Real valued functions Exercises

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Real valued functions of a real variable

Exercise 1. Let $f : [0, \infty) \to \mathbb{R}$ be the function defined by $f(x) = \sqrt{x}$.
Then f is increasing.

Proof. Let $a, b \in [0, \infty)$ such that $a < b$.
Since $a \in [0, \infty)$, then $a \geq 0$.
Since $0 \leq a$ and $a < b$, then $0 \leq a < b$, so $\sqrt{a} < \sqrt{b}$.
Hence, $f(a) < f(b)$, so f is increasing. □

Exercise 2. Let $f : \mathbb{N} \to \mathbb{Z}$ be the function defined by $f(n) = n^2 + n$.
Then f is increasing.

Proof. Let $a, b \in \mathbb{N}$ such that $a < b$.
Since $a \in \mathbb{N}$, then $a > 0$, so $0 < a < b$.
Hence, $a^2 < b^2$.
Since $a^2 < b^2$ and $a < b$, then $a^2 + a < b^2 + b$, so $f(a) < f(b)$.
Therefore, f is increasing. □

Exercise 3. Let $g : \mathbb{Z} \to \mathbb{R}$ be the function defined by $g(n) = n^3 - n$.
Then g is not increasing.

Proof. Since $0 < 1$ and $g(0) = 0 = g(1)$, then g is not increasing. □

Exercise 4. Let $h : \mathbb{R} \to \mathbb{R}$ be the function defined by $h(x) = |x|$.
Then h is not increasing and h is not decreasing.

Proof. Since $-1 < 1$, but $h(-1) = 1 = h(1)$, then h is not increasing and h is not decreasing. □

Exercise 5. Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by $f(x) = x^2 - 2x$.
Then f is not one to one(injective).

Proof. Since $0 \neq 2$ and $f(0) = 0 = f(2)$, then f is not one to one. □

Exercise 6. Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by $f(x) = x^3$.
Then f is increasing.

Proof. Let $a, b \in \mathbb{R}$ such that $a < b$.
Since $a \in \mathbb{R}$, then either $a > 0$ or $a = 0$ or $a < 0$.
We consider these cases separately.
Case 1: Suppose $a > 0$.
Since $0 < a$ and $a < b$, then $0 < a < b$, so $0 < a^2 < b^2$.
Since $0 < a < b$ and $0 < a^2 < b^2$, then $0 < a^3 < b^3$, so $a^3 < b^3$.
Therefore, $f(a) = a^3 < b^3 = f(b)$, so $f(a) < f(b)$.
Case 2: Suppose $a = 0$.
Since $a = 0$ and $a < b$, then $0 < b$.
Since $b > 0$, then $b^2 > 0$, so $b^3 > 0$.
Thus, $f(a) = f(0) = 0^3 = 0 < b^3 = f(b)$, so $f(a) < f(b)$.
Case 3: Suppose $a < 0$.
Since $b \in \mathbb{R}$, then either $b > 0$ or $b = 0$ or $b < 0$.
We consider these cases separately.
Case 3a: Suppose $b > 0$.
Since $a < 0$, then $a^2 > 0$, so $a^3 < 0$.
Since $b > 0$, then $b^2 > 0$, so $b^3 > 0$.
Thus, $f(a) = a^3 < 0 < b^3 = f(b)$, so $f(a) < f(b)$.
Case 3b: Suppose $b = 0$.
Since $a < 0$, then $a^2 > 0$, so $a^3 < 0$.
Thus, $f(a) = a^3 < 0 = 0^3 = b^3 = f(b)$, so $f(a) < f(b)$.
Case 3c: Suppose $b < 0$.
Since $a < 0$, then $a^2 > 0$.
Since $b < 0$, then $b^2 > 0$.
Thus, $a^2 + b^2 > 0$.
Since $a < 0$ and $b < 0$, then $ab > 0$.
Since $a^2 + b^2 > 0$ and $ab > 0$, then $a^2 + b^2 + ab > 0$.
Since $a < b$, then $0 < b - a$.
Since $b - a > 0$ and $a^2 + b^2 + ab > 0$, then $(b - a)(a^2 + b^2 + ab) > 0$, so $(b - a)(a^2 + ab + b^2) > 0$.
Hence, $b^3 - a^3 > 0$, so $b^3 > a^3$.
Thus, $f(b) > f(a)$, so $f(a) < f(b)$.

