\delta) \subset S \cup T$ for some positive δ.

Thus, in all cases, there is some positive δ such that the δ neighborhood of p is contained in $S \cup T$, as desired. □

Exercise 11. If S_1, S_2 and S_3 are open subsets of $\mathbb{R}$, then $S_1 \cup S_2 \cup S_3$ is an open subset of $\mathbb{R}$.

Solution.

To prove $S_1 \cup S_2 \cup S_3$ is an open subset of $\mathbb{R}$, we must prove every point in $S_1 \cup S_2 \cup S_3$ is an interior point of $S_1 \cup S_2 \cup S_3$. Thus, let p be an arbitrary point in $S_1 \cup S_2 \cup S_3$. To prove p is an interior point of $S_1 \cup S_2 \cup S_3$, we must show there exists some $\delta > 0$ such that the δ neighborhood of p is contained entirely in $S_1 \cup S_2 \cup S_3$; that is, we must show $N(p;\delta) \subset S_1 \cup S_2 \cup S_3$ for some $\delta > 0$.

Since $p \in S_1 \cup S_2 \cup S_3$, then either $p \in S_1$ or $p \in S_2$ or $p \in S_3$.

We consider these cases separately.

Case 1: Suppose $p \in S_1$.

Since S_1 is an open set, then every point in S_1 is an interior point of S_1. Hence, in particular, p is an interior point of S_1. Thus, there exists a positive δ_1 such that the δ_1 neighborhood of p is contained entirely in S_1; that is, $N(p;\delta_1) \subset S_1$ for some $\delta_1 > 0$.

Let $\delta = \delta_1$. Then $\delta > 0$.

We must prove $N(p;\delta) \subset S_1 \cup S_2 \cup S_3$.

Since $\delta = \delta_1$, then $N(p;\delta) = N(p;\delta_1)$. Since $N(p;\delta_1) \subset S_1$, then $N(p;\delta) \subset S_1$. We want to prove $N(p;\delta) \subset S_1 \cup S_2 \cup S_3$, so if we could prove $S_1 \subset S_1 \cup S_2 \cup S_3$, then we'd be done because of transitivity of $\subset$ relation.

We know that for any set X, X is a subset of a finite union of X and any number of sets, so in particular, S is a subset of a finite union of 3 sets, S_1, S_2, and S_3. Therefore, $S_1 \subset S_1 \cup S_2 \cup S_3$.

Since $N(p;\delta) \subset S_1$ and $S_1 \subset S_1 \cup S_2 \cup S_3$, then $N(p;\delta) \subset S_1 \cup S_2 \cup S_3$.

Hence, $N(p;\delta) \subset S_1 \cup S_2 \cup S_3$ for some positive δ.

Case 2: Suppose $p \in S_2$.

Since S_2 is an open set, then every point in S_2 is an interior point of S_2. Hence, in particular, p is an interior point of S_2. Thus, there exists a positive δ_2 such that the δ_2 neighborhood of p is contained entirely in S_2; that is, $N(p;\delta_2) \subset S_2$ for some $\delta_2 > 0$.

Let $\delta = \delta_2$. Then $\delta > 0$.

We must prove $N(p;\delta) \subset S_1 \cup S_2 \cup S_3$.

Since $\delta = \delta_2$, then $N(p;\delta) = N(p;\delta_2)$. Since $N(p;\delta_2) \subset S_2$, then $N(p;\delta) \subset S_2$.

Since $N(p;\delta) \subset S_2$ and $S_2 \subset S_1 \cup S_2 \cup S_3$, then $N(p;\delta) \subset S_1 \cup S_2 \cup S_3$.

Hence, $N(p;\delta) \subset S_1 \cup S_2 \cup S_3$ for some positive δ.

Case 3: Suppose $p \in S_3$.

Since S_3 is an open set, then every point in S_3 is an interior point of S_3. Hence, in particular, p is an interior point of S_3. Thus, there exists a positive δ_3 such that the δ_3 neighborhood of p is contained entirely in S_3; that is, $N(p;\delta_3) \subset S_3$ for some $\delta_3 > 0$.

Let $\delta = \delta_3$. Then $\delta > 0$.

We must prove $N(p;\delta) \subset S_1 \cup S_2 \cup S_3$.

