Real valued functions Notes

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May 25, 2023

Sets of Numbers

$\mathbb{R} =$ set of all real numbers
$\mathbb{R}^+ = \{x \in \mathbb{R} : x > 0\} = (0, \infty) =$ set of all positive real numbers
$\mathbb{R}^* = \{x \in \mathbb{R} : x \neq 0\} = (-\infty, 0) \cup (0, \infty) =$ set of all nonzero real numbers

Real valued functions of a real variable

Definition 1. real valued function
A **real valued function** is a function f such that $rngf \subset \mathbb{R}$.

Definition 2. real valued function of a real variable
A **real valued function of a real variable** is a real valued function f such that $domf \subset \mathbb{R}$.

Example 3. identity function
Let $I : \mathbb{R} \to \mathbb{R}$ be a function defined by $I(x) = x$ for all $x \in \mathbb{R}$.
Then I is called the **identity function**.

Example 4. reciprocal function
Let $f : \mathbb{R}^* \to \mathbb{R}$ be a function defined by $f(x) = \frac{1}{x}$ for all $x \neq 0$.
Then f is called the **reciprocal function**.

Example 5. square root function
Let $f : \mathbb{R}_+ \to \mathbb{R}$ be a function defined by $f(x) = \sqrt{x}$ for all $x \geq 0$.
Then f is called the **square root function**.

Example 6. absolute value function
Let $f : \mathbb{R} \to \mathbb{R}$ be a function defined by $f(x) = |x|$ for all $x \in \mathbb{R}$.
Then f is called the **absolute value function**.

Example 7. greatest integer function
Let $x \in \mathbb{R}$.
Let $\lfloor x \rfloor$ denote the greatest integer n such that $n \leq x$.
Then $\lfloor x \rfloor = \max\{n \in \mathbb{Z} : n \leq x\}$.
Let $f : \mathbb{R} \to \mathbb{R}$ be a function defined by $f(x) = \lfloor x \rfloor$ for all $x \in \mathbb{R}$.
Then f is called the **greatest integer function**.

Example 8. rational test function

Let $f : \mathbb{R} \to \mathbb{R}$ be a function defined by

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ -1 & \text{if } x \notin \mathbb{Q} \end{cases}$$

for all $x \in \mathbb{R}$.

Then f is called the **rational test function**.

Definition 9. zero of a function

Let f be a real valued function.

An element $x \in dom f$ is a **zero of** f iff $f(x) = 0$.

Definition 10. even/odd function

Let f be a real valued function of a real variable. Then f is said to be

i. **even** iff $f(-x) = f(x)$ for all $x \in dom f$.

ii. **odd** iff $f(-x) = -f(x)$ for all $x \in dom f$.

An even function has line symmetry about the y axis.

An odd function has point symmetry about the origin.

Definition 11. increasing/decreasing function

Let f be a real valued function of a real variable.

Let S be a subset of the domain of f.

Then f is said to be

i. **constant on** S iff $(\forall a, b \in S)[f(a) = f(b)]$.

ii. **increasing on** S iff $(\forall a, b \in S)[a < b \to f(a) < f(b)]$.

iii. **decreasing on** S iff $(\forall a, b \in S)[a < b \to f(a) > f(b)]$.

Definition 12. monotonic function in $\mathbb{R}$

Let f be a real valued function of a real variable.

Let $S \subset dom f$.

Then f is said to be

i. **strictly increasing on** S iff $a < b$ implies $f(a) < f(b)$ for all $a, b \in S$.

ii. **(monotonic) increasing on** S iff $a < b$ implies $f(a) \leq f(b)$ for all $a, b \in S$.

iii. **strictly decreasing on** S iff $a < b$ implies $f(a) > f(b)$ for all $a, b \in S$.

iv. **(monotonic) decreasing on** S iff $a < b$ implies $f(a) \geq f(b)$ for all $a, b \in S$.

v. **monotonic** iff f is either monotonic increasing or monotonic decreasing.

Definition 13. monotonic function

Let f be a real valued function of a real variable.

Then f is **monotonic** iff f is either nondecreasing or non increasing on an interval of its domain.

We say that f is **strictly monotonic on an interval of its domain** iff f is either increasing on the entire interval or decreasing on the entire interval.

Theorem 14. *Strictly monotonic functions are injective.*

Let f be a real valued function of a real variable.
Let S be a subset of the domain of f.
1. If f is strictly increasing on S, then f is one to one on S.
2. If f is strictly decreasing on S, then f is one to one on S.