In all cases, we conclude $f(a) < f(b)$, so f is increasing. □

Exercise 7. Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by $f(x) = \sin x$.
Then f is not increasing.

Proof. Observe that $0 < \pi$, but $\sin 0 = 0 = \sin \pi$, so f is not decreasing. □

Exercise 8. The product of an even function and an odd function is odd.

Proof. Let f be an even function and g be an odd function.
We must prove the product function fg is odd.
Let $x \in dom(fg)$.
Then $x \in dom f$ and $x \in dom g$.
Since $x \in dom f$ and f is even, then $f(-x) = f(x)$.

Since $x \in domg$ and g is odd, then $g(-x) = -g(x)$.
Observe that

$$\begin{aligned} (fg)(-x) &= f(-x)g(-x) \\ &= f(x)[-g(x)] \\ &= -[f(x)][g(x)] \\ &= -(fg)(x). \end{aligned}$$

Therefore, $(fg)(-x) = -(fg)(x)$, so the function fg is odd. □

Exercise 9. Let n be a fixed positive integer.
The function given by $f(x) = a_{2n}x^{2n} + a_{2n-2}x^{2n-2} + ... + a_2x^2 + a_0$ is even.

Proof. Let $x \in domf$.
Then

$$\begin{aligned} f(-x) &= a_{2n}(-x)^{2n} + a_{2n-2}(-x)^{2n-2} + ... + a_2(-x)^2 + a_0 \\ &= a_{2n}(-x)^{2n} + a_{2n-2}(-x)^{2(n-1)} + ... + a_2(-x)^{2*1} + a_0 \\ &= a_{2n}[(-x)^2]^n + a_{2n-2}[(-x)^2]^{n-1} + ... + a_2[(-x)^2]^1 + a_0 \\ &= a_{2n}(x^2)^n + a_{2n-2}(x^2)^{n-1} + ... + a_2(x^2)^1 + a_0 \\ &= a_{2n}x^{2n} + a_{2n-2}x^{2n-2} + ... + a_2x^2 + a_0 \\ &= f(x). \end{aligned}$$

Therefore, $f(-x) = f(x)$, so f is even. □

Exercise 10. Let $f : [2, \infty) \to \mathbb{R}$ be the function defined by $f(x) = x^2 + 5$.
Then the image of $[2, \infty)$ under f is the interval $[9, \infty)$.

Proof. Let S be the image of the interval $[2, \infty)$ under f.
Let T be the interval $[9, \infty)$.
Then $S = f([2, \infty)) = \{f(x) : x \in [2, \infty)\}$ and $T = [9, \infty)$.
We must prove $S = T$.

We first prove $S \subset T$.
Let $y \in S$.
Then $y = f(x)$ for some $x \in [2, \infty)$.
Thus, $y = x^2 + 5$ for some $x \geq 2$.
Since $x \geq 2$, then $x^2 \geq 4$, so $y = x^2 + 5 \geq 4 + 5 = 9$.
Hence, $y \geq 9$, so $y \in T$.
Therefore, $S \subset T$.

We next prove $T \subset S$.
Let $y \in T$.
Then $y \geq 9$.
Let $x = \sqrt{y-5}$.
Since $y \geq 9$, then $y - 5 \geq 4$, so $\sqrt{y-5} \geq 2$.
Hence, $x \geq 2$, so $x \in [2, \infty)$.
Thus, $f(x) = x^2 + 5 = (\sqrt{y-5})^2 + 5 = (y-5) + 5 = y$.
Thus, there exists $x \in [2, \infty)$ such that $y = f(x)$, so $y \in S$.
Since $y \in T$ implies $y \in S$, then $T \subset S$.