Since $\delta = \delta_3$, then $N(p;\delta) = N(p;\delta_3)$. Since $N(p;\delta_3) \subset S_3$, then $N(p;\delta) \subset S_3$. Since $N(p;\delta) \subset S_3$ and $S_3 \subset S_1 \cup S_2 \cup S_3$, then $N(p;\delta) \subset S_1 \cup S_2 \cup S_3$.

Hence, $N(p;\delta) \subset S_1 \cup S_2 \cup S_3$ for some positive δ.

Therefore, in all cases, there is some positive δ such that the δ neighborhood of p is contained in $S_1 \cup S_2 \cup S_3$. □

Proof. To prove $S_1 \cup S_2 \cup S_3$ is an open subset of $\mathbb{R}$, we must prove every point in $S_1 \cup S_2 \cup S_3$ is an interior point of $S_1 \cup S_2 \cup S_3$.

Let p be an arbitrary point in $S_1 \cup S_2 \cup S_3$.

To prove p is an interior point of $S_1 \cup S_2 \cup S_3$, we must find some positive δ such that the δ neighborhood of p is contained entirely in $S_1 \cup S_2 \cup S_3$; that is, we must show $N(p;\delta) \subset S_1 \cup S_2 \cup S_3$ for some $\delta > 0$.

Since $p \in S_1 \cup S_2 \cup S_3$, then either $p \in S_1$ or $p \in S_2$ or $p \in S_3$.

We consider these cases separately.

Case 1: Suppose $p \in S_1$.

Since S_1 is an open set, then every point in S_1 is an interior point of S_1. Hence, in particular, p is an interior point of S_1. Thus, there is some positive δ_1 such that the δ_1 neighborhood of p is contained entirely in S_1; that is, $N(p;\delta_1) \subset S_1$ for some $\delta_1 > 0$.

Let $\delta = \delta_1$. Then $\delta > 0$.

Since $\delta = \delta_1$ and $N(p;\delta_1) \subset S_1$, then $N(p;\delta) \subset S_1$.

Since $N(p;\delta) \subset S_1$ and $S_1 \subset S_1 \cup S_2 \cup S_3$, then $N(p;\delta) \subset S_1 \cup S_2 \cup S_3$.

Hence, $N(p;\delta) \subset S_1 \cup S_2 \cup S_3$ for some positive δ.

Case 2: Suppose $p \in S_2$.

Since S_2 is an open set, then every point in S_2 is an interior point of S_2. Hence, in particular, p is an interior point of S_2. Thus, there is some positive δ_2

such that the δ_2 neighborhood of p is contained entirely in S_2; that is, $N(p;\delta_2) \subset S_2$ for some $\delta_2 > 0$.

Let $\delta = \delta_2$. Then $\delta > 0$.

Since $\delta = \delta_2$ and $N(p;\delta_2) \subset S_2$, then $N(p;\delta) \subset S_2$.

Since $N(p;\delta) \subset S_2$ and $S_2 \subset S_1 \cup S_2 \cup S_3$, then $N(p;\delta) \subset S_1 \cup S_2 \cup S_3$.

Hence, $N(p;\delta) \subset S_1 \cup S_2 \cup S_3$ for some positive δ.

Case 3: Suppose $p \in S_3$.

Since S_3 is an open set, then every point in S_3 is an interior point of S_3. Hence, in particular, p is an interior point of S_3. Thus, there is some positive δ_3 such that the δ_3 neighborhood of p is contained entirely in S_3; that is, $N(p;\delta_3) \subset S_3$ for some $\delta_3 > 0$.

Let $\delta = \delta_3$. Then $\delta > 0$.

Since $\delta = \delta_3$ and $N(p;\delta_3) \subset S_3$, then $N(p;\delta) \subset S_3$.

Since $N(p;\delta) \subset S_3$ and $S_3 \subset S_1 \cup S_2 \cup S_3$, then $N(p;\delta) \subset S_1 \cup S_2 \cup S_3$.

Hence, $N(p;\delta) \subset S_1 \cup S_2 \cup S_3$ for some positive δ.

Therefore, in all cases, there is some positive δ such that the δ neighborhood of p is contained in $S_1 \cup S_2 \cup S_3$. $\square$

Proposition 12. *The union of any finite collection of open subsets of* $\mathbb{R}$ *is open.*

Solution.

The statement means:

Let $n \in \mathbb{Z}, n \geq 0$.