Let f be a real valued function of a real variable.
Let $domf$ be the domain of f.
If f is strictly increasing on $domf$, then f is one to one on $domf$.
If f is strictly decreasing on $domf$, then f is one to one on $domf$.
Therefore, a strictly increasing function is one to one and a strictly decreasing function is one to one.

Definition 15. bounded real valued function

A real valued function f is said to be
i. **bounded above** iff the range of f is bounded above in $\mathbb{R}$.
ii. **bounded below** iff the range of f is bounded below in $\mathbb{R}$.
iii. **bounded** iff the range of f is a bounded set in $\mathbb{R}$.
iv. **unbounded** iff f is not bounded.

Let f be a real valued function.
Let $rng(f)$ be the range of f.
Then $rng(f) = \{f(x) \in \mathbb{R} : x \in dom(f)\}$.
Hence, $rng(f) = \{f(x) : x \in dom(f)\}$.
A set S is bounded above in $\mathbb{R}$ iff there exists $M \in \mathbb{R}$ such that $x \leq M$ for all $x \in S$.
Hence, f is bounded above iff there exists $M \in \mathbb{R}$ such that $f(x) \leq M$ for all $x \in dom(f)$.
Therefore, f is bounded above iff $(\exists M \in \mathbb{R})(\forall x \in dom(f))(f(x) \leq M)$.
Similarly, f is bounded below iff $(\exists m \in \mathbb{R})(\forall x \in dom(f))(m \leq f(x))$.
A set S is bounded in $\mathbb{R}$ iff S is bounded above and below in $\mathbb{R}$.
Hence, a set S is bounded in $\mathbb{R}$ iff there exist $m, M \in \mathbb{R}$ such that $m \leq x \leq M$ for all $x \in S$.
Thus, f is bounded in $\mathbb{R}$ iff there exist $m, M \in \mathbb{R}$ such that $m \leq f(x) \leq M$ for all $x \in dom(f)$.
Therefore, f is bounded in $\mathbb{R}$ iff $(\exists m, M \in \mathbb{R})(\forall x \in dom(f))(m \leq f(x) \leq M)$.
Equivalently, a set S is bounded in $\mathbb{R}$ iff $(\exists b \in \mathbb{R})(\forall x \in S)(|x| \leq b)$.
Therefore, f is bounded in $\mathbb{R}$ iff $(\exists b \in \mathbb{R})(\forall x \in dom(f))(|f(x)| \leq b)$.
Therefore, f is unbounded in $\mathbb{R}$ iff $(\forall b \in \mathbb{R})(\exists x \in dom(f))(|f(x)| > b)$.

Suppose f is a bounded real valued function.
Then there exists $b \in \mathbb{R}$ such that $|f(x)| \leq b$ for all $x \in dom(f)$.
Hence, there exists $b > 0$ such that $|f(x)| < b$ for all $x \in dom(f)$.

Extrema of a real valued function

Definition 16. maxima/minima of a function

Let f be a real valued function.

If there exists $c \in domf$ such that $f(c) \geq f(x)$ for all $x \in domf$, then we say $f(c)$ is a **maximum of f on** $domf$.

Equivalently, we say $f(c)$ is an **absolute maximum of f on** $domf$.

If there exists $c \in domf$ such that $f(c) \leq f(x)$ for all $x \in domf$, then we say $f(c)$ is a **minimum of f on** $domf$.

Equivalently, we say $f(c)$ is an **absolute minimum of f on** $domf$.

A function value is an **extreme value** iff it is either a maximum or minimum value.

Definition 17. relative maxima/minima of a function

Let $E \subset \mathbb{R}$.

Let $f : E \to \mathbb{R}$ be a function.

Let $c \in E$.

Then $f(c)$ is a **relative maximum of f** iff there exists $\delta > 0$ such that $N(c;\delta) \subset E$ and $f(c) \geq f(x)$ for all $x \in N(c;\delta)$.

Then $f(c)$ is a **relative minimum of f** iff there exists $\delta > 0$ such that $N(c;\delta) \subset E$ and $f(c) \leq f(x)$ for all $x \in N(c;\delta)$.

Therefore, $f(c)$ is a relative maximum of f on E iff there is a δ neighborhood of c in E such that $f(c) \geq f(x)$ for every $x \in N(c;\delta)$.