Since $S \subset T$ and $T \subset S$, then $S = T$, as desired. □

Exercise 11. Let $f : [1, 4] \to \mathbb{R}$ be the function defined by $f(x) = 2x^2 + \sqrt{x}$.
Then f is increasing and the pre-image of 0 is the empty set.

Proof. We prove f is increasing.
Let $a, b \in [1, 4]$ such that $a < b$.
Since $a \in [1, 4]$, then $1 \leq a \leq 4$, so $1 \leq a$.
Hence, $a \geq 1 > 0$, so $a > 0$.
Since $0 < a$ and $a < b$, then $0 < a < b$, so $0 < a^2 < b^2$ and $0 < \sqrt{a} < \sqrt{b}$.
Since $0 < \sqrt{a} < \sqrt{b}$, then $\sqrt{a} < \sqrt{b}$.
Since $0 < a^2 < b^2$, then $a^2 < b^2$, so $2a^2 < 2b^2$.
Since $2a^2 < 2b^2$ and $\sqrt{a} < \sqrt{b}$, then $2a^2 + \sqrt{a} < 2b^2 + \sqrt{b}$, so $f(a) < f(b)$.
Therefore, f is increasing. □

Proof. We prove the pre-image of 0 is the empty set.
Let A be the pre-image of 0.
Then $A = f^{-1}(0) = \{x \in [1, 4] : f(x) = 0\}$.
We must prove $A = \emptyset$.
We prove by contradiction.
Suppose $A \neq \emptyset$.
Then there exists $x \in A$.
Thus, $x \in [1, 4]$ and $f(x) = 0$.
Since $x \in [1, 4]$, then $1 \leq x \leq 4$, so $1 \leq x$.
Since $x \geq 1$, then $\sqrt{x} \geq 1$.
Since $x \geq 1$, then $x^2 \geq 1$, so $2x^2 \geq 2$.
Since $2x^2 \geq 2$ and $\sqrt{x} \geq 1$, then $2x^2 + \sqrt{x} \geq 3 > 0$, so $f(x) > 0$.
Thus, we have $f(x) > 0$ and $f(x) = 0$, a violation of trichotomy of $\mathbb{R}$.
Therefore, $A = \emptyset$. □

Exercise 12. Let $f : [1, 3] \to \mathbb{R}$ be the function defined by $f(x) = x + \frac{6}{x}$.
Then f is not decreasing and determine if f is not one to one.

Proof. Since $2 < 3$, but $f(2) = 2 + \frac{6}{2} = 5 = 3 + \frac{6}{3} = f(3)$, then f is not decreasing and f is not one to one. □

Exercise 13. Let $f : [0, \infty) \to \mathbb{R}$ be the function defined by $f(x) = \frac{1}{1+x}$.
Then f is bounded.

Proof. Let $x \in [0, \infty)$.

Then $x \geq 0$.

Since $1 + x \geq 1 + 0 = 1 > 0$, then $1 + x \geq 1$ and $1 + x > 0$.

Observe that

$$\begin{aligned} 0 \leq 1 \leq 1 + x \quad &\Rightarrow \quad 0 \leq \frac{1}{1+x} \leq 1 \\ &\Leftrightarrow \quad 0 \leq f(x) \leq 1. \end{aligned}$$

Thus, $0 \leq f(x) \leq 1$ for every $x \in [0, \infty)$.

Since 0 and 1 are real numbers such that $0 \leq f(x) \leq 1$ for every $x \in [0, \infty)$, then f is bounded. □

Proof. Let $x \in [0, \infty)$.

Then $x \geq 0$, so $1 + x \geq 1 > 0$.

Hence, $1 + x > 0$.