Let $\{S_1, S_2, S_3, ..., S_n\}$ be a finite collection of n open subsets of $\mathbb{R}$.

Then each S_i is an open subset of $\mathbb{R}$, for $i = 1, 2, 3, ..., n$.

We must prove $\cup_{i=1}^{n} S_i$ is an open subset of $\mathbb{R}$.

Let $\mathbb{N}_n = \{1, 2, 3, ..., n\}$ be the domain of discourse.

Define predicate $p(n) : \cup_{i=1}^{n} S_i$ is an open subset of $\mathbb{R}$ over $\mathbb{N}_n$.

To prove $\cup_{i=1}^{n} S_i$ is an open subset of $\mathbb{R}$, we must prove $(\forall n \in \mathbb{N}_n)(p(n))$.

Let T be the truth set of $p(n)$.

Then $T = \{n \in \mathbb{N}_n : p(n)\}$, so $T \subset \mathbb{N}_n$.

To prove $(\forall n \in \mathbb{N}_n)(p(n))$, we must prove $T = \mathbb{N}_n$.

To prove $T = \mathbb{N}$, we use an induction argument.

Basis:

To prove $1 \in T$, we must prove $p(1)$ is true. Thus, we must prove $\cup_{i=1}^{1} S_i$ is an open subset of $\mathbb{R}$.

Since $\cup_{i=1}^{1} S_i = S_1$ and S_1 is an open subset of $\mathbb{R}$, then $\cup_{i=1}^{1} S_i$ is an open subset of $\mathbb{R}$, as desired.

Induction:

To prove T is inductive, we must prove $(\forall n \in \mathbb{N}_n)(n \in T \rightarrow n + 1 \in T)$. Thus, we let $k \in \mathbb{N}_n$ be arbitrary such that $k \in T$. To prove $k + 1 \in T$, we must prove $p(k + 1)$ is true; that is, we must prove $\cup_{i=1}^{k+1} S_i$ is an open subset of $\mathbb{R}$.

Since $k \in T$, then $p(k)$ is true; that is, $\cup_{i=1}^{k} S_i$ is an open subset of $\mathbb{R}$. Hence, the union of a collection of k open subsets of $\mathbb{R}$ is open.

Observe that $\cup_{i=1}^{k+1} S_i = S_1 \cup (\cup_{i=2}^{k+1} S_i)$.

We know S_1 is an open subset of $\mathbb{R}$. The set $\cup_{i=2}^{k+1} S_i$ is a collection of $k+1-2+1 = k$ open subsets of $\mathbb{R}$. Hence, $\cup_{i=2}^{k+1} S_i$ is open.

Since the union of two open subsets of $\mathbb{R}$ is open, then $S_1 \cup (\cup_{i=2}^{k+1} S_i)$ is open. Hence, the set $\cup_{i=1}^{k+1} S_i$ is open, as desired. ☐

Proof. Let $n \in \mathbb{Z}, n \geq 0$.

Let $\{S_1, S_2, S_3, ..., S_n\}$ be a finite collection of n open subsets of $\mathbb{R}$.

Then $\{S_1, S_2, S_3, ..., S_n\}$ is a collection of either zero open subsets of $\mathbb{R}$ or of 1 or more open subsets of $\mathbb{R}$.

Since a collection of zero open subsets of $\mathbb{R}$ is empty, then we must prove two statements:

1. The empty set is an open subset of $\mathbb{R}$.

2. The set $\cup_{i=1}^{n} S_i$ is an open subset of $\mathbb{R}$, where each S_i is an open subset of $\mathbb{R}$, for $i = 1, 2, 3, ..., n$.

We prove the empty set is an open subset of $\mathbb{R}$.

Since the empty set is an open subset of $\mathbb{R}$, then we are done.

To prove the set $\cup_{i=1}^{n} S_i$ is an open subset of $\mathbb{R}$, we define the predicate $p(n) : \cup_{i=1}^{n} S_i$ is an open subset of $\mathbb{R}$.

To prove $\cup_{i=1}^{n} S_i$ is an open subset of $\mathbb{R}$, we must prove $(\forall n \in \mathbb{N}_n)(p(n))$.

Let T be the truth set of $p(n)$.

Then $T = \{n \in \mathbb{N}_n : p(n)\}$, so $T \subset \mathbb{N}_n$.

To prove $(\forall n \in \mathbb{N}_n)(p(n))$, we must prove $T = \mathbb{N}_n$.

