Real valued functions Theory

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Real valued functions of a real variable

Theorem 1. ***Strictly monotonic functions are injective.***
Let f be a real valued function of a real variable.
Let S be a subset of the domain of f.
1. If f is strictly increasing on S, then f is one to one on S.
2. If f is strictly decreasing on S, then f is one to one on S.

Proof. We prove 1.
Suppose f is strictly increasing on S.
Let $a, b \in S$ such that $a \neq b$.
Then either $a < b$ or $a > b$.
Without loss of generality, we may assume $a < b$.
Since f is strictly increasing on S, then $f(a) < f(b)$, so $f(a) \neq f(b)$.
Therefore, $a \neq b$ implies $f(a) \neq f(b)$, so f is one to one. □

Proof. We prove 2.
Suppose f is strictly decreasing on S.
Let $a, b \in S$ such that $a \neq b$.
Then either $a < b$ or $a > b$.
Without loss of generality, we may assume $a < b$.
Since f is strictly decreasing on S, then $f(a) > f(b)$, so $f(a) \neq f(b)$.
Therefore, $a \neq b$ implies $f(a) \neq f(b)$, so f is one to one. □

Propositions involving algebra of functions

Proposition 2. *Let f and g each be real valued functions of a real variable.*
1. If f and g are even, then $f + g$, $f - g$, fg, and $g \circ f$ are even.
2. If f and g are odd, then $f + g$, $f - g$, and $g \circ f$ are odd and fg is even.

Proof. We prove 1.
Suppose f and g are even.
Since f is even, then $f(-x) = f(x)$ for all $x \in domf$.
Since g is even, then $g(-x) = g(x)$ for all $x \in domg$.

We prove $f + g$ is even.
Let $x \in dom(f+g)$.
Then $x \in domf$ and $x \in domg$.
Since $x \in domf$, then $f(-x) = f(x)$.
Since $x \in domg$, then $g(-x) = g(x)$.
Observe that

$$\begin{aligned} (f+g)(-x) &= f(-x) + g(-x) \\ &= f(x) + g(x) \\ &= (f+g)(x). \end{aligned}$$

Therefore, $(f+g)(-x) = (f+g)(x)$, so the function $f+g$ is even.

We prove $f - g$ is even.
Let $x \in dom(f-g)$.
Then $x \in domf$ and $x \in domg$.
Since $x \in domf$, then $f(-x) = f(x)$.
Since $x \in domg$, then $g(-x) = g(x)$.
Observe that

$$\begin{aligned} (f-g)(-x) &= f(-x) - g(-x) \\ &= f(x) - g(x) \\ &= (f-g)(x). \end{aligned}$$

Therefore, $(f-g)(-x) = (f-g)(x)$, so the function $f-g$ is even.

We prove fg is even.
Let $x \in dom(fg)$.
Then $x \in domf$ and $x \in domg$.
Since $x \in domf$, then $f(-x) = f(x)$.
Since $x \in domg$, then $g(-x) = g(x)$.
Observe that

$$\begin{aligned} (fg)(-x) &= f(-x)g(-x) \\ &= f(x)g(x) \\ &= (fg)(x). \end{aligned}$$

Therefore, $(fg)(-x) = (fg)(x)$, so the function fg is even.

We prove $g \circ f$ is even.
Let $x \in dom(g \circ f)$.
Then $x \in domf$ and $f(x) \in domg$.
Since $x \in domf$, then $f(-x) = f(x)$.

Observe that

$$
\begin{aligned}
(g \circ f)(-x) &= g(f(-x)) \\
&= g(f(x)) \\
&= (g \circ f)(x).
\end{aligned}
$$

Therefore, $(g \circ f)(-x) = (g \circ f)(x)$, so the function $g \circ f$ is even. $\square$

Proof. We prove 2.
Suppose f and g are odd.
Since f is odd, then $f(-x) = -f(x)$ for all $x \in domf$.
Since g is odd, then $g(-x) = -g(x)$ for all $x \in domg$.

We prove $f + g$ is odd.
Let $x \in dom(f + g)$.
Then $x \in domf$ and $x \in domg$.
Since $x \in domf$, then $f(-x) = -f(x)$.
Since $x \in domg$, then $g(-x) = -g(x)$.
Observe that

$$
\begin{aligned}
(f + g)(-x) &= f(-x) + g(-x) \\
&= -f(x) - g(x) \\
&= -[f(x) + g(x)] \\
&= -(f + g)(x).
\end{aligned}
$$

Therefore, $(f + g)(-x) = -(f + g)(x)$, so the function $f + g$ is odd.

We prove $f - g$ is odd.
Let $x \in dom(f - g)$.
Then $x \in domf$ and $x \in domg$.
Since $x \in domf$, then $f(-x) = -f(x)$.
Since $x \in domg$, then $g(-x) = -g(x)$.
Observe that

$$
\begin{aligned}
(f - g)(-x) &= f(-x) - g(-x) \\
&= -f(x) - (-g(x)) \\
&= -f(x) + g(x) \\
&= -[f(x) - g(x)] \\
&= -(f - g)(x).
\end{aligned}
$$

Therefore, $(f - g)(-x) = -(f - g)(x)$, so the function $f - g$ is odd.

We prove fg is even.
Let $x \in dom(fg)$.
Then $x \in dom f$ and $x \in dom g$.
Since $x \in dom f$, then $f(-x) = -f(x)$.
Since $x \in dom g$, then $g(-x) = -g(x)$.
Observe that

$$\begin{aligned}
(fg)(-x) &= f(-x)g(-x) \\
&= [-f(x)][-g(x)] \\
&= f(x)g(x) \\
&= (fg)(x).
\end{aligned}$$

Therefore, $(fg)(-x) = (fg)(x)$, so the function fg is even.

We prove $g \circ f$ is odd.
Let $x \in dom(g \circ f)$.
Then $x \in dom f$ and $f(x) \in dom g$.
Since $x \in dom f$, then $f(-x) = -f(x)$.
Since $f(x) \in dom g$, then $g(-f(x)) = -g(f(x))$.
Observe that

$$\begin{aligned}
(g \circ f)(-x) &= g(f(-x)) \\
&= g(-f(x)) \\
&= -g(f(x)) \\
&= -(g \circ f)(x).
\end{aligned}$$

Therefore, $(g \circ f)(-x) = -(g \circ f)(x)$, so the function $g \circ f$ is odd. □