Observe that

$$\begin{aligned} 0 \leq x \quad &\Leftrightarrow \quad 1 \leq 1 + x \\ &\Rightarrow \quad \frac{1}{1+x} \leq 1 \\ &\Rightarrow \quad \frac{|1|}{|1+x|} \leq 1 \\ &\Leftrightarrow \quad |\frac{1}{1+x}| \leq 1 \\ &\Leftrightarrow \quad |f(x)| \leq 1. \end{aligned}$$

Thus, $|f(x)| \leq 1$ for every $x \in [0, \infty)$.

Since 1 is a real number such that $|f(x)| \leq 1$ for every $x \in [0, \infty)$, then f is bounded. □

Proof. We prove by contradiction.

Suppose f is not bounded.

Then for every $b \in \mathbb{R}$ there exists $x \in [0, \infty)$ such that $|f(x)| > b$.

Let $b = 1$.

Then there exists $x \in [0, \infty)$ such that $|f(x)| > 1$.

Thus, $x \geq 0$ and $|\frac{1}{1+x}| > 1$.

Since $x \geq 0$, then $1 + x \geq 1 > 0$, so $1 + x > 0$.

Observe that

$$\begin{aligned} |\frac{1}{1+x}| > 1 \quad &\Leftrightarrow \quad \frac{|1|}{|1+x|} > 1 \\ &\Rightarrow \quad \frac{1}{1+x} > 1 \\ &\Rightarrow \quad 1 > 1 + x \\ &\Leftrightarrow \quad 0 > x. \end{aligned}$$

Since $|\frac{1}{1+x}| > 1$ and $|\frac{1}{1+x}| > 1$ implies $0 > x$, then we conclude $0 > x$.
Hence, $x \geq 0$ and $x < 0$, a violation of trichotomy in $\mathbb{R}$.
Therefore, f must be bounded. □

Exercise 14. Let $f : \mathbb{Z}^+ \to \mathbb{R}$ be the function defined by $f(n) = \frac{n}{n+1}$.
Then f is bounded.

Proof. Let $n \in \mathbb{Z}^+$.
Then $n \geq 1$.
Thus, $n + 1 \geq 2 > 0$, so $n + 1 > 0$.
Observe that

$$\begin{aligned} 1 \leq n \text{ and } 0 < 1 &\Leftrightarrow n + 1 \leq 2n \text{ and } n < n + 1 \\ &\Rightarrow \frac{1}{2} \leq \frac{n}{n+1} \text{ and } \frac{n}{n+1} < 1 \\ &\Rightarrow \frac{1}{2} \leq \frac{n}{n+1} < 1 \\ &\Rightarrow \frac{1}{2} \leq f(n) < 1. \end{aligned}$$

Thus, $1 \leq n$ and $0 < 1$ implies $\frac{1}{2} \leq f(n) < 1$.
Since $1 \leq n$ and $0 < 1$, then we conclude $\frac{1}{2} \leq f(n) < 1$.
Since $\frac{1}{2} \leq f(n) < 1 \leq 1$, then $\frac{1}{2} \leq f(n) \leq 1$.
Hence, $\frac{1}{2} \leq f(n) \leq 1$ for every $n \in \mathbb{Z}^+$.
Since $\frac{1}{2}$ and 1 are real numbers such that $\frac{1}{2} \leq f(n) \leq 1$ for every $n \in \mathbb{Z}^+$, then f is bounded. □

Proof. Let $n \in \mathbb{Z}^+$.
Then $n \geq 1$.
Since $n \geq 1 > 0$, then $n > 0$.
Since $n \geq 1$, then $n + 1 \geq 2 > 0$, so $n + 1 > 0$.
Observe that

$$\begin{aligned} 0 < 1 &\Leftrightarrow n < n + 1 \\ &\Rightarrow \frac{n}{n+1} < 1 \\ &\Rightarrow \frac{|n|}{|n+1|} < 1 \\ &\Leftrightarrow |\frac{n}{n+1}| < 1 \\ &\Leftrightarrow |f(n)| < 1. \end{aligned}$$