To prove $T = \mathbb{N}_n$, we use an induction argument.

Basis:

To prove $1 \in T$, we must prove $p(1)$ is true. Thus, we must prove $\cup_{i=1}^{1} S_i$ is an open subset of $\mathbb{R}$.

Since $\cup_{i=1}^{1} S_i = S_1$ and S_1 is an open subset of $\mathbb{R}$, then $\cup_{i=1}^{1} S_i$ is an open subset of $\mathbb{R}$, as desired.

Induction:

To prove T is inductive, we must prove $(\forall n \in \mathbb{N}_n)(n \in T \to n+1 \in T)$. Thus, we let k be an arbitrary natural number such that $k \in T$. To prove $k+1 \in T$, we must prove $p(k+1)$ is true, so we must prove the set $\cup_{i=1}^{k+1} S_i$ is an open subset of $\mathbb{R}$.

Since $k \in T$, then $p(k)$ is true, so the set $\cup_{i=1}^{k} S_i$ is an open subset of $\mathbb{R}$. Hence, the union of a collection of k open subsets of $\mathbb{R}$ is open.

Observe that the union of a collection of $k+1$ open subsets of $\mathbb{R}$ is $\cup_{i=1}^{k+1} S_i = S_1 \cup (\cup_{i=2}^{k+1} S_i)$.

We know the set S_1 is an open subset of $\mathbb{R}$. The set $\cup_{i=2}^{k+1} S_i$ is a collection of $k+1-2+1 = k$ open subsets of $\mathbb{R}$. Hence, the set $\cup_{i=2}^{k+1} S_i$ is open.

Since the union of two open subsets of $\mathbb{R}$ is open, then $S_1 \cup (\cup_{i=2}^{k+1} S_i)$ is open. Hence, the set $\cup_{i=1}^{k+1} S_i$ is open, as desired. ☐

Proposition 13. *Let $\{S_i : i \in \mathbb{N}\}$ be a family of open subsets of $\mathbb{R}$ indexed by $\mathbb{N}$. Then $\cup_{i=1}^{\infty} S_i$ is open.*

Solution.

Let $\{S_i : i \in \mathbb{N}\} = \{S_1, S_2, S_3, ...\}$ be a family of open subsets of $\mathbb{R}$.

Then each S_i is an open subset of $\mathbb{R}$, for $i \in \mathbb{N}$.

We must prove the set $\cup_{i=1}^{\infty} S_i$ is an open subset of $\mathbb{R}$.

We can't use an induction argument because we can't define a predicate in terms of a natural number.

We need to use a different approach, such as the one used to prove $S \cup T$ is open.

To prove $\cup_{i=1}^{\infty} S_i$ is an open set, we must prove every point in $\cup_{i=1}^{\infty} S_i$ is an interior point of $\cup_{i=1}^{\infty} S_i$. Thus, let $p \in \cup_{i=1}^{\infty} S_i$ be arbitrary. Then $(\exists i \in \mathbb{N})(p \in S_i)$.

Hence, let S_i be an arbitrary open subset of $\mathbb{R}$ such that $p \in S_i$ for some $i \in \mathbb{N}$.

To prove p is an interior point of $\cup_{i=1}^{\infty} S_i$, we must prove $(\exists \delta > 0)(N(p; \delta) \subset \cup_{i=1}^{\infty} S_i)$.

How do we prove $N(p; \delta) \subset \cup_{i=1}^{\infty} S_i$?

We assume $x \in N(p; \delta)$. To prove $x \in \cup_{i=1}^{\infty} S_i$, we must prove $(\exists i \in \mathbb{N})(x \in S_i)$.

Since S_i is an arbitrary open subset of $\mathbb{R}$, then every point in S_i is an interior point of S_i.

Hence, in particular, since $p \in S_i$, then p must be an interior point of S_i. Therefore, $(\exists \delta_1 > 0)(N(p; \delta_1) \subset S_i)$.

If we could show $x \in N(p; \delta_1)$, then we could argue $x \in N(p; \delta_1) \subset S_i \Rightarrow x \in S_i$. Since $x \in N(p; \delta)$ and we want $x \in N(p; \delta_1)$, we choose $\delta = \delta_1$.

We know δ will be a function of p (indirectly it is, because δ is in terms of δ_1 which is in terms of p).