Therefore, $f(c)$ is a relative minimum of f on E iff there is a δ neighborhood of c in E such that $f(c) \leq f(x)$ for every $x \in N(c;\delta)$.

Curves in $\mathbb{R}^2$

Definition 18. symmetry of curves

Let $C \subset \mathbb{R}^2$.

The curve C is **symmetric with respect to the x axis** iff $(\forall x \in \mathbb{R})(\forall y \in \mathbb{R})[(x, y) \in C \to (x, -y) \in C]$.

The curve C is **symmetric with respect to the y axis** iff $(\forall x \in \mathbb{R})(\forall y \in \mathbb{R})[(x, y) \in C \to (-x, y) \in C]$.

The curve C is **symmetric with respect to the origin** iff $(\forall x \in \mathbb{R})(\forall y \in \mathbb{R})[(x, y) \in C \to (-x, -y) \in C]$.

The curve C is **symmetric with respect to the point** (h, k) iff $(\forall x \in \mathbb{R})(\forall y \in \mathbb{R})[(x + h, y + k) \in C \to (-x + h, -y + k) \in C]$.

Algebra of real valued functions

Definition 19. function addition

Let f and g be real valued functions.

Let $f + g$ be the function defined by $(f + g)(x) = f(x) + g(x)$ for all $x \in domf \cap domg$.

We call $f + g$ the **sum of f and g**.

The domain of $f + g$ is the set $domf \cap domg$.
Therefore, $dom(f + g) = domf \cap domg$.

Definition 20. Let f be a real valued function.
Let $-f$ be the function defined by $(-f)(x) = -f(x)$ for all $x \in domf$.

The domain of $-f$ is the set $domf$.
Therefore, $dom(-f) = domf$.

Definition 21. function subtraction
Let f and g be real valued functions.
Let $f - g$ be the function defined by $f - g = f + (-g)$.
We call $f - g$ the **difference of f and g**.

Let f and g be real valued functions.
Then $(f - g)(x) = (f + (-g))(x) = f(x) + (-g)(x) = f(x) - g(x)$.
The domain of $f - g$ is the set $domf \cap dom(-g) = domf \cap domg$.
Therefore, $dom(f - g) = domf \cap domg$.
Therefore, $f - g$ is the function defined by $(f - g)(x) = f(x) - g(x)$ for all $x \in domf \cap domg$.

Definition 22. function multiplication
Let f and g be real valued functions.
Let fg be the function defined by $(fg)(x) = f(x)g(x)$ for all $x \in domf \cap domg$.
We call fg the **product of f and g**.

The domain of fg is the set $domf \cap domg$.
Therefore, $dom(fg) = domf \cap domg$.

Definition 23. Let f be a real valued function.
Let $\frac{1}{f}$ be the function defined by $(\frac{1}{f})(x) = \frac{1}{f(x)}$ for all $x \in domf$ such that $f(x) \neq 0$.

The domain of $\frac{1}{f}$ is the set $\{x \in domf : f(x) \neq 0\}$.
Therefore, $dom\frac{1}{f} = \{x \in domf : f(x) \neq 0\}$.

Definition 24. function division
Let f and g be real valued functions.
Let $\frac{f}{g}$ be the function defined by $\frac{f}{g} = f \cdot \frac{1}{g}$.
We call $\frac{f}{g}$ the **quotient of f and g**.

Let f and g be real valued functions.
Then $(\frac{f}{g})(x) = (f\frac{1}{g})(x) = f(x)(\frac{1}{g})(x) = f(x)\frac{1}{g(x)} = \frac{f(x)}{g(x)}$.
The domain of $\frac{f}{g}$ is the set $domf \cap dom\frac{1}{g} = domf \cap \{x \in domg : g(x) \neq 0\}$.
Therefore, $dom\frac{f}{g} = domf \cap \{x \in domg : g(x) \neq 0\}$.
Therefore, $\frac{f}{g}$ is the function defined by $(\frac{f}{g})(x) = \frac{f(x)}{g(x)}$ for all $x \in domf \cap domg$ such that $g(x) \neq 0$.

Definition 25. function scalar multiplication

Let λ be a constant.

Let f be a real valued function.

Let λf be the function defined by $(\lambda f)(x) = \lambda f(x)$ for all $x \in domf$.

We call λf the **scalar multiple of λ and f**.

The domain of λf is the set $domf$.

Therefore, $dom(\lambda f) = domf$.