Thus, $0 < 1$ implies $|f(n)| < 1$.
Since $0 < 1$, then $|f(n)| < 1$.
Since $|f(n)| < 1 \leq 1$, then $|f(n)| \leq 1$, so $|f(n)| \leq 1$ for all $n \in \mathbb{Z}^+$.
Since 1 is a real number such that $|f(n)| \leq 1$ for all $n \in \mathbb{Z}^+$ then f is bounded. □

Proof. We prove by contradiction.
Suppose f is unbounded.
Then for every $b \in \mathbb{R}$, there exists $n \in \mathbb{Z}^+$ such that $|f(n)| > b$.
Let $b = 1$.
Then there exists $n \in \mathbb{Z}^+$ such that $|f(n)| > 1$.
Since $n \in \mathbb{Z}^+$, then $n \geq 1$.
Since $n \geq 1 > 0$, then $n > 0$.
Since $n \geq 1$, then $n + 1 \geq 2 > 0$, so $n + 1 > 0$.
Thus, $1 < |f(n)| = |\frac{n}{n+1}| = \frac{|n|}{|n+1|} = \frac{n}{n+1}$, so $1 < \frac{n}{n+1}$.
Since $n + 1 > 0$, then we multiply by $n + 1$ to obtain $n + 1 < n$.
We subtract n to obtain $1 < 0$, a contradiction.
Therefore, f is bounded. $\square$

Exercise 15. Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by $f(x) = 1 + 2|x|$.
A. The image of f is the interval $[1, \infty)$.
B. The pre-image of 5 is the set $\{-2, 2\}$.
C. The inverse image of the interval $(0, 5)$ is the interval $(-2, 2)$.

Proof. We first prove the statement : The image of f is the interval $[1, \infty)$.
Let S be the image of f and let T be the interval $[1, \infty)$.
Then $S = f(\mathbb{R}) = \{f(x) : x \in \mathbb{R}\}$ and $T = [1, \infty)$.
We must prove $S = T$.

We first prove $S \subset T$.
Let $y \in S$.
Then $y = f(x)$ for some $x \in \mathbb{R}$.
Since $x \in \mathbb{R}$, then $|x| \geq 0$, so $2|x| \geq 0$.
Thus, $y = 1 + 2|x| \geq 1$, so $y \geq 1$.
Hence, $y \in [1, \infty)$, so $S \subset [1, \infty)$.
Therefore, $S \subset T$.

We next prove $T \subset S$.
Let $y \in T$.
Then $y \geq 1$.
Let $x = \frac{y-1}{2}$.
Then $2x = y - 1$, so $2x + 1 = y$.
Since $y \geq 1$, then $y - 1 \geq 0$, so $\frac{y-1}{2} \geq 0$.
Hence, $x \geq 0$, so $|x| = x$.
Thus, $y = 2x + 1 = 2|x| + 1 = 1 + 2|x| = f(x)$.
Therefore, there is a real number x such that $y = f(x)$, so $y \in S$.
Since $y \in T$ implies $y \in S$, then $T \subset S$.

Since $S \subset T$ and $T \subset S$, then $S = T$, as desired. □

Proof. We prove the statement: The pre-image of 5 is the set $\{-2, 2\}$.

Let A be the pre-image of 5 and let B be the set $\{-2, 2\}$.

Then $A = f^{-1}(5) = \{x \in \mathbb{R} : f(x) = 5\}$ and $B = \{-2, 2\}$.

We must prove $A = B$.

We first prove $B \subset A$.

Since $f(2) = 1 + 2|2| = 1 + 2 * 2 = 5$, then $2 \in A$.

Since $f(-2) = 1 + 2|-2| = 1 + 2 * 2 = 5$, then $-2 \in A$.