Thus, let $\delta = \delta_1$. $\square$

Proof. Let $\{S_i : i \in \mathbb{N}\}$ be a family of open subsets of $\mathbb{R}$.

Then each set S_i is an open subset of $\mathbb{R}$, for $i \in \mathbb{N}$.

To prove the set $\cup_{i=1}^{\infty} S_i$ is an open subset of $\mathbb{R}$, we must prove every point in $\cup_{i=1}^{\infty} S_i$ is an interior point of $\cup_{i=1}^{\infty} S_i$.

Let $p \in \cup_{i=1}^{\infty} S_i$ be arbitrary.

To prove p is an interior point of $\cup_{i=1}^{\infty} S_i$, we must prove $(\exists \delta > 0)(N(p; \delta) \subset \cup_{i=1}^{\infty} S_i)$.

Since $p \in \cup_{i=1}^{\infty} S_i$, then $(\exists i \in \mathbb{N})(p \in S_i)$.

Hence, let S_i be an arbitrary open subset of $\mathbb{R}$ such that $p \in S_i$ for some $i \in \mathbb{N}$.

Since S_i is open, then every point in S_i is an interior point of S_i.

Hence, in particular, since $p \in S_i$, then p must be an interior point of S_i.

Therefore, $(\exists \delta_1 > 0)(N(p; \delta_1) \subset S_i)$.

Let $\delta = \delta_1$.

Then $\delta = \delta_1 > 0$, so $\delta > 0$, as desired.

To prove $N(p; \delta) \subset \cup_{i=1}^{\infty} S_i$, we assume $x \in N(p; \delta)$. To prove $x \in \cup_{i=1}^{\infty} S_i$, we must prove $(\exists i \in \mathbb{N})(x \in S_i)$.

Since $x \in N(p; \delta) = N(p; \delta_1) \subset S_i$, then $x \in S_i$, as desired. $\square$

Proposition 14. *The empty set is closed.*

Solution.

The conclusion is: $\emptyset$ is closed.

A set S is closed iff every point of accumulation of S is in S. A set S is not closed iff some point of accumulation of S is not in S.

To prove $\emptyset$ is closed or not, we must know whether $\emptyset$ has any points of accumulation.

Let's suppose $\emptyset$ has some point of accumulation.

Let a be an arbitrary point of accumulation of $\emptyset$. Then every deleted δ neighborhood of a intersects $\emptyset$. Hence, for every $\delta > 0$, $N'(a;\delta) \cap \emptyset \neq \emptyset$.

But, we know $A \cap \emptyset = \emptyset$ for every set A. Thus, the set $N'(a;\delta) \cap \emptyset$ must equal $\emptyset$. Hence, it cannot be the case that $\emptyset$ has some point of accumulation. Therefore, $\emptyset$ has no points of accumulation.

We use proof by contradiction. Suppose $\emptyset$ is not closed. Then there exists some point of accumulation that is not in $\emptyset$. But, $\emptyset$ has no points of accumulation, so $\emptyset$ cannot be not closed. Therefore, $\emptyset$ must be closed. □

Proof. Suppose the empty set has a point of accumulation. Let a be an arbitrary point of accumulation of $\emptyset$. Then every deleted δ neighborhood of a intersects $\emptyset$. Hence, for every $\delta > 0$, $N'(a;\delta) \cap \emptyset \neq \emptyset$. Let δ be an arbitrary positive real number. Then $N'(a;\delta) \cap \emptyset \neq \emptyset$.

Since $A \cap \emptyset = \emptyset$ for every set A, then in particular, the set $N'(a;\delta) \cap \emptyset$ must equal $\emptyset$. Hence, it cannot be the case that $\emptyset$ has some point of accumulation. Therefore, $\emptyset$ has no points of accumulation.

We use proof by contradiction. Suppose $\emptyset$ is not closed. Then there exists some point of accumulation of $\emptyset$ that is not in $\emptyset$. But, $\emptyset$ has no points of accumulation, so there cannot exist some point of accumulation of $\emptyset$ that is not in $\emptyset$. Hence, $\emptyset$ cannot be not closed. Therefore, $\emptyset$ must be closed. □

Proposition 15. $\mathbb{R}$ *is a closed subset of itself.*

Solution.

To prove $\mathbb{R}$ is closed, we must prove every point of accumulation of $\mathbb{R}$ is in $\mathbb{R}$.