Propositions involving algebra of functions

Proposition 26. *Let f and g each be real valued functions of a real variable.*

1. If f and g are even, then $f + g$, $f - g$, fg, and $g \circ f$ are even.

2. If f and g are odd, then $f + g$, $f - g$, and $g \circ f$ are odd and fg is even.

Classes of real valued functions

Definition 27. constant function

A function f is a **constant function** iff there exists a number k such that $f(x) = k$ for all $x \in domf$.

Let $k \in \mathbb{R}$.

Let $f : \mathbb{R} \to \mathbb{R}$ be the constant function defined by $f(x) = k$ for all $x \in \mathbb{R}$.

The graph of f is the constant linear function $f(x) = k$ with slope 0.

The distance between any function values is zero because all of the terms are equal.

Thus, if $x_1, x_2 \in \mathbb{R}$, then $d(f(x_1), f(x_2)) = d(k, k) = |k - k| = 0$.

Definition 28. polynomial function

A function p of a real variable is a **polynomial function** iff there exist a nonnegative integer n and real numbers $a_0, a_1, ..., a_n$ such that $p(x) = \sum_{k=0}^{n} a_k x^k$ for all $x \in domp$.

If $a_n \neq 0$, we say n is the **degree of the polynomial** p.

The numbers a_i are the **coefficients** and a_n is the **leading coefficient** and a_0 is the **constant term**.

Example 29. Let p be a polynomial function of a real variable.

Then there exist a nonnegative integer n and real numbers $a_0, a_1, ..., a_n$ such that $p(x) = a_0 + a_1 x + a_2 x^2 + ... + a_n x^n$.

The domain of p is $\mathbb{R}$.

Therefore, $domp = \mathbb{R}$, so the domain of any polynomial function is $\mathbb{R}$.

Let f be a polynomial function with degree $n = 0$.

Then $f(x) = \sum_{k=0}^{0} a_k x^k = a_0 x^0 = a_0$, so $f(x) = a_0$ and f is a constant function.

Therefore, every constant function is a polynomial function.

Let f be a polynomial function of degree $n = 1$.

Then $f(x) = \sum_{k=0}^{1} a_k x^k = a_0 x^0 + a_1 x^1 = a_0 + a_1 x$ and $a_1 \neq 0$, so $f(x) = a_1 x + a_0$ and f is a linear function.

Therefore, f is a linear polynomial.

Let f be a polynomial function of degree $n = 2$.

Then $f(x) = \sum_{k=0}^{2} a_k x^k = a_0 x^0 + a_1 x^1 + a_2 x^2 = a_0 + a_1 x + a_2 x^2$ and $a_2 \neq 0$, so $f(x) = a_2 x^2 + a_1 x + a_0$ and f is a quadratic function.

Therefore, f is a quadratic polynomial.

Let f be a polynomial function of degree $n = 3$.

Then $f(x) = \sum_{k=0}^{3} a_k x^k = a_0 x^0 + a_1 x^1 + a_2 x^2 + a_3 x^3 = a_0 + a_1 x + a_2 x^2 + a_3 x^3$ and $a_3 \neq 0$, so $f(x) = a_3 x^3 + a_2 x^2 + a_1 x + a_0$ and f is a cubic function.

Therefore, f is a cubic polynomial.

Let f be a polynomial function of degree $n = 4$.

Then $f(x) = \sum_{k=0}^{4} a_k x^k = a_0 x^0 + a_1 x^1 + a_2 x^2 + a_3 x^3 + a_4 x^4 = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4$ and $a_4 \neq 0$, so $f(x) = a_4 x^4 + a_3 x^3 + a_2 x^2 + a_1 x + a_0$ and f is a quartic function.

Therefore, f is a quartic polynomial.

Let f be a polynomial function of degree $n = 5$.

Then $f(x) = \sum_{k=0}^{5} a_k x^k = a_0 x^0 + a_1 x^1 + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5$ and $a_5 \neq 0$, so $f(x) = a_5 x^5 + a_4 x^4 + a_3 x^3 + a_2 x^2 + a_1 x + a_0$ and f is a quintic function.

Therefore, f is a quintic polynomial.

A rational function is a ratio of two polynomial functions.

Definition 30. rational function

A function r is a **rational function** iff there exist polynomial functions p, q such that $dom\ \mathrm{r} = \mathbb{R} - \{x : q(x) = 0\}$ and $r(x) = \frac{p(x)}{q(x)}$ for all $x \in dom\ \mathrm{r}$.