Since $2 \in A$ and $-2 \in A$, then every element of set B is contained in set A, so $B \subset A$.

We next prove $A \subset B$.

Let $x \in A$.

Then $x \in \mathbb{R}$ and $f(x) = 5$.

Observe that

$$\begin{aligned} f(x) = 5 \quad &\Leftrightarrow \quad 1 + 2|x| = 5 \\ &\Leftrightarrow \quad 2|x| = 4 \\ &\Leftrightarrow \quad |x| = 2 \\ &\Leftrightarrow \quad x = 2 \text{ or } x = -2 \\ &\Leftrightarrow \quad x \in \{-2, 2\}. \end{aligned}$$

Since $f(x) = 5$ and $f(x) = 5$ if and only if $x \in \{-2, 2\}$, then $x \in \{-2, 2\}$, so $x \in B$.

Hence, $x \in A$ implies $x \in B$, so $A \subset B$.

Since $A \subset B$ and $B \subset A$, then $A = B$, as desired. □

Proof. We prove the statement: The inverse image of the interval $(0, 5)$ is the interval $(-2, 2)$.

Let S be the inverse image of the interval $(0, 5)$ and let T be the interval $(-2, 2)$.

Then $S = f^{-1}((0, 5)) = \{x \in \mathbb{R} : f(x) \in (0, 5)\}$ and $T = (-2, 2)$.

We must prove $S = T$.

We first prove $S \subset T$.

Let $x \in S$.

Then $x \in \mathbb{R}$ and $f(x) \in (0, 5)$.

Observe that

$$\begin{aligned} f(x) \in (0,5) &\Leftrightarrow 1+2|x| \in (0,5) \\ &\Leftrightarrow 0 < 1+2|x| < 5 \\ &\Leftrightarrow -1 < 2|x| < 4 \\ &\Leftrightarrow \frac{-1}{2} < |x| < 2 \\ &\Leftrightarrow \frac{-1}{2} < |x| \text{ and } |x| < 2. \end{aligned}$$

Since $|x| \geq 0 > \frac{-1}{2}$, then $|x| > \frac{-1}{2}$ is always true, so $|x| < 2$.
Hence, $-2 < x < 2$, so $x \in (-2,2)$.
Thus, $x \in T$.
Since $x \in S$ implies $x \in T$, then $S \subset T$.

We next prove $T \subset S$.
Let $x \in T$.
Then $x \in (-2,2)$, so $-2 < x < 2$.
Hence, $|x| < 2$.
Since $|x| \geq 0 > \frac{-1}{2}$, then $|x| > \frac{-1}{2}$, so $\frac{-1}{2} < |x| < 2$.
Observe that

$$\begin{aligned} \frac{-1}{2} < |x| < 2 &\Leftrightarrow -1 < 2|x| < 4 \\ &\Leftrightarrow 0 < 1+2|x| < 5 \\ &\Leftrightarrow 0 < f(x) < 5 \\ &\Leftrightarrow f(x) \in (0,5). \end{aligned}$$

Thus, $f(x) \in (0,5)$, so $x \in S$.
Since $x \in T$ implies $x \in S$, then $T \subset S$.
Since $S \subset T$ and $T \subset S$, then $S = T$, as desired. □

Exercise 16. Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by $f(x) = x^2$.
A. The pre-image of 4 is the set $\{-2, 2\}$.
B. The inverse image of the closed interval $[4,5]$ is the set $[-\sqrt{5}, -2] \cup [2, \sqrt{5}]$.

Proof. We prove: The pre-image of 4 is the set $\{-2, 2\}$.
Let A be the pre-image of 4 and let B be the set $\{-2, 2\}$.
Then $A = f^{-1}(4) = \{x \in \mathbb{R} : f(x) = 4\}$ and $B = \{-2, 2\}$.
We must prove $A = B$.