Let p be an arbitrary point of accumulation of $\mathbb{R}$.

We must prove $p \in \mathbb{R}$.

By definition of point of accumulation, p is a real number. □

Proof. To prove $\mathbb{R}$ is closed, we must prove every point of accumulation of $\mathbb{R}$ is in $\mathbb{R}$.

Let p be an arbitrary point of accumulation of $\mathbb{R}$.

We must prove $p \in \mathbb{R}$.

By definition of point of accumulation, p is a real number, so $p \in \mathbb{R}$, as desired. □

Proposition 16. *Any interval of the form* $[a, b]$*, where* $a \neq b$*, is a closed subset of* $\mathbb{R}$*.*

Solution. The statement to prove is:

Every interval of the form $[a, b]$ with $a \neq b$ is a closed subset of $\mathbb{R}$.

This statement has the form $\forall x.p(x)$.

Thus, let $[a, b]$ be an arbitrary interval such that $a \neq b$.

To prove $[a, b]$ is a closed subset of $\mathbb{R}$, we must prove every point of accumulation of $[a, b]$ is in $[a, b]$.

Hence, we must prove $(\forall p)(p \in [a, b])$, where p is a point of accumulation of $[a, b]$.

Therefore, let p be an arbitrary point of accumulation of $[a, b]$.

We must prove $p \in [a, b]$.

Since $a \neq b$, then either $a < b$ or $a > b$.

We consider these cases separately.

Case 1: Suppose $a < b$.

Either $p < a$ or $p \in [a, b]$ or $p > b$.

We can draw a picture of this scenario.

Suppose $p < a$.

Since p is a point of accumulation of $[a, b]$, then every deleted δ neighborhood of p intersects $[a, b]$.

Hence, for every $\delta > 0$, $N'(p; \delta) \cap [a, b] \neq \emptyset$.

Let $\delta = a - p$.

Since $p < a$, then $a > p$, so $a - p > 0$, so $\delta > 0$, as desired.

Since the set $N'(p; \delta) \cap [a, b]$ is not empty, then there exists some element x in both $N'(p; \delta)$ and $[a, b]$.

Let x be an arbitrary element of both $N'(p; \delta)$ and $[a, b]$.

Since $x \in N'(p; \delta)$, then $p - \delta < x < p + \delta$, so $p - (a - p) < x < p + (a - p)$. Thus, $2p - a < x < a$, so $x < a$.

Since $x \in [a, b]$, then $a \leq x \leq b$, so $a \leq x$.

Hence, we have $x < a$ and $x \geq a$, which violates the trichotomy law of $\mathbb{R}$. Therefore, p cannot be less than a.

Suppose $p > b$. Let $\delta = p - b$. Since $p > b$, then $p - b > 0$, so $\delta > 0$, as desired. Since the set $N'(p; \delta) \cap [a, b]$ is not empty, then there exists some element x in both $N'(p; \delta)$ and $[a, b]$.

Let x be an arbitrary element of both $N'(p; \delta)$ and $[a, b]$.

Since $x \in N'(p; \delta)$, then $p - \delta < x < p + \delta$, so $p - (p - b) < x < p + (p - b)$. Thus, $b < x < 2p - b$, so $b < x$.

Since $x \in [a, b]$, then $a \leq x \leq b$, so $x \leq b$.

Hence, we have $x > b$ and $x \leq b$, which violates the trichotomy law of $\mathbb{R}$. Therefore, p cannot be greater than b.

Thus, since neither p can be less than a nor greater than b, then the only possibility is that p must be in $[a, b]$. Therefore, $p \in [a, b]$, as desired.

□

Proof. To prove every interval of the form $[a, b]$ with $a \neq b$ is a closed subset of $\mathbb{R}$, let $[a, b]$ be an arbitrary interval such that $a \neq b$ for real numbers a and b.

To prove $[a, b]$ is a closed subset of $\mathbb{R}$, we must prove every point of accumulation of $[a, b]$ is in $[a, b]$.

Let p be an arbitrary point of accumulation of $[a, b]$.

We must prove $p \in [a, b]$.

Since $a \neq b$, then either $a < b$ or $a > b$.

We consider these cases separately.

Case 1: Suppose $a < b$.

Either $p < a$ or $p \in [a, b]$ or $p > b$.

