Real valued functions Examples

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Real valued functions of a real variable

Example 1. greatest integer function

Let $x \in \mathbb{R}$.

Let $\lfloor x \rfloor$ denote the greatest integer n such that $n \leq x$.

Then $\lfloor x \rfloor = \max\{n \in \mathbb{Z} : n \leq x\}$.

Let $f : \mathbb{R} \to \mathbb{R}$ be a function defined by $f(x) = \lfloor x \rfloor$ for all $x \in \mathbb{R}$.

Then f is called the **greatest integer function**.

Proof. Let $x \in \mathbb{R}$.

Let $S = \{n \in \mathbb{Z} : n \leq x\}$.

We prove $\max S$ exists, thereby justifying the existence of the greatest integer function.

Since each real number lies between two consecutive integers, then there exists an integer m such that $m \leq x < m + 1$.

We prove $m = \max S$.

Since $m \leq x < m + 1$, then $m \leq x$ and $x < m + 1$.

Since $m \in \mathbb{Z}$ and $m \leq x$, then $m \in S$.

Let $s \in S$.

Then $s \in \mathbb{Z}$ and $s \leq x$.

Suppose for the sake of contradiction $s > m$.

Since $m < s$ and $s \leq x$ and $x < m + 1$, then $m < s \leq x < m + 1$, so $m < s < m + 1$.

Since $m, s \in \mathbb{Z}$ and $m < s < m + 1$, then s is an integer between the two consecutive integers m and $m + 1$.

This contradicts the fact that there is no integer between two consecutive integers.

Hence, $s \leq m$, so $s \leq m$ for all $s \in S$.

Therefore, m is an upper bound of S.

Since $m \in S$ and m is an upper bound of S, then $m = \max S$, as desired. □

Classes of real valued functions

Example 2. Let p be a polynomial function of a real variable.

Then there exist a nonnegative integer n and real numbers $a_0, a_1, ..., a_n$ such that $p(x) = a_0 + a_1x + a_2x^2 + ... + a_nx^n$.

The domain of p is $\mathbb{R}$.

Therefore, $domp = \mathbb{R}$, so the domain of any polynomial function is $\mathbb{R}$.

Proof. Let $domp$ be the domain of p.

Then $domp = \{x \in \mathbb{R} : (\exists y \in \mathbb{R})(y = p(x))\}$.

Hence, $domp \subset \mathbb{R}$.

Let $x \in \mathbb{R}$.

Let $y = a_0 + a_1 + a_2x^2 + ... + a_nx^n$.

Then $y = p(x)$.

Since $\mathbb{R}$ is closed under addition and multiplication, then $y \in \mathbb{R}$.

Hence, there exists $y \in \mathbb{R}$ such that $y = p(x)$, so $x \in domp$.

Thus, $\mathbb{R} \subset domp$.

Since $domp \subset \mathbb{R}$ and $\mathbb{R} \subset domp$, then $domp = \mathbb{R}$.

Therefore, the domain of a polynomial function is $\mathbb{R}$. □