We first prove $B \subset A$.
Since $2 \in \mathbb{R}$ and $f(2) = 2^2 = 4$, then $2 \in A$.
Since $-2 \in \mathbb{R}$ and $f(-2) = (-2)^2 = 4$, then $-2 \in A$.
Since $2 \in A$ and $-2 \in A$, then every element of B is contained in A, so $B \subset A$.

We next prove $A \subset B$.

Let $x \in A$.

Then $x \in \mathbb{R}$ and $f(x) = 4$.

Observe that

$$\begin{aligned} f(x) = 4 &\Leftrightarrow x^2 = 4 \\ &\Leftrightarrow (x^2 - 4) = 0 \\ &\Leftrightarrow (x+2)(x-2) = 0 \\ &\Leftrightarrow x + 2 = 0 \text{ or } x - 2 = 0 \\ &\Leftrightarrow x = -2 \text{ or } x = 2 \\ &\Leftrightarrow x \in \{-2, 2\}. \end{aligned}$$

Since $f(x) = 4$ and $f(x) = 4$ if and only if $x \in \{-2, 2\}$, then $x \in \{-2, 2\}$, so $x \in B$.

Thus, $x \in A$ implies $x \in B$, so $A \subset B$.

Since $A \subset B$ and $B \subset A$, then $A = B$, as desired. ☐

Proof. We next prove the statement: The inverse image of the closed interval $[4, 5]$ is the set $[-\sqrt{5}, -2] \cup [2, \sqrt{5}]$.

Let S be the inverse image of the interval $[4, 5]$ and let T be the set $[-\sqrt{5}, -2] \cup [2, \sqrt{5}]$.

Then $S = f^{-1}[4, 5] = \{x \in \mathbb{R} : f(x) \in [4, 5]\}$ and $T = [-\sqrt{5}, -2] \cup [2, \sqrt{5}]$.

We must prove $S = T$.

We first prove $T \subset S$.

Let $x \in T$.

Then either $x \in [-\sqrt{5}, -2]$ or $x \in [2, \sqrt{5}]$.

We consider these cases separately.

Case 1: Suppose $x \in [2, \sqrt{5}]$.

Then $2 \leq x \leq \sqrt{5}$, so $0 < 2 \leq x \leq \sqrt{5}$.

Thus, $4 \leq x^2 \leq 5$, so $x^2 \in [4, 5]$.

Therefore, $f(x) \in [4, 5]$, so $x \in S$.

Case 2: Suppose $x \in [-\sqrt{5}, -2]$.

Then $-\sqrt{5} \leq x \leq -2$, so $-\sqrt{5} \leq x \leq -2 < 0$.

Thus, $(-\sqrt{5})^2 \geq x^2 \geq (-2)^2$, so $5 \geq x^2 \geq 4$.

Hence, $4 \leq x^2 \leq 5$, so $x^2 \in [4, 5]$.

Thus, $f(x) \in [4, 5]$, so $x \in S$.

In either case, $x \in S$, so $x \in T$ implies $x \in S$.

Therefore, $T \subset S$.

We next prove $S \subset T$.

Let $x \in S$.

Then $x \in \mathbb{R}$ and $f(x) \in [4, 5]$, so $x^2 \in [4, 5]$.

Hence, $4 \leq x^2 \leq 5$.

Since $0 < 4 \leq x^2 \leq 5$, then $2 \leq |x| \leq \sqrt{5}$, so $2 \leq |x|$ and $|x| \leq \sqrt{5}$.

Since $|x| \leq \sqrt{5}$, then $-\sqrt{5} \leq x \leq \sqrt{5}$, so $x \in [-\sqrt{5}, \sqrt{5}]$.
Since $|x| \geq 2$, then either $x \geq 2$ or $x \leq -2$.
We consider these cases separately.
Case 1: Suppose $x \geq 2$.
Then $x \in [2, \infty)$.
Since $x \in [2, \infty)$ and $x \in [-\sqrt{5}, \sqrt{5}]$, then $x \in [2, \infty) \cap [-\sqrt{5}, \sqrt{5}] = [2, \sqrt{5}]$.
Case 2: Suppose $x \leq -2$.
Then $x \in (-\infty, -2]$.
Since $x \in (-\infty, -2]$ and $x \in [-\sqrt{5}, \sqrt{5}]$, then $x \in (-\infty, -2] \cap [-\sqrt{5}, \sqrt{5}] = [-\sqrt{5}, -2]$.