Since p is a point of accumulation of $[a, b]$, then every deleted δ neighborhood of p intersects $[a, b]$.

Hence, for every $\delta > 0$, $N'(p; \delta) \cap [a, b] \neq \emptyset$.

Suppose $p < a$.

Let $\delta = a - p$.

Since $p < a$, then $a > p$, so $a - p > 0$, so $\delta > 0$, as desired.

Since the set $N'(p; \delta) \cap [a, b]$ is not empty, then there exists some element x in both $N'(p; \delta)$ and $[a, b]$.

Let x be an arbitrary element of both $N'(p; \delta)$ and $[a, b]$.

Since $x \in N'(p; \delta)$, then $p - \delta < x < p + \delta$, so $p - (a - p) < x < p + (a - p)$. Thus, $2p - a < x < a$, so $x < a$.

Since $x \in [a, b]$, then $a \leq x \leq b$, so $a \leq x$.

Hence, we have $x < a$ and $x \geq a$, which violates the trichotomy law of $\mathbb{R}$. Therefore, p cannot be less than a.

Suppose $p > b$.

Let $\delta = p - b$.

Since $p > b$, then $p - b > 0$, so $\delta > 0$, as desired.

Since the set $N'(p; \delta) \cap [a, b]$ is not empty, then there exists some element x in both $N'(p; \delta)$ and $[a, b]$.

Let x be an arbitrary element of both $N'(p; \delta)$ and $[a, b]$.

Since $x \in N'(p; \delta)$, then $p - \delta < x < p + \delta$, so $p - (p - b) < x < p + (p - b)$. Thus, $b < x < 2p - b$, so $b < x$.

Since $x \in [a, b]$, then $a \leq x \leq b$, so $x \leq b$.

Hence, we have $x > b$ and $x \leq b$, which violates the trichotomy law of $\mathbb{R}$. Therefore, p cannot be greater than b.

Thus, since p can be neither less than a nor greater than b, then the only possibility is that p must be in $[a, b]$. Therefore, $p \in [a, b]$, as desired.

Case 2: Suppose $a > b$.

Suppose the interval $[a, b]$ is not empty. Then there exists some element in $[a, b]$. Let x be an arbitrary real number in $[a, b]$. Then $a \leq x$ and $x \leq b$. Therefore, $a \leq b$.

By trichotomy, it is impossible that $a \leq b$ and $a > b$. Therefore $[a, b]$ must be empty, so $[a, b] = \emptyset$.

Since the empty set is a closed set of $\mathbb{R}$, then $[a, b]$ is also a closed set of $\mathbb{R}$, as desired. $\square$

Proposition 17. *If S is a closed subset of $\mathbb{R}$, then its complement is an open subset of $\mathbb{R}$.*

Solution. Hypothesis is: $S \subset \mathbb{R}$ and S is a closed set.

Conclusion is: $\overline{S}$ is an open set.

Using a direct approach does not lead anywhere, especially because it is difficult to use the hypothesis.

We must try a different approach. We'll try an indirect proof using contradiction.

Thus, we'll assume the complement is not open.

Suppose for the sake of contradiction that $\overline{S}$ is not open. Then there is some point of $\overline{S}$ that is not an interior point of $\overline{S}$.

Let p be an arbitrary point of $\overline{S}$ that is not an interior point of $\overline{S}$. Then no δ neighborhood of p is contained in $\overline{S}$. Hence, every δ neighborhood of p is not contained in $\overline{S}$. Thus, every δ neighborhood of p contains at least some point not in $\overline{S}$, so every δ neighborhood of p contains some point in S. Since $p \notin S$, then every deleted δ neighborhood of p contains some point in S. Hence, p is a point of accumulation of S. Since S is closed, then every point of accumulation of S must be contained in S. Hence, p must be contained in S. Thus, $p \in S$ and $p \in \overline{S}$, so $p \in S \cap \overline{S}$. Hence, $p \in \emptyset$, a contradiction. Therefore, $\overline{S}$ must be open. □

Proof. Suppose for the sake of contradiction that the complement of S,$\overline{S}$, is not open. Then there is some point in $\overline{S}$ that is not an interior point of $\overline{S}$.