Hence, either $x \in [2, \sqrt{5}]$ or $x \in [-\sqrt{5}, -2]$, so $x \in [2, \sqrt{5}] \cup [-\sqrt{5}, -2] = [-\sqrt{5}, -2] \cup [2, \sqrt{5}]$.
Thus, $x \in T$.
Since $x \in S$ implies $x \in T$, then $S \subset T$.
Since $S \subset T$ and $T \subset S$, then $S = T$, as desired. ∎

Exercise 17. Let $f : (0, 4) \to \mathbb{R}$ be the function defined by $f(x) = \sqrt{x}$.
Then f is increasing and f is bounded.

Proof. We first prove f is increasing.
Let $a, b \in (0, 4)$ such that $a < b$.
Since $a \in (0, 4)$, then $0 < a < 4$, so $0 < a$.
Since $0 < a$ and $a < b$, then $0 < a < b$, so $0 < \sqrt{a} < \sqrt{b}$.
Thus, $\sqrt{a} < \sqrt{b}$, so $f(a) < f(b)$.
Therefore, f is increasing. ∎

Proof. We prove f is bounded.
Let $x \in (0, 4)$.
Then $0 < x < 4$, so $0 < \sqrt{x} < 2$.
Hence, $0 < \sqrt{x}$ and $\sqrt{x} < 2$.
Since $\sqrt{x} > 0$, then $|\sqrt{x}| = \sqrt{x}$.
Thus, $|f(x)| = |\sqrt{x}| = \sqrt{x} < 2 \leq 2$, so $|f(x)| \leq 2$.
Therefore, $|f(x)| \leq 2$ for all $x \in (0, 4)$, so f is bounded. ∎

Exercise 18. Any real valued function on a finite set is bounded.

Proof. Let A be a finite set.
Let $f : A \to \mathbb{R}$ be a function.
To prove f is bounded, we must prove the range of f is a bounded set.
Let S be the range of f.
Then $S = \{f(x) \in \mathbb{R} : x \in A\}$.
We must prove the set S is bounded.
Since A is a finite set, then there are n elements in A for some integer $n \geq 0$.
Since f is a function, then for every $x \in A$, there is exactly one $f(x) \in \mathbb{R}$, so there are n elements in S.
Hence, S is a finite set of n real numbers.

Since $n \geq 0$, then either $n > 0$ or $n = 0$.

We consider these cases separately.

Case 1: Suppose $n > 0$.

Then S contains at least one real number.

Since the elements of S are ordered, let $S = \{a_i \in \mathbb{R} : a \in \{1, .., n\}\}$ and $a_1 \leq ..., \leq a_n$.

Since $a_1 \in S$ and $a_1 \leq a_i$ for every $i \in \{1, ..., n\}$, then a_1 is the least element of S, so $s \geq a_1$ for all $s \in S$.

Since $a_n \in S$ and $a_i \leq a_n$ for every $i \in \{1, ..., n\}$, then a_n is the greatest element of S, so $a_n \geq s$ for all $s \in S$.

Thus, $a_1 \leq s \leq a_n$ for every $s \in S$, so S is bounded.

Case 2: Suppose $n = 0$.

Then S contains no real numbers, so $S = \emptyset$.

Since every real number is both an upper and lower bound for the empty set, then $\emptyset$ is bounded in $\mathbb{R}$, so S is bounded.

In all cases, we conclude the set S is bounded, so the function f must be bounded. $\square$