Let p be an arbitrary point in $\overline{S}$ that is not an interior point of $\overline{S}$. Then no δ neighborhood of p is contained in $\overline{S}$, so every δ neighborhood of p is not contained in $\overline{S}$. Thus, every δ neighborhood of p contains at least some point not in $\overline{S}$, so every δ neighborhood of p contains some point in S. Since p is not in S, then every deleted δ neighborhood of p contains some point in S. Hence, p is a point of accumulation of S. Since S is closed, then every point of accumulation of S must be contained in S. Hence, p must be contained in S, so $p \in S$. Since $p \in S$ and $p \in \overline{S}$, then $p \in S \cap \overline{S}$. Hence, $p \in \emptyset$, a contradiction. Therefore, $\overline{S}$ must be open. □

Theorem 18. ***Fermat's Theorem***
Let f be a real valued function defined on open interval (a, b). If $f(x_0)$ is a relative maximum on (a, b) and $f'(x_0)$ exists, then $f'(x_0) = 0$.

Solution.

Let us define propositions:

Let $H_1 : f(x_0)$ is a relative maximum on (a, b).

Let $H_2 : f'(x_0)$ exists.

Let $C : f'(x_0) = 0$.

We must prove: $(H_1 \wedge H_2) \to C$.

There does not seem to be a way to deduce C directly. So, let's try an indirect approach. Let's use proof by contradiction.

Suppose $f'(x_0) \neq 0$. Then either $f'(x_0) > 0$ or $f'(x_0) < 0$.

We consider these cases separately.

Case 1: Suppose $f'(x_0) > 0$.

How can we derive a contradiction given that $f(x_0)$ is a relative maximum on (a, b)?

We have $f'(x_0) > 0$ and if we draw picture, then we intuitively see there is some $x < x_0$ such that $f(x) < f(x_0)$. Further, there if $x > x_0$ then $f(x) > f(x_0)$. So, we should prove this and use this result.

□

Proof. Suppose for the sake of contradiction that $f'(x_0) \neq 0$. Then either $f'(x_0) > 0$ or $f'(x_0) < 0$.

We consider these cases separately.

Case 1: Suppose $f'(x_0) > 0$.

□

Proof. How do we know we can even choose $\delta > 0$?

Let $\triangle$ be a positive real number less than δ.

We consider the limit from the right.

Let x be an arbitrary real number such that $x = x_0 + \triangle x$. Since $f(x_0)$ is a relative maximum, then $f(x_0) \geq f(x)$ for every x in $N(x_0; \delta)$. Thus, $f(x_0) - f(x) \geq 0$, so $f(x) - f(x_0) \leq 0$. Since $\triangle x > 0$, then $x - x_0 > 0$. Hence, $\frac{f(x)-f(x_0)}{x-x_0} \leq 0$. Thus, we need to fill in details to show that $\lim_{x \to x_0^+} \frac{f(x)-f(x_0)}{x-x_0} \leq 0$. Let $L_1 = \lim_{x \to x_0^+} \frac{f(x)-f(x_0)}{x-x_0} \leq 0$.

We consider the limit from the left. Let x be an arbitrary real number such that $x = x_0 - \triangle x$. Since $f(x_0)$ is a relative maximum, then $f(x_0) \geq f(x)$ for every x in $N(x_0; \delta)$. Thus, $f(x_0) - f(x) \geq 0$, so $f(x) - f(x_0) \leq 0$. Since $\triangle x > 0$, then $x_0 - x > 0$, so $x - x_0 < 0$. Hence, $\frac{f(x)-f(x_0)}{x-x_0} \geq 0$. Thus, we need to fill in details to show that $\lim_{x \to x_0^+} \frac{f(x)-f(x_0)}{x-x_0} \geq 0$. Let $L_2 = \lim_{x \to x_0^-} \frac{f(x)-f(x_0)}{x-x_0} \geq 0$.

Since the $f'(x_0)$ exists and $f'(x_0) = \lim_{x \to x_0} \frac{f(x)-f(x_0)}{x-x_0}$,

let $L = \lim_{x \to x_0} \frac{f(x)-f(x_0)}{x-x_0}$.

Since the left and right limits must exist and equal L, then $L_1 = L_2 = L$. Since $L_1 \leq 0$ and $L_1 = L$, then $L \leq 0$. Since $L_2 \geq 0$ and $L_2 = L$, then $L \geq 0$. Since $L \leq 0$ and $0 \leq L$, then by the antisymmetric property of $\leq$, $L = 0$. Therefore, $f'(x_0) = 0$, as desired. □